Journal of Applied Analysis
Vol. 3, No. 2 (1997), pp. 225–237

THE GRADIENT METHOD FOR NON–DIFFERENTIABLE OPERATORS IN PRODUCT HILBERT SPACES AND APPLICATIONS TO ELLIPTIC SYSTEMS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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Received November 25, 1996 and, in revised form, April 28, 1997

Abstract. The gradient method is considered in Hilbert spaces. Earlier results on linear convergence are extended to systems of equations with certain non-differentiable operators. The method includes the approximate solution of elliptic systems of quasilinear boundary value problems.

0. Introduction

The gradient method, a classical approximation method for equations in $\mathbb{R}^n$, was first applied in Hilbert space by Kantorovich to linear equations via minimizing the quadratic functional ([2], [3]). Later this result was extended to any uniformly convex smooth functional ([5]), thus allowing

1991 *Mathematics Subject Classification.* Primary 46N20, 35J65, 49M10.
Key words and phrases. Gradient method, systems of operator equations, elliptic systems of quasilinear boundary value problems.
This research was supported by Hungarian National Research Foundation OTKA Grant No. T 019460

ISSN 1425-6908

the approximate solution of nonlinear operator equations of the following type:

Theorem (see e.g. [4]). *Let H be a real Hilbert space and $F : H \to H$ have the following properties:*

(i) *F is Gâteaux differentiable;*
(ii) *for any $u, k, w, h \in H$ the mapping $s, t \mapsto F'(u+sk+tw)h$ is continuous from $\mathbb{R}^2$ to H;*
(iii) *for any $u \in H$ the operator $F'(u)$ is self-adjoint;*
(iv) *there are constants $M \geq m > 0$ such that for all $u, h \in H$*

$$m\|h\|^2 \leq \langle F'(u)h, h\rangle \leq M\|h\|^2 .$$

Then for any $b \in H$ the equation

$$F(u) = b$$

has a unique solution $u^* \in H$ and for any $u^0 \in H$ the sequence

$$u^{k+1} := u^k - \frac{2}{M+m}(F(u^k) - b) \quad (k \in \mathbb{N})$$

converges linearly to u^*, namely,

$$\|u^k - u^*\| \leq \frac{1}{m}\|F(u_0) - b\| \left(\frac{M-m}{M+m}\right)^k \quad (k \in \mathbb{N}) . \tag{1}$$

In [4] this method has been generalized to a class of non–differentiable operators which can be suitably transformed into Gâteaux differentiable operators. The result gives rise to the approximate solution of quasilinear elliptic differential equations.

The aim of this paper is to extend the above methods to systems of equations with non–differentiable operators. The obtained method, developped in product Hilbert spaces, allows the approximate solution of elliptic systems of quasilinear differential equations, in an approach different from discretization methods. Namely, it yields a direct theoretical approximation of the studied system in the corresponding Sobolev space, and reduces computational problems to Laplacian type auxiliary linear equations.

The extension of the earlier result [4] to systems allows the approximate solution of stationary states of reaction–diffusion type systems. (For the RDE itself, the obtained result might be a starting point of methods that discretize time.)

1. Approximate solution of systems in product Hilbert spaces

A real Hilbert space with scalar product $\langle\ ,\ \rangle$ and corresponding norm $\|\ \ \|$ will be considered. For any $r \in \mathbb{N}^+$ denote by $H^r := H \times H \times \ldots \times H$ (r times) the product space and for any subspace $D \subset H$ let $D^r := D \times D \times \ldots \times D$. For $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r)$ in H^r the scalar product is $[u, v] := \sum_{i=1}^r \langle u_i, v_i \rangle$ and the corresponding norm is $[[u]] := \left(\sum_{i=1}^r \|u_i\|^2\right)^{1/2}$.

