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# $L_\infty$ – ESTIMATE FOR QUALITATIVELY BOUNDED WEAK SOLUTIONS OF NONLINEAR DEGENERATE DIAGONAL PARABOLIC SYSTEMS

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*Abstract.* The Dirichlet problem to nonlinear degenerate diagonal parabolic system with some special right–hand sides but still satisfying the maximal growth conditions is considered. Applying the idea of Stampacchia a $L_\infty$ – estimate in terms of data for a priori bounded weak solutions is found.

**1. Introduction.** In this paper we show only boundedness of qualitatively bounded weak solutions to the following Dirichlet problem for a diagonal parabolic system

$$\begin{aligned} & u_{it} - \operatorname{div}\,(a_i(x,t,u,\nabla u)\cdot\nabla u_i) = b_i(x,t,u,\nabla u) && \text{in } \Omega^T, \\ & u_i|_{t=0} = u_{0i} && \text{in } \Omega, \\ & u_i = u_{bi} && \text{on } S^T, \end{aligned} \tag{1.1}$$

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where $i = 1, \dots, m$, $\Omega \subset \mathbb{R}^n$, $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$, $S$ is the boundary of $\Omega$, and dot denotes the scalar product in $\mathbb{R}^n$.

We assume the following growth conditions

$$a_i(x,t,u,\nabla u) \cdot \nabla u_i \cdot \nabla u_i \geq \alpha_0 |\nabla u|^{p-2} |\nabla u_i|^2 - \phi_{1i}(x,t)\,, \tag{1.2}$$

$$b_i(x,t,u,\nabla u) \leq \beta_0 |\nabla u|^{p-2} |\nabla u_i|^2 + \phi_{2i}(x,t)\,,$$

where $i = 1, \dots, m$, $\alpha_0$, $\beta_0$ are positive constants, and $\phi_{1i}$, $\phi_{2i}$ are positive functions.

By a bounded weak solution of (1.1) we mean a function

$$u \in L_\infty(0,T; L_2(\Omega; \mathbb{R}^m)) \cap L_p(0,T; W_p^1(\Omega; \mathbb{R}^m)) \cap L_\infty(\Omega^T; \mathbb{R}^m)$$

satisfying the integral identity

$$\sum_{i=1}^m \int_{\Omega^{T-h}} (u_{iht}\phi_i + (a_i \nabla u_i)_h \cdot \nabla \phi_i - b_{ih}\phi_i)\, dx\, dt = 0\,, \tag{1.3}$$

which holds for any $\phi \in L_p(0,T; \overset{\circ}{W}{}_p^1(\Omega; \mathbb{R}^m)) \cap L_\infty(\Omega^T; \mathbb{R}^m)$, where $u_h = \frac{1}{h}\int_t^{t+h} u(x,\tau)\, d\tau$ is the Steklov average.

Applying the idea of Stampacchia for one equation (see [1, Ch.5, Sect.17]) an $L_\infty$ – estimate in terms of data is found for a priori qualitatively bounded weak solutions to nonlinear degenerate diagonal parabolic systems. For nondegenerate diagonal parabolic system such result is proved in [4], where additionally the Hölder continuity and global existence are proved.

By $W_p^1(Q, \mathbb{R}^m)$, $L_p(Q; \mathbb{R}^m)$ we denote the Sobolev spaces for functions with values in $\mathbb{R}^m$. We denote also $|u|_{p,Q} = \| u \|_{L_p(Q)}$.

**2. Boundedness of solutions.** First we prove

**Lemma 2.1.** *Assume that $u_0 \in L_\infty(\Omega; \mathbb{R}^m)$, $u_b \in L_\infty(S^T; \mathbb{R}^m)$, $u \in L_q(\Omega^T; \mathbb{R}^m)$, $q \geq 1$. Assume that $u$ is a qualitatively bounded weak solution of (1.3). Assume the growth conditions (1.2). Let $\phi_1 \in L_{p_1}(\Omega^T; \mathbb{R}^m)$, $\phi_2 \in L_{p_2}(\Omega^T; \mathbb{R}^m)$, $p_1 > \frac{n+p}{p}$, $p_2 > \frac{1}{1 - \frac{n}{p(n+2)} - \frac{n}{2(n+p)}}$, $p \geq 2$.*

