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COUNTABLE DECOMPOSITION OF DERIVATIVES AND BAIRE 1 FUNCTIONS

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Abstract. We give conditions under which Baire 1 functions, Darboux Baire 1 functions and derivatives are not countably continuous.

Recall that $f : A \to B$ is countably continuous iff there exists $\{A_n\}_{n=1}^{\infty}$ such that $\cup_{n=1}^{\infty} A_n = A$ and $f|A_n$, the restriction of f to A_n, is continuous. If A, B are Polish spaces and f, in addition to being countably continuous, has some sort of regularity property, then A can be decomposed in a way so that A_n's have appropriate regularity property as well. For example, if A, B are Polish spaces and a countably continuous function f is of Borel class 1, then A can be decomposed so that A_n's are G_δ sets. To see this, let $\{A_n\}$ be an arbitrary decomposition of A so that $f|A_n$ is continuous for all n. Now, we may choose an extension $f^*|A_n^*$ of $f|A_n$ so that $f^*|A_n^*$ is continuous and A_n^* is a G_δ set containing A_n. Since f is of Borel class 1 on A, and f^* is continuous on the G_δ set A_n^*, we have the set of points of A_n^* where $f^* = f$ is a G_δ set. Hence we have extended A_n to a G_δ set where f is continuous.

It was conjectured by Jackson and Mauldin [3] that if X is the the Banach space of real-valued bounded Baire 1 functions defined on $[0, 1]$ or the Banach space of bounded derivatives defined on $[0, 1]$, then $D_\omega(X)$, the set of functions in X which are countably continuous, is meager in X. It was

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shown by van Mill and Pol that indeed such is the case [5]. We give a general condition (Theorem 2) under which Darboux Baire 1 functions and derivatives are not countably continuous. From this condition we obtain Theorem 3 which shows $D_\omega(X)$ is nowhere dense in X where X is either the Banach space of bounded derivatives or the Banach space of bounded Darboux Baire 1 functions. It also follows (Corollary 2) from Theorem 2 that no Pompeiu derivative is countably continuous.

Jackson and Mauldin [3] also showed using some notions of recursion theory that the Lebesgue measure viewed as a function defined on the set of compact subsets of $[0,1]$ is an upper semicontinuous function which is not countably continuous. In the paper mentioned earlier [5], van Mill and Pol gave a direct argument for this result. We give a condition (Theorem 1) under which a Baire 1 function from a Polish space into Polish space is not countably continuous. As a corollary to Theorem 1, we obtain even a simpler proof of the fact that the Lebesgue measure is not countably continuous.

At this point the author would like to thank Micha Morayne, Abhijit Dasgupta, and Roman Pol for many stimulating conversations and correspondence on the subject. The reader is referred to [2] for several interesting remarks and observations resulting from this collaboration as well as an expository treatment of the subject.

We now introduce some definitions and terminology most of which may be found in [1] and [4].

A function $f : [0,1] \to \mathbb{R}$ is Darboux means that it satisfies the intermediate value property. We will use standard facts about Darboux Baire 1 functions which may be found in Chapter 2 of [1]. More specifically, recall that a Baire 1 function $f : [0,1] \to \mathbb{R}$ is Darboux iff f has a bilateral road at each $x \in (0,1)$ and a unilateral road at 0 and 1. (We say that f has a bilateral road at x if there is a set M such that x is a two-sided limit point of M and $f|M$ is continuous at x. A unilateral road is defined in an analogous manner.)

We let bDB_1 ($b\Delta$) denote the Banach space of bounded Darboux Baire 1 functions (bounded derivatives) equipped with the sup norm. We let DB_1 denote the linear space consisting of Darboux Baire 1 functions defined on $[0,1]$. Note that $b\Delta \subseteq bDB_1 \subseteq DB_1$. If X is some space of functions, then we use $D_\omega(X)$ to denote the set of functions in X which are countably continuous.

We say that $f : [0,1] \to \mathbb{R}$ is a Pompeiu derivative iff f is a derivative and f is zero on a dense set without identically vanishing. Recall that Zahorski type functions which are "lifted" on dense, first category, F_σ sets (Thm 6.5 in [1]) and the derivatives of differentiable nowhere monotone functions are Pompeiu derivatives.

We let $\mathcal{K}(I)$ denote the set of all closed subsets of the unit interval $[0,1]$. Recall that $\mathcal{K}(I)$ forms a compact metric space when endowed with the Hausdorff metric. If $M \subseteq \mathbb{R}$ is measurable, then we use $\lambda(M)$ to denote the Lebesgue measure of M. Also recall that $\lambda : \mathcal{K}(I) \to [0,1]$ is an upper semicontinuous function.