Theorem 1. *Let H be a real Hilbert space and $D \subset H$ a dense subspace. Let $T_i : D^r \to H \quad (i = 1, \ldots, r)$ be non–differentiable operators. We consider the system*

$$T_i(u_1, \ldots, u_r) = g_i \quad (i = 1, \ldots, r) \tag{2}$$

with $g = (g_1, \ldots, g_r) \in H^r$. Let $B : D \to H$ be a symmetric linear operator with lower bound $p > 0$ and denote by H_B the energy space of B, i.e. the completion of D with respect to the scalar product $\langle x, y \rangle_B := \langle Bx, y \rangle \ (x, y \in D)$. (Denote by $\|\ \ \|_B$ the corresponding norm in H_B.) Assume that the following conditions hold:

(i) *$R(B) \supset R(T_i) \quad (i = 1, \ldots, r)$;*

(ii) *for any $i = 1, \ldots, r$ the operators $B^{-1}T_i$ (whose domain $D(B^{-1}T_i) = D^r$ is dense in the product space H_B^r) have Gâteaux differentiable extensions $F_i : H_B^r \to H_B$, respectively;*

(iii) *for any $u, k, w, h \in H_B^r$ the mappings $s, t \mapsto F_i'(u + sk + tw)h$ are continuous from $\mathbb{R}^2$ to H_B;*

(iv) *for any $u, h, k \in H_B^r$*

$$\sum_{i=1}^r \langle F_i'(u)h, k_i \rangle_B = \sum_{i=1}^r \langle h_i, F_i'(u)k \rangle_B\,;$$

(v) *there are constants $M \geq m > 0$ such that for all $u, h \in H_B^r$*

$$m \sum_{i=1}^r \|h_i\|_B^2 \leq \sum_{i=1}^r \langle F_i'(u)h, h_i \rangle_B \leq M \sum_{i=1}^r \|h_i\|_B^2\,. \tag{3}$$

Let $g_i \in R(B) \quad (i = 1, \ldots, r)$. Then

(1) *the system (2) has a unique generalized solution $u^* = (u_1^*, \ldots, u_r^*) \in H_B^r$, i.e. which satisfies*

$$\langle F_i(u^*), v \rangle_B = \langle g_i, v \rangle \qquad (v \in H_B,\ i = 1, \ldots, r); \tag{4}$$

(2) *for any $u^0 \in D^r$ the sequence $u^k := (u_1^k, \ldots, u_r^k)_{k \in \mathbb{N}}$ given by the coordinate sequences*

$$u_i^{k+1} := u_i^k - \frac{2}{M+m} B^{-1}(T_i(u^k) - g_i) \quad (i = 1, \ldots, r;\ k \in \mathbb{N}) \tag{5}$$

converges linearly to u^*. *Namely,*

$$\left(\sum_{i=1}^{r} \|u_i^k - u_i^*\|_B^2\right)^{1/2} \leq \frac{1}{m\sqrt{p}} \left(\sum_{i=1}^{r} \|T_i(u^0) - g_i\|^2\right)^{1/2} \left(\frac{M-m}{M+m}\right)^k \quad (k \in \mathbb{N})\,. \tag{6}$$

Using notation

$$[[u]]_B := \left(\sum_{i=1}^{r} \|u_i\|_B^2\right)^{1/2}$$

for $u \in H_B^r$, *estimate* (6) *is written briefly as*

$$[[u^k - u^*]]_B \leq \frac{1}{m\sqrt{p}} [[T(u^0) - g]] \left(\frac{M-m}{M+m}\right)^k \quad (k \in \mathbb{N})\,. \tag{7}$$

Proof. The scalar product of the product space H_B^r will be denoted by

$$[u,v]_B := \sum_{i=1}^{r} \langle u_i, v_i \rangle_B \quad (u = (u_1, \dots, u_r),\ v = (v_1, \dots, v_r) \in H_B^r)$$

and the corresponding norm by $[[u]]_B$ as given in the theorem.

We introduce the operator $F := (F_1, \dots, F_r) : H_B^r \to H_B^r$, i.e.

$$F(u) := (F_1(u), \dots, F_r(u)) \in H_B^r \quad (u \in H_B^r).$$

Let $g_i \in R(B)$ and $b_i \in B^{-1}g_i \quad (i = 1, \dots, r)$, further, $b := (b_1, \dots, b_r)$. Instead of the system (2) we consider equations

$$F_i(u) = b_i\,.$$

Let $u^0 \in D^r$ and

$$u_i^{k+1} := u_i^k - \frac{2}{M+m}(F_i(u^k) - b_i) \quad (i = 1, \dots, r;\ k \in \mathbb{N})\,. \tag{8}$$

Since $b \in D^r$ and from assumption (i) we have $R\left(F_{i|D}\right) = R(B^{-1}T_i) \subset D$, it follows by induction that $u^k \in D^r \quad (k \in \mathbb{N})$. Thus

$$u_i^{k+1} = u_i^k - \frac{2}{M+m} B^{-1}(T_i(u^k) - g_i)\,,$$

that is, the sequence (5) coincides with (8).