*Then there exists a positive increasing function $F_1$ such that*

$$|u_i|_{\infty,\Omega^T} \leq F_1\left(|\phi_{1i}|_{p_1,\Omega^T}, |\phi_{2i}|_{p_2,\Omega^T}\right) \max\left\{|u_{0i}|_{\infty,\Omega}\,,\ |u_{bi}|_{\infty,\Omega^T}\,,\ |u_i|_{q,\Omega^T}\right\} \tag{2.1}$$

*where $i = 1, \dots, m$, $q \geq 1$.*

*Proof.* Assume that $M$ is the essential supremum of $u_i$ in $\Omega^T$, $i=1,\dots,m$, and $u_i$, $i=1,\dots,m$, are nonnegative. We put the testing function $(u_{ih}-k)_+$, $i=1,\dots,m$, where $k<M$ into (1.3). Performing calculations in the first term and passing with $h$ to zero we get

$$\int_\Omega \frac{1}{2}(u_{ih}-k)_+^2\,dx\Big|_{t=0}^{t=t}+\int_{\Omega^t} a_i\cdot\nabla(u_{ih}-k)_+\cdot\nabla(u_{ih}-k)_+\,dx\,dt \tag{2.2}$$
$$-\int_{\Omega^t} b_i(u_{ih}-k)_+\,dx\,dt=0\,,\quad i=1,\dots,m\,,$$

where the sum was dropped because we took as testing functions

$$\phi=(\phi_i,\dots,\phi_m)$$

such that $\phi_j=0$, $j\neq i$, $\phi_i=(u_{ih}-k)_+$. Moreover, we used that $k>u_{bi}$, $i=1,\dots,m$.

In view of the growth conditions (1.2) we have

$$\frac{1}{2}\int_\Omega (u_i-k)_+^2\,dx+\alpha_0\int_{\Omega^t}|\nabla u|^{p-2}\,|\nabla(u_i-k)_+|^2\,dx\,dt\le \tag{2.3}$$
$$\le\beta_0\int_{\Omega^t}|\nabla u|^{p-2}\,|\nabla(u_i-k)_+|^2(u_i-k)_+\,dx\,dt+$$
$$+\int_{A^+_{k,i}}\phi_{1i}\,dx\,dt+\int_{\Omega^t}\phi_{2i}\,(u_i-k)_+\,dx\,dt\,,\quad i=1,\dots,m\,,$$

where we used that $k>u_{0i}$, $i=1,\dots,m$, and

$$A^+_{k,i}=\Big\{(x,t)\in\Omega^T: u_i(x,t)>k\Big\}\,. \tag{2.4}$$

In view of the Hölder inequality the second term on the r.h.s. of (2.3) is bounded by

$$|\phi_{1i}|_{p_1,\Omega^T}|A^+_{k,i}|^{1/p_1'}\,, \tag{2.5}$$

where $\frac{1}{p_1}+\frac{1}{p_1'}=1$, $|A^+_{k,i}|=\operatorname{meas}A^+_{k,i}$, and the last term on the r.h.s. of (2.3) is estimated by

$$|\phi_{2i}|_{p_2,\Omega^T}\left(\int_{\Omega^t}(u_{ih}-k)_+^{p_2'}\right)^{1/p_2'}\equiv I_1\,,$$

where $\frac{1}{p_2}+\frac{1}{p_2'}=1$. Continuing, we have

$$I_1\le|\phi_{2i}|_{p_2,\Omega^T}|A^+_{k,i}|^{1/\lambda_1 p_2'}\left(\int_{A^+_{k,i}}(u_i-k)^{\lambda_2 p_2'}\,dx\,dt\right)^{1/\lambda_2 p_2'}\equiv I_2\,,$$

where $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$. Assuming that $\lambda_2 p_2'=p\frac{n+2}{n}$ and using the imbedding theorem for the parabolic norm (see [1, Ch.1, Sect.3]) we have