Finally, we use $cl(M)$ and $bd(M)$ to denote the closure of M and the boundary of M, respectively.

Theorem 1. *Suppose A, B are Polish spaces, $f : A \to B$ is a Borel class 1 function, and $\mathcal{U}$ is an uncountable collection of open subsets of B such that*

1. *if $U, V \in \mathcal{U}$ and $U \neq V$, then $cl(U) \subseteq V$ or $cl(V) \subseteq U$, and*
2. *for each $U \in \mathcal{U}$, $P_U \setminus f^{-1}(U)$ is dense in P_U where $P_U = cl(f^{-1}(U))$.*

Then f is not countably continuous.

Proof. To obtain a contradiction, assume that f is countably continuous and that $\{A_n\}$ is a sequence of G_δ sets such that $f|A_n$ is continuous and $\bigcup A_n = A$. Since f is Borel class 1, we have that for all $U \in \mathcal{U}$, $P_U \setminus f^{-1}(U)$ is a dense G_δ subset of P_U and $f|P_U$ is continuous on some dense G_δ subset of P_U. Intersecting the points of continuity of $f|P_U$ and $P_U \setminus f^{-1}(U)$, we obtain a G_δ set $Q_U \subseteq P_U \setminus f^{-1}(U)$ such that Q_U is dense in P_U and $f|P_U$ is continuous on Q_U. As $f^{-1}(U)$ is dense in P_U, we have that $f(Q_U) \subseteq bd(U)$.

Now let $\{O_n\}$ be a countable basis for A and for each $n, m \in \mathbb{N}$, let

$$H(n,m) = \{U \in \mathcal{U} : P_U \cap O_m \neq \emptyset, \text{ and } A_n \cap P_U \text{ is dense in } P_U \cap O_m\}.$$

As $\{A_n\}$ is a sequence of G_δ sets whose union contains the set P_U, utilizing the Baire category theorem we have that every $U \in \mathcal{U}$ belongs to some $H(n,m)$. Hence $\bigcup H(n,m) = \mathcal{U}$. Now choose $n', m' \in \mathbb{N}$ so that for some $U, V \in \mathcal{U}$ we have that $cl(U) \subseteq V$ and $U, V \in H(n', m')$. Note that $P_U \subseteq P_V$. As $A_{n'}$ is G_δ and dense in $P_U \cap O_{m'}$, we have that $A_{n'} \cap Q_U \cap O_{m'}$ is a dense G_δ subset of $P_U \cap O_{m'}$ and $f(A_{n'} \cap Q_U \cap O_{m'}) \subseteq bd(U)$. Similarly, we have that $A_{n'} \cap Q_V \cap O_{m'}$ is a dense G_δ subset of $P_V \cap O_{m'}$ and $f(A_{n'} \cap Q_V \cap O_{m'}) \subseteq bd(V)$. Now consider $y \in A_{n'} \cap Q_U \cap O_{m'}$. Then, $f(y) \in bd(U)$. However, y is a limit point of $A_{n'} \cap Q_V \cap O_{m'}$ and $f(A_{n'} \cap Q_V \cap O_{m'}) \subseteq bd(V)$. Since $bd(U) \cap bd(V) = \emptyset$, we have a contradiction. $\square$

Corollary 1. *The function $\lambda : \mathcal{K}(I) \to [0,1]$ is an upper semicontinuous function which is not countably continuous.*

Proof. As stated earlier, it is easy to verify that λ is upper semicontinuous. Hence, λ is of Borel class 1. We now use Theorem 1 to show that λ is not countably continuous. Let $\mathcal{U} = \{(1/2 - \epsilon, 1/2 + \epsilon) : 0 < \epsilon < 1/2)\}$. Then $\mathcal{U}$ clearly satisfies condition (1) of Theorem 1. Let us now show that condition (2) of Theorem 1 is satisfied to conclude the proof of the corollary. Let $U = (1/2 - \epsilon, 1/2 + \epsilon) \in \mathcal{U}$. Then, $P_U = cl(\lambda^{-1}(U))$ is just $\{M \in \mathcal{K}(I) : \lambda(M) \geq 1/2 - \epsilon\}$. Since for every $M \in P_U$ there exists, arbitrarily close to M, $N \in \mathcal{K}(I)$ such that $\lambda(N) = 1/2 - \epsilon$, we have that $P_U \setminus \lambda^{-1}(U)$ is dense in P_U. $\square$