It follows from the assumptions that conditions (i)–(iv) of the theorem quoted in the Introduction hold for F in the space H_B^r. Indeed:

(i) F is Gâteaux differentiable since all F_i are Gâteaux differentiable.

(ii) For any $u, k, w, h \in H_B^r$ the mapping $s, t \mapsto F'(u + sk + tw)h$ is continuous from $\mathbb{R}^2$ to H_B^r since for all $i = 1, \dots, r$ the mappings $s, t \mapsto F_i'(u + sk + tw)h$ are continuous from $\mathbb{R}^2$ to H_B.

(iii) For any $u, h, v \in H_B^r$

$$[F'(u)h, v]_B = \sum_{i=1}^{r} \langle F_i'(u)h, v_i \rangle_B = \sum_{i=1}^{r} \langle h_i, F_i'(u)v \rangle_B = [h, F'(u)v]_B.$$

Since the operator $F'(u)$ is bounded linear and symmetric, hence it is self–adjoint.

(iv) For any $u, h \in H_B^r$ $\quad m[[h]]_B^2 \le [F'(u)h, h]_B \le M[[h]]_B^2$ (obtained from (3) and the definition of $[\cdot,\cdot]_B$).

Hence the quoted theorem applies to equation $F(u) = b$ in H_B^r. This means that the system $F_i(u) = b_i$ $(i = 1, \dots, r)$ has a unique solution $u^* = (u_1^*, \dots, u_r^*) \in H_B^r$ and estimate (1) holds for the sequence (u^k) defined by (8):

$$[[u^k - u^*]]_B \le \frac{1}{m}[[F(u^0) - b]]_B \left(\frac{M-m}{M+m}\right)^k \quad (k \in \mathbb{N}). \tag{9}$$

This u^* is the generalized solution of (2) since

$$\langle F_i(u^*), v \rangle_B = \langle b_i, v \rangle_B := \langle Bb_i, v \rangle = \langle g_i, v \rangle \quad (v \in D,\ i = 1, \dots, r)$$

and the equality also holds for any $v \in H_B$ since D is dense in H_B.

Finally, estimate (9) is transformed to (7) in the following way: for any $w \in D$ we have $\|w\|_B := \sqrt{\langle Bw, w \rangle} \ge \sqrt{p}\|w\|$, hence $\|w\|_B^2 = \langle Bw, w \rangle \le \|Bw\|\|w\| \le \frac{1}{\sqrt{p}}\|Bw\|\|w\|_B$, i.e. $\|w\|_B \le \frac{1}{\sqrt{p}}\|Bw\|$. Applying this to $F_i(u^0) - b_i \in D$, we have for all $i = 1, \dots, r$

$$\|F_i(u^0) - b_i\|_B \le \frac{1}{\sqrt{p}}\|BF_i(u^0) - Bb_i\| = \frac{1}{\sqrt{p}}\|T_i(u^0) - g_i\|,$$

hence

$$\left(\sum_{i=1}^{r} \|F_i(u^0) - b_i\|_B^2\right)^{1/2} \le \frac{1}{\sqrt{p}} \left(\sum_{i=1}^{r} \|T_i(u^0) - g_i\|^2\right)^{1/2} = \frac{1}{\sqrt{p}}[[T(u^0) - g]],$$

i.e. (7) is proved. $\square$

Remark 1. The sequence (u^k) converges to u^* also in the norm of H^r:

$$[[u^k - u^*]] \le \frac{1}{\sqrt{p}}[[u^k - u^*]]_B \le \frac{1}{mp}[[T(u^0) - g]] \left(\frac{M-m}{M+m}\right)^k \quad (k \in \mathbb{N}).$$

2. Applications to elliptic systems of differential equations

In this section the approximate solution of uniformly elliptic systems consisting of r quasilinear boundary value problems is developed, using the result of Section 1.