$$
\begin{aligned}
I_2 &\le c_1|\phi_{2i}|_{p_2,\Omega^T}|A^+_{k,i}|^{1/\lambda_1 p_2'}\Bigg[\bigg(\operatorname{ess\,sup}_t\int_\Omega (u_i-k)_+^2\,dx\bigg)^{1/2}+ \qquad (2.6)\\
&+\bigg(\int_{\Omega^t}|\nabla(u_i-k)_+|^p\,dx\,dt\bigg)^{1/p}\Bigg]\le\\
&\le c_1|\phi_{2i}|_{p_2,\Omega^T}\bigg[\frac{\varepsilon_1}{2}\operatorname{ess\,sup}_t\int_\Omega (u_i-k)_+^2\,dx+\frac{1}{2\varepsilon_1}|A^+_{k,i}|^{2/\lambda_1 p_2'}\bigg]+\\
&+c_1|\phi_{2i}|_{p_2,\Omega^T}\bigg[\frac{\varepsilon_2^p}{p}\int_{\Omega^t}|\nabla(u_i-k)_+|^p\,dx\,dt+\frac{1}{p'\varepsilon_2^{p'}}|A^+_{k,i}|^{p'/\lambda_1 p_2'}\bigg]\equiv\\
&\equiv I_3\,,
\end{aligned}
$$

where $c_1$ is the constant from the imbedding, $\frac{1}{p}+\frac{1}{p'}=1$, $\varepsilon_1,\ \varepsilon_2\in(0,1)$.

Using (2.5) and (2.6) in (2.3) and assuming that

$$
\varepsilon_1=\frac{1}{c_1|\phi_{2i}|_{p_2,\Omega^T}}\,,\quad \varepsilon_2=\bigg(\frac{p}{2}\frac{\alpha_0}{c_1|\phi_{2i}|_{p_2,\Omega^T}}\bigg)^{1/p}
$$

we obtain

$$
\begin{aligned}
&\frac14\operatorname{ess\,sup}_t\int_\Omega (u_i-k)_+^2\,dx+\frac{\alpha_0}{2}\int_{\Omega^t}|\nabla u|^{p-2}|\nabla(u_i-k)_+|^2\,dx\,dt\le \qquad (2.7)\\
&\le\beta_0\int_{\Omega^t}|\nabla u|^{p-2}|\nabla(u_i-k)_+|^2(u_i-k)_+\,dx\,dt+\\
&+|\phi_{1i}|_{p_1,\Omega^T}|A^+_{k,i}|^{\frac{1}{p_1'}}+\frac{c_1^2}{2}|\phi_{2i}|^2_{p_2,\Omega^T}|A^+_{k,i}|^{\frac{2}{p_2'}-\frac{2n}{p(n+2)}}+\\
&+\Big(c_1|\phi_{2i}|_{p_2,\Omega^T}\Big)^{\frac{p}{p-1}}\frac{p-1}{p}\bigg(\frac{2}{p\alpha_0}\bigg)^{\frac{1}{p-1}}|A^+_{k,i}|^{\frac{p}{(p-1)p_2'}-\frac{n}{(p-1)(n+2)}}\,.
\end{aligned}
$$

Let

$$
\begin{aligned}
\frac14 c_2(p_1,p_2)&=|\phi_{1i}|_{p_1,\Omega^T}+\frac{c_1^2}{2}|\phi_{2i}|^2_{p_2,\Omega^T}+ \qquad (2.8)\\
&+\Big(c_1|\phi_{2i}|_{p_2,\Omega^T}\Big)^{\frac{p}{p-1}}\frac{p-1}{p}\bigg(\frac{2}{p\alpha_0}\bigg)^{\frac{1}{p-1}}\,,
\end{aligned}
$$

and assume that $k = M - 2\varepsilon$ where $2\varepsilon \le \frac{\alpha_0}{4\beta_0}$, and

$$M - 2\varepsilon \ge \sup_i \sup\{|u_{0i}|_{\infty,\Omega}, |u_{bi}|_{\infty,S^T}\}\,, \quad i = 1, \dots, m\,.$$

Then from (2.7) we obtain

$$\text{ess sup}_t \int_\Omega (u_i - k)_+^2\, dx + \alpha_0 \int_{\Omega^t} |\nabla u|^{p-2} |\nabla (u_i - k)_+|^2\, dx\, dt \le \tag{2.9}$$

$$\le c_2 \left( |A^+_{k,i}|^{\frac{1}{p_1'}} + |A^+_{k,i}|^{\frac{2}{p_2'} - \frac{2n}{p(n+2)}} + |A^+_{k,i}|^{\frac{p}{(p-1)p_2'} - \frac{n}{(p-1)(n+2)}} \right) .$$