Theorem 2. *Suppose $f \in DB_1$ such that $graph(f|C(f))$, the graph of f restricted to the points of continuity of f, is not dense in $graph(f)$. Then f is not countably continuous.*

Proof. Let I, J be open intervals such that $graph(f) \cap (J \times I) \neq \emptyset$ and $graph(f|C(f)) \cap (J \times I) = \emptyset$. Since f is Darboux and $graph(f|C(f)) \cap (J \times I) = \emptyset$, $f(J)$ is an interval. Hence, $f(J) \cap I$ is an interval. Let $t \in f(J) \cap I$ and $\epsilon > 0$ be such that $(t - \epsilon, t + \epsilon) \subseteq f(J) \cap I$ and $f(0), f(1) \notin (t - \epsilon, t + \epsilon)$. Let $\mathcal{U} = \{(t - \delta, t + \delta) : 0 < \delta < \epsilon\}$, $A = J$ and $B = I$. Then $\mathcal{U}$ clearly satisfies condition (1) of Theorem 1. Let us show that it satisfies condition (2) as well. Let $U \in \mathcal{U}$. Since $graph(f|C(f)) \cap (J \times I) = \emptyset$, and $cl(U) \subseteq I$, we have that $f^{-1}(cl(U)) \cap J$ is nowhere dense in J. Therefore, $cl(f^{-1}(U)) \cap J$ is nowhere dense in J. Since $f \in DB_1$, f has a bilateral road at every point of $(0, 1)$. Therefore, all endpoints of connected components of $[0, 1] \setminus cl(f^{-1}(U)) \cap J$ map outside U under f. Hence, condition (2) of Theorem 1 is satisfied as well and we have that f is not countably continuous. $\square$

Corollary 2. *Let $f : [0, 1] \to \mathbb{R}$ be a Pompeiu derivative. Then f is not countably continuous.*

Proof. Let I be an open interval such that $0 \notin cl(I)$ and $f([0, 1]) \cap I \neq \emptyset$. Since f is zero on a dense set, $graph(f|C(f)) \cap ((0, 1) \times I) = \emptyset$. Hence it follows from Theorem 2 that f is not countably continuous. $\square$

Theorem 3. *If $X = bDB_1$ or $X = b\Delta$, then $D_\omega(X)$ is nowhere dense in X.*

Proof. Let

$$\mathcal{G} = \{f \in bDB_1 : graph(f|C(f)) \text{ is dense in } graph(f)\} \text{ and}$$

$$\mathcal{H} = \{f \in b\Delta : graph(f|C(f)) \text{ is dense in } graph(f)\}.$$

We will show that $\mathcal{G}$ and $\mathcal{H}$ are closed and nowhere dense in bDB_1 and $b\Delta$, respectively. This fact, together with Theorem 2, yields a proof of the theorem. □

Let us now show that $\mathcal{H}$ is a nowhere dense closed subset of $b\Delta$. (An argument very similar to the one that follows will show that $\mathcal{G}$ is a nowhere dense closed subset of bDB_1 as well.) It is easy to verify that $\mathcal{H}$ is closed. To show that $\mathcal{H}$ is nowhere dense in $b\Delta$, it suffices to show that $b\Delta \setminus \mathcal{H}$ is dense in $b\Delta$. To this end, let $f \in b\Delta$ and let $\epsilon > 0$. Let $p \in (0,1)$ be a point of continuity of f. Let $\delta > 0$ be such that $(p-\delta, p+\delta) \subseteq (0,1)$ and if $x \in [0,1]$ and $|p-x| \leq \delta$, then $|f(x) - f(p)| < \frac{\epsilon}{3}$. Now, let l be the line that goes through $(p-\delta, f(p-\delta))$ and $(p+\delta, f(p+\delta))$ and let $u : [p-\delta, p+\delta] \to [0, \frac{\epsilon}{3}]$ be an approximately continuous function such that $u(p-\delta) = u(p+\delta) = 0$ and $graph(u|C(u))$ is not dense in $graph(u)$. Such a function u maybe constructed using Thm 6.5 of [1]. Define $g : [0,1] \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, p-\delta] \\ l(x) + u(x) & \text{if } x \in (p-\delta, p+\delta) \\ f(x) & \text{if } x \in [p+\delta, 1]. \end{cases}$$

Then, $g \in b\Delta$, $d(f,g) < \epsilon$, and $graph(g|C(g))$ is not dense in $graph(g)$. Hence, $b\Delta \setminus \mathcal{H}$ is dense in $b\Delta$.

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