The following notations will be used: denote by d the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ where $|\alpha| := \alpha_1 + \dots + \alpha_N \le n$. We write any $\xi \in \mathbb{R}^{rd}$ as $\xi = (\xi_{i,\alpha})_{i=1,\dots,r,\ |\alpha|\le n} = \Big((\xi_{1,\alpha})_{|\alpha|\le n}, \dots, (\xi_{r,\alpha})_{|\alpha|\le n} \Big)$ with $\xi_{i,\alpha} \in \mathbb{R} \quad (i = 1, \dots, r, |\alpha| \le n)$. Differentiation with respect to $\xi_{i,\alpha}$ is denoted by $\partial_{\xi_{i,\alpha}} \quad (i = 1, \dots, r,\ |\alpha| \le n)$. Further, let $D^{(n)}u := (\partial^\alpha u_i)_{i=1,\dots,r,\ |\alpha|\le n}$ for any $u = (u_1, \dots, u_r) \in H_0^n(\Omega)^r$.

The following system is considered with given functions $f_{i,\alpha} : \overline{\Omega} \times \mathbb{R}^{rd} \to \mathbb{R}$ and $g_j : \overline{\Omega} \to \mathbb{R} \quad (i = 1, \dots, r,\ |\alpha| \le n)$:

$$\begin{cases} T_i(u_1, \dots, u_r) := \sum\limits_{|\alpha|\le n} (-1)^{|\alpha|} \partial^\alpha \Big(f_{i,\alpha}(x, D^{(n)}u) \Big) = g_i \quad (i = 1, \dots, r) \\ \partial^\alpha u_{i|\partial\Omega} = 0 \quad (i = 1, \dots, r,\ |\alpha| \le n-1) \end{cases} \tag{10}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with $\partial\Omega \in C^{2n,\nu}$ (where $N, n \in \mathbb{N}^+$, $0 < \nu < 1$).

The scalar product of the real Hilbert space $H_0^n(\Omega)$ is defined by

$$\langle w, v \rangle_{H_0^n(\Omega)} := \int_\Omega \sum_{i_1,\dots,i_n=1}^{N} (\partial_{i_1} \dots \partial_{i_n} w)(\partial_{i_1} \dots \partial_{i_n} v)$$

and that of the space $H_0^n(\Omega)^r$ by $\langle w, v \rangle_{H_0^n(\Omega)^r} := \sum_{i=1}^r \langle w_i, v_i \rangle_{H_0^n(\Omega)}$. We introduce other scalar products $\langle w, v \rangle^*_{H_0^n(\Omega)} := \int_\Omega \sum_{|\alpha|\le n} (\partial^\alpha w)(\partial^\alpha v)$ and $\langle w, v \rangle'_{H_0^n(\Omega)} := \int_\Omega \sum_{|\alpha|=n} (\partial^\alpha w)(\partial^\alpha v)$ on $H_0^n(\Omega)$, which define equivalent norms to the original one, namely, there exist $\omega_1, \omega_2 > 0$ such that

$$\omega_1 \|w\|_{H_0^n(\Omega)} \le \|w\|'_{H_0^n(\Omega)} \le \|w\|^*_{H_0^n(\Omega)} \le \omega_2 \|w\|_{H_0^n(\Omega)} \,. \tag{11}$$

The result on the approximation of (10) can now be formulated.

Theorem 2. *Let the functions* $f_{i,\alpha} : \overline{\Omega} \times \mathbb{R}^{rd} \to \mathbb{R}$ *satisfy the following conditions:*

(a) $f_{i,\alpha} \in C^{|\alpha|,\nu}(\overline{\Omega} \times \mathbb{R}^{rd}) \quad (i = 1, \dots, r,\ 0 < |\alpha| \le n)$,
$f_{i,0} \in C^1(\overline{\Omega} \times\ \times \mathbb{R}^{rd}) \quad (i = 1, \dots, r)$;