Consider the sequence of increasing levels

$$k_s = M - \varepsilon - \frac{\varepsilon}{2^s}\,, \quad s = 0, 1, 2, \dots\,, \tag{2.10}$$

and corresponding family of sets

$$A^+_{k_s,i} = \left\{ (x,t) \in \Omega^T : u_i(x,t) > k_s \right\} . \tag{2.11}$$

Using (2.10) and (2.11) we write (2.9) in the form

$$\text{ess sup}_t \int_\Omega (u_i - k_s)_+^2\, dx + \alpha_0 \int_{\Omega^t} |\nabla u|^{p-2} |\nabla (u_i - k_s)_+|^2\, dx\, dt \le \tag{2.12}$$

$$\le c_2 \left( |A^+_{k_s,i}|^{\frac{1}{p_1'}} + |A^+_{k_s,i}|^{\frac{2}{p_2'} - \frac{2n}{p(n+2)}} + |A^+_{k_s,i}|^{\frac{p}{(p-1)p_2'} - \frac{n}{(p-1)(n+2)}} \right) .$$

From the multiplicative inequality (see [1, Ch.1, Sect.3]) we get

$$\left( \frac{\varepsilon}{2^{s+1}} \right)^{p\frac{(n+2)}{n}} |A^+_{k_{s+1},i}| \le \tag{2.13}$$

$$\le \int_{A^+_{k_{s+1},i}} (u_i - k_s)_+^{p\frac{(n+2)}{n}}\, dx\, dt \le \int_{\Omega^T} (u_i - k_s)_+^{p\frac{(n+2)}{n}}\, dx\, dt \le$$

$$\le c_3 \left( \text{ess sup}_t \int_\Omega (u_i - k_s)_+^2\, dx \right)^{p/n} \int_{\Omega^T} |\nabla (u_i - k_s)_+|^p\, dx\, dt\,,$$

where $c_3$ is the constant from the imbedding theorem.

Now from (2.12) and (2.13) we have

$$\left( \frac{\varepsilon}{2^{s+1}} \right)^{p\frac{n+2}{n}} |A^+_{k_{s+1},i}| \le c_4 \left( |A^+_{k_s,i}|^{\frac{1}{p_1'}\left(1+\frac{p}{n}\right)} + \right. \tag{2.14}$$

$$\left. + |A^+_{k_s,i}|^{\left(\frac{2}{p_2'} - \frac{2n}{p(n+2)}\right)\left(1+\frac{p}{n}\right)} + |A^+_{k_s,i}|^{\left(\frac{p}{(p-1)p_2'} - \frac{n}{(p-1)(n+2)}\right)\left(1+\frac{p}{n}\right)} \right) .$$

To apply the standard technique (see [1, Ch.5]) we have to check that exponents in all terms on the r.h.s. of (2.14) are larger than 1. This implies some restrictions on $p_1$ and $p_2$. Considering the first term we have to assume that $\frac{1}{p_1'}\left(1+\frac{p}{n}\right)>1$, so

$$p_1 > \frac{n+p}{p}\,. \tag{2.15}$$

In the second term we get the restriction

$$\left(\frac{2}{p_2'} - \frac{2n}{p(n+2)}\right)\left(1+\frac{p}{n}\right) > 1\,, \tag{2.16}$$

which is satisfied if

$$0 < \frac{1}{p_2} < 1 - \frac{n}{p(n+2)} - \frac{n}{2(n+p)}\,. \tag{2.17}$$

The above inequality implies another restriction

$$f(p) := \frac{1}{p(n+2)} + \frac{1}{2(n+p)} < \frac{1}{n}\,. \tag{2.18}$$

For $p=2$ we have that $f(2)=\frac{1}{n+2}$ so (2.18) holds. Moreover, $f'(p)<0$, so (2.18) is satisfied for $p\ge 2$.

However, we shall not consider the case $p<2$, we examine the inequality (2.18) for $p\in[1,2)$ also. Put $p=1$ into (2.18). Then we get

$$\frac{1}{1+\frac{2}{n}} + \frac{1}{2\left(1+\frac{1}{n}\right)} < 1\,.$$

The above inequality is satisfied for $n\le 3$. For $n\to\infty$ the l.h.s. tends to $\frac{3}{2}$. Therefore for $n>3$ there exists $p=p(n)\in(1,2)$ such that

$$\frac{1}{p(n+2)} + \frac{1}{2(n+p)} = \frac{1}{n}\,.$$

Hence (2.16) does not hold for $p\le p(n)$.

Finally, we consider the exponent in the third term on the r.h.s. of (2.14). The necessary condition takes the form

$$\left(\frac{p}{(p-1)p_2'} - \frac{n}{(p-1)(n+2)}\right)\left(1+\frac{p}{n}\right) > 1\,, \tag{2.19}$$

which is satisfied if

$$0 < \frac{1}{p_2} < 1 - \frac{n}{p(n+2)} - \frac{n(p-1)}{p(n+p)}\,, \tag{2.20}$$

so we have to have that

$$f_0(p) := \frac{1}{p(n+2)} + \frac{p-1}{p(n+p)} < \frac{1}{n}\,. \tag{2.21}$$

Since $f_0(2) = \frac{1}{n+2} < \frac{1}{2}$ and $f_0'(p) < 0$ we see that (2.21) holds for $p \geq 2$, so (2.19) is also valid.

For $p \in (1,2)$ the considerations are more complicated than in the previous case, so we omit them.

Since $|A^+_{k,i}| \leq |\Omega^T|$ instead of (2.14) we obtain

$$\left(\frac{\varepsilon}{2^{s+1}}\right)^{p\frac{n+2}{n}} |A^+_{k_{s+1},i}| \leq c_4 c_5 \left(|\Omega^T|\right) |A^+_{k_s,i}|^{1+\kappa_0}\,, \tag{2.22}$$

where $c_5$ is an increasing function of its argument, $\kappa_0 > 0$ and

$$1+\kappa_0 = \min\left\{\frac{1}{p_1'}\left(1+\frac{p}{n}\right),\ \left(\frac{2}{p_2'} - \frac{2n}{p(n+2)}\right)\left(1+\frac{p}{n}\right),\right. \tag{2.23}$$
$$\left.\left(\frac{p}{(p-1)p_2'} - \frac{n}{(p-1)(n+2)}\right)\left(1+\frac{p}{n}\right)\right\}.$$

Therefore, (2.22) implies

$$|A^+_{k_{s+1},i}| \leq c_0 b^s \varepsilon^{-p\frac{n+2}{n}} |A^+_{k_s,i}|^{1+\kappa_0}\,, \tag{2.24}$$

where $c_0 = c_4 c_5 2^{\frac{p(n+2)}{n}}$, $b = 2^{\frac{p(n+2)}{n}}$.

From (2.24) and either Lemma 4.1 from [1, Ch.1], or Lemma 5.6 from [2, Ch.2], or Lemma 4.7 from [3, Ch.2], it follows that $|A^+_{k_s,i}| \to 0$ as $s \to \infty$ if

$$|A^+_{k_0,i}| \leq \gamma_* \equiv \left(\frac{\varepsilon^{p\frac{n+2}{n}}}{c_0}\right)^{1/\kappa_0} b^{-1/\kappa_0^2}\,. \tag{2.25}$$

In this case we would have that

$$u \leq M - \varepsilon \quad \text{a.e. in} \quad \Omega^T\,,$$

with contradicts the definition of $M$.

Since $k_s = M - \varepsilon - \frac{\varepsilon}{2^s}$, $s \geq 0$, we have that $k_0 = M - 2\varepsilon$ and we can take $\varepsilon$ so small that $k_0 > \frac{M}{2}$. Then we have

$$\left(\frac{M}{2}\right)^q |A^+_{k_0,i}| \leq \left(\frac{M}{2}\right)^q |A^+_{\frac{M}{2},i}| \leq \int_{\Omega^T} |u_i|^q\,dx\,dt\,.$$

Hence,

$$|A^+_{k_0,i}| \leq \left(\frac{2}{M}\right)^q \int_{\Omega^T} |u_i|^q\,dx\,dt\,. \tag{2.26}$$