(b) $\partial_{\xi_{i,\alpha}} f_{j,\beta} = \partial_{\xi_{j,\beta}} f_{i,\alpha} \quad (i, j = 1, \dots, r;\ |\alpha|, |\beta| \le n)$;

(c) *there exist constants* $0 < \mu_1 \le \mu_2$ *such that for any* $(x, \xi) \in \Omega \times \mathbb{R}^{rd}$ *and* $\zeta \in \mathbb{R}^{rd}$

$$\mu_1 \sum_{i=1}^r \sum_{|\alpha|=n} |\zeta_{i,\alpha}|^2 \le \sum_{i,j=1}^r \sum_{|\alpha|,|\beta|\le n} \partial_{\xi_{j,\beta}} f_{i,\alpha}(x, \xi) \zeta_{j,\beta} \zeta_{i,\alpha} \le \mu_2 \sum_{i=1}^r \sum_{|\alpha|\le n} |\zeta_{i,\alpha}|^2 .$$

Further, let $g_i \in C^{0,\nu}(\overline{\Omega}) \quad (i=1,\dots,r)$.

Then

(1) *system* (10) *has a generalized solution* $u^* = (u_1^*,\dots,u_r^*) \in H_0^n(\Omega)^r$, *i.e. for which*

$$\int_\Omega \sum_{|\alpha|\le n} f_{i,\alpha}(x, D^{(n)}u^*)\partial^\alpha v\ dx = \int_\Omega g_i v\ dx \qquad (i=1,\dots,r,\ v\in H_0^n(\Omega)) \quad (12)$$

holds.

(2) *Let* $D := \left\{ u \in C^{2n,\nu}(\overline{\Omega}) : \partial^\alpha u_{|\partial\Omega} = 0\,, \quad |\alpha| \le n-1 \right\}$. *Then for any* $u^0 \in D^r$ *the sequence* $u^k := (u_1^k,\dots,u_r^k)_{k\in\mathbb{N}}$ *given by the coordinate sequences*

$$u_i^{k+1} := u_i^k - \frac{2}{M+m} z_i^k \quad (i=1,\dots,r,\ k\in\mathbb{N})\,, \tag{13}$$

where $z_i^k \in C^{2n,\nu}(\overline{\Omega})$ *is the solution of the linear equation*

$$\begin{cases} \triangle^n z_i^k = (-1)^n (T_i(u^k) - g_i) & \\ \partial^\alpha z^k_{i|\partial\Omega} = 0 & (|\alpha| \le n-1)\,, \end{cases} \tag{14}$$

converges linearly to u^* *according to the estimate*

$$\left(\sum_{i=1}^r \|u_i^k - u_i^*\|^2_{H_0^n(\Omega)} \right)^{1/2} \le \frac{1}{m\sqrt{p}} \left(\sum_{i=1}^r \|T_i(u^0) - g_i\|^2_{L^2(\Omega)} \right)^{1/2} \left(\frac{M-m}{M+m} \right)^k \qquad (k\in\mathbb{N}) \quad (15)$$

(where $m := \mu_1\omega_1^2$, $M := \mu_2\omega_2^2$ *and* p *stands for the smallest eigenvalue of* $B := (-1)^n\triangle^n$ *on* D*).*

Proof.

(1) First we remark the following facts:

(i) Assumption (c) implies that for all $i,j = 1,\dots,r$, $|\alpha|, |\beta| \le n$ and $(x,\xi) \in \Omega\times\mathbb{R}^{rd}$

$$\left| \partial_{\xi_{j,\beta}} f_{i,\alpha}(x,\xi) \right| \le \mu_2\,.$$

(ii) Lagrange's inequality yields that for all $i = 1,\dots,r$, $|\alpha| \le n$, $(x,\xi) \in \Omega\times\mathbb{R}^{rd}$

$$|f_{i,\alpha}(x,\xi)| \le |f_{i,\alpha}(x,0)| + \mu_2 \sum_{j=1}^r \sum_{|\beta|\le n} |\xi_{j,\beta}|.$$