If the r.h.s. is less than $\gamma_*$ we have a contradiction. Thus,

$$\text{ess sup}_{\Omega^T} u_i \le 2\gamma_*^{-1/q} \left( \int_{\Omega^T} |u_i|^q \, dx \, dt \right)^{1/q}, \tag{2.27}$$

$$i = 1, \dots, m, \quad q \ge 1.$$

If $u_i < 0$ we have to introduce the cut–off function

$$(u_i - k)_- = \max\{-(u_i - k), 0\}, \quad k < 0.$$

Then we obtain a similar estimate from below. This concludes the proof. $\square$

Finally we have to obtain an estimate for $|u_i|_{q,\Omega^T}$, $q \ge 1$. Therefore, we have

**Lemma 2.2.** *Let $u_0 \in L_\infty(\Omega; \mathbb{R}^m)$, $u_b \in L_\infty(S^T; \mathbb{R}^m)$, $\phi_i \in L_q(\Omega^T; \mathbb{R}^m)$, where $q > \frac{n+2}{2}$, $i = 1, 2$.*
*Then the following estimate holds*

$$|u_i|_{\frac{n+2}{2},\Omega^T} \le F_2 \left( |u_{0i}|_{\infty,\Omega}, \, |u_{bi}|_{\infty,S^T}, \, |\phi_{1i}|_{q,\Omega^T}, \, |\phi_{2i}|_{q,\Omega^T} \right), \tag{2.28}$$

$$i = 1, \dots, m,$$

*where $F_2$ is an increasing positive function of its arguments.*

*Proof.* Let $k_* = \max_i\{\text{ess sup}_\Omega |u_{0i}|, \text{ess sup}_{S^T} |u_{bi}|\}$, and let the test function in (1.3) be such that $\phi_j = 0$ for $j \ne i$, $\phi_i = (u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+}$. Then (1.3) takes the form

$$\begin{aligned} &\int_\Omega u_{iht}(u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \, dx + \\ &+ \int_\Omega (a_i(x,t,u,\nabla u) \cdot \nabla u_i)_h \cdot \nabla \left( (u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \right) dx = \\ &= \int_\Omega (b_{ih}(x,t,u,\nabla u))_h \, (u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \, dx. \end{aligned} \tag{2.29}$$

The first term in (2.29) we treat in the following way

$$\int_\Omega u_{iht}(u_{ih}-k_*)_+ e^{\alpha(u_{ih}-k_*)_+}\,dx = \tag{2.30}$$

$$= \int_{A^+_{k_*,i}} u_{iht}(u_{ih}-k_*)e^{\alpha(u_{ih}-k_*)}\,dx = \frac{1}{\alpha}\int_{A^+_{k_*,i}} (u_{ih}-k_*)\partial_t e^{\alpha(u_{ih}-k_*)}\,dx =$$

$$= \frac{1}{\alpha}\int_{A^+_{k_*,i}} \left[\partial_t\left((u_{ih}-k_*)e^{\alpha(u_{ih}-k_*)}\right) - \partial_t(u_{ih}-k_*)e^{\alpha(u_{ih}-k_*)}\right]\,dx =$$

$$= \frac{1}{\alpha}\int_\Omega \left[\partial_t\left((u_{ih}-k_*)_+ e^{\alpha(u_{ih}-k_*)_+}\right) - \frac{1}{\alpha}\partial_t e^{\alpha(u_{ih}-k_*)_+}\right]\,dx =$$

$$= \frac{1}{\alpha}\partial_t\int_\Omega \left[(u_{ih}-k_*)_+ - \frac{1}{\alpha}\right] e^{\alpha(u_{ih}-k_*)_+}\,dx\,.$$

Now inserting (2.30) into (2.29), integrating the result with respect to time and passing with $h$ to 0 yields

$$\frac{1}{\alpha}\int_\Omega \left[(u_i-k_*)_+ - \frac{1}{\alpha}\right] e^{\alpha(u_i-k_*)_+}\,dx + \tag{2.31}$$

$$+\int_{\Omega^t} a_i\cdot\nabla u_i\cdot\nabla u_i(1+\alpha(u_i-k_*)_+)e^{\alpha(u_i-k_*)_+}\,dx\,dt \le$$

$$\le \int_{\Omega^t} b_i(u_i-k_*)_+ e^{\alpha(u_i-k_*)_+}\,dx\,dt\,,$$

where we performed calculations in the second term on the l.h.s. of (2.29) and used the fact that

$$\left[(u_i-k_*)_+ - \frac{1}{\alpha}\right] e^{\alpha(u_i-k_*)_+}\Big|_{t=0} = -\frac{1}{\alpha} < 0\,.$$