(2) Now we prove that the assertions (i)–(v) of Theorem 1 are fulfilled in $H := L^2(\Omega)$ with $D(T) := D := \Big\{ u \in C^{2n,\nu}(\overline{\Omega}) : \partial^\alpha u_{|\partial\Omega} = 0\,,\ |\alpha| \le n-1 \Big\}$ and with

$$B := (-1)^n \triangle^n$$

on $D(B) := D$. The operator B is symmetric and has lower bound $p > 0$. Further, note that for $w, v \in D$ the divergence theorem yields

$$\langle w, v \rangle_{H_0^n(\Omega)} = \int_\Omega (-1)^n \left(\sum_{i_1,\dots,i_n=1}^{N} \partial_{i_1}^2 \dots \partial_{i_n}^2 w \right) v = \int_\Omega (Bw) v\,,$$

i.e. $H_0^n(\Omega)$ is the energy space of B.

(i) For any $u \in D^r$ the smoothness $f_{i,\alpha} \in C^{|\alpha|,\nu}(\overline{\Omega} \times \mathbb{R}^{rd})$ $(i = 1, \dots, r,$ $|\alpha| \le n)$ implies that $T_i(u) \in C^{0,\nu}(\overline{\Omega})$, hence $R(T) \subset C^{0,\nu}(\overline{\Omega})$. The assumption $\partial\Omega \in C^{2n,\nu}$ implies that $R(B) = C^{0,\nu}(\overline{\Omega})$ (see [1]), therefore $R(B) \supset R(T_i) \quad (i = 1, \dots, r)$.

(ii) For any $u \in D^r$, $v \in D$ and $i = 1, \dots, r$ we have (from the divergence theorem)

$$\begin{aligned}\Big\langle (B^{-1}T_i)(u), v \Big\rangle_{H_0^n(\Omega)} &= \int_\Omega BB^{-1}T_i(u) v = \int_\Omega T_i(u) v \\ &= \int_\Omega \sum_{|\alpha| \le n} f_{i,\alpha}(x, D^{(n)}u) \partial^\alpha v \; dx\,.\end{aligned}$$

If we put arbitrary $u \in H_0^n(\Omega)^r$, $v \in H_0^n(\Omega)$ in the obtained integral then its modulus can be estimated by

$$\begin{aligned}&\int_\Omega \sum_{|\alpha| \le n} \left(|f_{i,\alpha}(x, 0)| + \mu_2 \sum_{j=1}^{r} \sum_{|\beta| \le n} |\partial^\beta u_j(x)| \right) |\partial^\alpha v(x)| dx \\ &\qquad \le \sqrt{d} \left(\max_{|\alpha| \le n} \|f_{i,\alpha}(id_\Omega, 0)\|_{L^2(\Omega)} + \mu_2 \sqrt{rd} \|u\|^*_{H_0^n(\Omega)^r} \right) \|v\|^*_{H_0^n(\Omega)}\,,\end{aligned}$$

i.e. for any fixed $u \in H_0^n(\Omega)^r$ the discussed integral defines a bounded linear functional in v on $H_0^n(\Omega)$. Hence, using Riesz's theorem, the formula

$$\langle F_i(u), v \rangle_{H_0^n(\Omega)} := \int_\Omega \sum_{|\alpha| \le n} f_{i,\alpha}(x, D^{(n)}u) \partial^\alpha v \; dx \quad (v \in H_0^n(\Omega))$$

defines $F_i(u) \in H_0^n(\Omega)$ for arbitrary $u \in H_0^n(\Omega)^r$. The same formula obtained above for $B^{-1}T_i$ when $u \in D^r$ shows that F_i is the desired extension of $B^{-1}T_i$ from D^r to $H_0^n(\Omega)^r$.

Now the desired properties of the F_i's are going to be checked, using the density of D in $H_0^n(\Omega)$ throughout the calculus. For any $u \in H_0^n(\Omega)^r$ let

$S_i(u) \in L\left(H_0^n(\Omega)^r, H_0^n(\Omega)\right)$ be the bounded linear operator defined by

$$\langle S_i(u)h, v\rangle_{H_0^n(\Omega)} := \int_\Omega \sum_{j=1}^r \sum_{|\alpha|,|\beta|\le n} \partial_{\xi_{j,\beta}} f_{i,\alpha}(x, D^{(n)}u)(\partial^\beta h_j)(\partial^\alpha v)\ dx$$
$$\left(h \in H_0^n(\Omega)^r, v \in H_0^n(\Omega)\right).$$

The existence of $S_i(u)$ is provided by Riesz's theorem, now using the estimate $\mu_2\sqrt{r}d\|h\|^*_{H_0^n(\Omega)^r}\|v\|^*_{H_0^n(\Omega)}$ for the right side integral. We will prove that F_i is Gâteaux differentiable ($i = 1,\dots,r$), namely,

$$F_i'(u) = S_i(u) \quad \left(u \in H_0^n(\Omega)^r\right).$$

Let $u, h \in H_0^n(\Omega)^r$ and $\mathcal{E} := \left\{v \in H_0^n(\Omega) : \|v\|_{H_0^n(\Omega)} = 1\right\}$. Then

$$\begin{aligned}
\delta^i_{u,h}(t) :=& \frac{1}{t}\|F_i(u+th) - F_i(u) - tS_i(u)h\|_{H_0^n(\Omega)} \\
=& \frac{1}{t}\sup_{v\in\mathcal{E}} \langle F_i(u+th) - F_i(u) - tS_i(u)h, v\rangle_{H_0^n(\Omega)} \\
=& \frac{1}{t}\sup_{v\in\mathcal{E}} \int_\Omega \sum_{|\alpha|\le n}\Big[f_{i,\alpha}\left(x, D^{(n)}u(x) + tD^{(n)}h(x)\right) - f_{i,\alpha}\left(x, D^{(n)}u(x)\right) \\
&- t\sum_{j=1}^r \sum_{|\alpha|,|\beta|\le n} \partial_{\xi_{j,\beta}} f_{i,\alpha}\left(x, D^{(n)}u(x)\right)\partial^\beta h_j(x)\Big]\partial^\alpha v(x)dx \\
=& \sup_{v\in\mathcal{E}} \int_\Omega \sum_{\alpha,\beta,j} [\partial_{\xi_{j,\beta}} f_{i,\alpha}\left(x, D^{(n)}u(x) + t\theta(x,t)D^{(n)}h(x)\right) \\
&- \partial_{\xi_{j,\beta}} f_{i,\alpha}\left(x, D^{(n)}u(x)\right)]\partial^\beta h_j(x)\partial^\alpha v(x)dx \\
\le& \sup_{v\in\mathcal{E}} \sum_{\alpha,\beta,j} \|(\partial_{\xi_{j,\beta}} f_{i,\alpha}(id, D^{(n)}u + t\theta D^{(n)}h) \\
&- \partial_{\xi_{j,\beta}} f_{i,\alpha}(id, D^{(n)}u))\partial^\beta h_j\|_{L^2(\Omega)} \times \|\partial^\alpha v\|_{L^2(\Omega)}.
\end{aligned}$$

Here $\|\partial^\alpha v\|_{L^2(\Omega)} \le \|v\|^*_{H_0^n(\Omega)} \le \omega_2$ $(|\alpha| \le n)$. Further, $|t\theta(x,t)D^{(n)}h(x)| \le |tD^{(n)}h(x)| \to 0$ (if $t \to 0$) almost everywhere on Ω, hence the continuity of $\partial_{\xi_{j,\beta}} f_{i,\alpha}$ implies that in each term of the sum the first factor converges a.e. to 0 when $t \to 0$. Since the integrands are majorated by $\left(2\mu_2|\partial^\beta h_j(x)|\right)^2$ (which belong to $L^1(\Omega)$) for any $|\beta| \le n$ and $j = 1,\dots,r$, Lebesgue's theorem yields that the obtained expression tends to 0 when $t \to 0$, thus

$$\lim_{t\to 0} \delta^i_{u,h}(t) = 0.$$

(iii) It can be proved similarly to (ii) that for fixed functions $u, k, w, h \in H_0^n(\Omega)^r$ the mapping