Using the structure condition (1.2) in (2.31), we obtain

$$\frac{1}{\alpha}\int_\Omega \left[(u_i-k_*)_+ - \frac{1}{\alpha}\right] e^{\alpha(u_i-k_*)_+}\,dx + \tag{2.32}$$

$$+\alpha_0\int_{\Omega^t} |\nabla u|^{p-2}|\nabla(u_i-k_*)_+|^2\,(1+\alpha(u_i-k_*)_+)\,e^{\alpha(u_i-k_*)_+}\,dx\,dt \le$$

$$\le \int_{\Omega^t}\Big[\phi_{2i}\,(u_i-k_*)_+ + \beta_0|\nabla u|^{p-2}|\nabla(u_i-k_*)_+|^2\,(u_i-k_*)_+ +$$

$$+\,\phi_{1i}\,(1+\alpha(u_i-k_*)_+)\Big]\,e^{\alpha(u_i-k_*)_+}\,dx\,dt\,.$$

Assuming that $\alpha > \frac{2\beta_0}{\alpha_0}$ we obtain

$$\frac{1}{\alpha}\int_\Omega\left[(u_i-k_*)_+ - \frac{1}{\alpha}\right]e^{\alpha(u_i-k_*)_+}\,dx + \tag{2.33}$$

$$+\alpha_0\int_{\Omega^t}|\nabla u|^{p-2}|\nabla(u_i-k_*)_+|^2\left(1+\frac{\alpha}{2}(u_i-k_*)_+\right)e^{\alpha(u_i-k_*)_+}\,dx\,dt \le$$

$$\le \int_{\Omega^t}\left[\phi_{1i}\,(1+\alpha(u_i-k_*)_+) + \phi_{2i}(u_i-k_*)_+\right]e^{\alpha(u_i-k_*)_+}\,dx\,dt\,.$$

To simplify notation we introduce the functions $v_i = (u_i-k_*)_+ > 0$ and $\phi_{0i} = \max\{\phi_{1i},\phi_{2i}\}$, $i=1,\dots,m$. Using also that $|\nabla u| \ge |\nabla u_i|$, we write (2.33) in the form

$$\text{ess sup}_t\int_\Omega\left(v-\frac{1}{\alpha}\right)e^{\alpha v}\,dx + \int_{\Omega^T}|\nabla v|^p\left(1+\frac{\alpha}{2}v\right)e^{\alpha v}\,dx\,dt \le \tag{2.34}$$

$$\le c\int_{\Omega^T}(1+\phi_0)(1+v)e^{\alpha v}\,dx\,dt\,,$$

where $v>0$ and the index $i$ was omitted for simplicity. Let

$$\omega = (v-1)_+^{1/p}e^{\frac{\alpha}{p}(v-1)_+}\,. \tag{2.35}$$

Let $\alpha > 1$ and $v > 1$. For $v \le 1$ we have the sup estimate for $v$ so there is nothing to prove. Then we have

$$\int_\Omega\left(v-\frac{1}{\alpha}\right)e^{\alpha v}\,dx = \int_\Omega(v-1)\,e^{\alpha v}\,dx + \int_\Omega\left(1-\frac{1}{\alpha}\right)e^{\alpha v}\,dx\,,$$

where

$$\int_\Omega(v-1)\,e^{\alpha v}\,dx \ge \int_\Omega(v-1)\,e^{\alpha(v-1)}\,dx = \int_\Omega\omega^p\,dx\,.$$

Therefore,

$$\int_\Omega\left(v-\frac{1}{\alpha}\right)e^{\alpha v}\,dx \ge \int_\Omega\omega^p\,dx\,. \tag{2.36}$$

Since $v>1$ we have that $\omega = (v-1)^{1/p}e^{\frac{\alpha}{p}(v-1)}$, so

$$\nabla\omega = \left(\frac{1}{p}(v-1)^{\frac{1}{p}-1} + \frac{\alpha}{p}(v-1)^{1/p}\right)\nabla v e^{\frac{\alpha}{p}(v-1)}\,,$$

and

$$|\nabla\omega|^p \le c\left[(v-1)^{1-p} + (v-1)\right]|\nabla v|^p e^{\alpha(v-1)} \le \tag{2.37}$$

$$\le c(1+v)|\nabla v|^p e^{\alpha(v-1)} \le c(v+1)|\nabla v|^p e^{\alpha v}\,.$$

Finally, since $\omega^p = (v-1)e^{\alpha(v-1)} \geq (v-1)e^{\alpha v}e^{-\alpha} = c(v-1)e^{\alpha v} =$ $= cve^{\alpha v} - ce^{\alpha v}$. Independently, since $v > 1$ we have that $\omega^p \geq ce^{\alpha v}$. Hence, $ve^{\alpha v} \leq c\omega^p + ce^{\alpha v} \leq c\omega^p$. Therefore, we have

$$(1+v)e^{\alpha v} \leq c\omega^p \,. \tag{2.38}$$

Using (2.36)–(2.38) in (2.34) implies

$$\text{ess sup}_t \int_\Omega \omega^p \, dx + \int_{\Omega^T} |\nabla \omega|^p \, dx \, dt \leq \tag{2.39}$$

$$\leq c_1 \int_{\Omega^T} (1+\phi_0)(1+\omega^p) \, dx \, dt \,.$$

Using the imbedding (3.1) from [1], Ch.1, Sect.3 for the space $V_0^{2,p}(\Omega^T) =$ $L_\infty(0,T;L_2(\Omega)) \cap L_p(0,T;\overset{\circ}{W}{}_p^1(\Omega))$ and the Hölder inequality on the r.h.s. of (2.39) we get

$$|\omega|_{\frac{p(n+2)}{n},\Omega^T} \leq c_2 + c_1^{1/p}\left(\int_{\Omega^T}(1+\phi_0)^{\lambda_1}\, dx\, dt\right)^{1/p\lambda_1} \tag{2.40}$$

$$\left(\int_{\Omega^T} |\omega|^{p\lambda_2}\, dx\, dt\right)^{1/p\lambda_2} \leq c_2 + c_3 |\Omega^T|^{\frac{1}{p\lambda_2} - \frac{n}{p(n+2)}} \, |\omega|_{\frac{p(n+2)}{n},\Omega^T} \,,$$

where $c_2 = c_1^{1/p}\left(\int_{\Omega^T}(1+\phi_0)\, dx\, dt\right)^{1/p}$, $c_3 = c_1^{1/p}\left(\int_{\Omega^T}(1+\phi_0)^{\lambda_1}\, dx\, dt\right)^{1/p\lambda_1}$, $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, $\lambda_2 < \frac{n+2}{n}$, $\lambda_1 > \frac{n+2}{n}$.

If $T$ is so small that

$$c_3|\Omega^T|^{\frac{1}{p\lambda_2} - \frac{n}{p(n+2)}} \leq \frac{1}{2}$$

we obtain the estimate

$$|\omega|_{\frac{p(n+2)}{n},\Omega^T} \leq 2c_2 \,. \tag{2.41}$$

For arbitrary $T$ the argument can be repeated up to covering the whole $[0,T]$ in a finite number of steps.

From the definition of $v_i$ and (2.35) we have that

$$((u_i - k_*)_+ - 1)_+^{1/p} = \omega_i e^{-\frac{\alpha}{p}(v_i - 1)_+} \,.$$

Hence, either

$$u_i \leq k_* + 1 \tag{2.42}$$

or $u_i > k_* + 1$. In the second case we have the inequality

$$u_i \leq c_4\left[(k_*+1) + \omega_i^p\right] \,.$$

Therefore,

$$|u_i|_{\frac{n+2}{n},\Omega^T} \le c_4(k_*+1)|\Omega^T|^{\frac{n}{n+2}} + c_4|\omega_i|^p_{\frac{p(n+2)}{n},\Omega^T} \le \tag{2.43}$$

$$\le c_4(k_*+1)|\Omega^T|^{\frac{n}{n+2}} + c_4(2c_2)^p\,.$$

From (2.42) and (2.43) we obtain (2.28). This concludes the proof. □

From Lemmas 2.1, 2.2, we have

**Theorem 2.3.** *Let the assumptions of Lemmas 2.1, 2.2 be satisfied. Then a qualitatively bounded solution of (1.1) is bounded in terms of data (see the inequalities (2.1) and (2.28)).*

## References

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