$$s, t \mapsto F_i'(u + sk + tw)h$$

is continuous from $\mathbb{R}^2$ to $H_0^n(\Omega)$. Namely,

$$\begin{aligned}\omega^i_{u,k,w,h}(s,t) :=& \|F_i'(u+sk+tw)h - F_i'(u)h\|_{H_0^n(\Omega)}\\ =& \sup_{v\in\mathcal{E}} \langle F_i'(u+sk+tw)h - F_i'(u)h, v\rangle_{H_0^n(\Omega)}\\ =& \sup_{v\in\mathcal{E}} \int_\Omega \sum_{\alpha,\beta,j} [\partial_{\xi_{j,\beta}} f_{i,\alpha}\left(x, D^{(n)}u(x) + sD^{(n)}k(x) + tD^{(n)}w(x)\right)\\ &- \partial_{\xi_{j,\beta}} f_{i,\alpha}\left(x, D^{(n)}u(x)\right)] \partial^\beta h_j(x) \partial^\alpha v(x)dx\,.\end{aligned}$$

Using the continuity of the functions $\partial_{\xi_{j,\beta}} f_{i,\alpha}$ and Lebesgue's theorem, we obtain just as above that

$$\lim_{s,t\to 0} \omega^i_{u,k,w,h}(s,t) = 0\,.$$

(iv) It follows from assumption (b) that for any $u,h,k \in H_0^n(\Omega)^r$

$$\begin{aligned}&\sum_{i=1}^r \langle F_i'(u)h, k_i\rangle_{H_0^n(\Omega)} = \int_\Omega \sum_{i,j} \sum_{|\alpha|,|\beta|} \partial_{\xi_{j,\beta}} f_{i,\alpha}(x, D^{(n)}u)(\partial^\beta h_j)(\partial^\alpha k_i)dx\\ &= \int_\Omega \sum_{i,j} \sum_{|\alpha|,|\beta|} \partial_{\xi_{i,\alpha}} f_{j,\beta}(x, D^{(n)}u)(\partial^\beta h_j)(\partial^\alpha k_i)dx = \sum_{j=1}^r \left\langle h_j, F_j'(u)k\right\rangle_{H_0^n(\Omega)}\,.\end{aligned}$$

(v) For any $u,h \in H_0^n(\Omega)^r$

$$\sum_{i=1}^r \langle F_i'(u)h, h_i\rangle_{H_0^n(\Omega)} = \int_\Omega \sum_{i,j} \sum_{\alpha,\beta} \partial_{\xi_{j,\beta}} f_{i,\alpha}(x, D^{(n)}u)(\partial^\beta h_j)(\partial^\alpha h_i)dx\,.$$

Hence from assumption (c) we have

$$\mu_1 \left(\|h\|'_{H_0^n(\Omega)^r}\right)^2 \le \sum_{i=1}^r \langle F_i'(u)h, h_i\rangle_{H_0^n(\Omega)} \le \mu_2 \left(\|h\|^*_{H_0^n(\Omega)}\right)^2 \quad (h \in H_0^n(\Omega)^r)\,,$$

therefore

$$m\|h\|^2_{H_0^n(\Omega)} \le \sum_{i=1}^r \langle F_i'(u)h, h_i\rangle_{H_0^n(\Omega)} \le M\|h\|^2_{H_0^n(\Omega)} \quad (h \in H_0^n(\Omega)^r)$$

with $m := \mu_1\omega_1^2$ and $M := \mu_2\omega_2^2$, using (11).

(3) We have seen in points (1)–(2) that $T_i \quad (i = 1,\dots,r)$ and B fulfil the conditions of Theorem 1 in the space $H := L^2(\Omega)$. Since $g_i \in C^{0,\nu}(\overline{\Omega}) = R(B) \quad (i = 1,\dots,r)$, the theorem yields the existence of $u^* \in H_0^n(\Omega)^r$ fulfilling (4), which means

$$\langle F_i(u^*), v\rangle_{H_0^n(\Omega)} = \int_\Omega g_i v \quad (i = 1,\dots,r,\ v \in H_0^n(\Omega))\,,$$

i.e. u^* is the generalized solution defined in (12). Further, (6) yields that the sequence (u^k) converges to u^* according to the estimate (15). □

Remark 2. We can see that the theorem quoted from [4] in the introduction applies theoretically to the generalized system $F_i(u) = b_i \quad (i = 1, \dots, r)$. However, in practice the functions $F_i(u^k) - b_i$ cannot be given explicitly for a general $u^k \in H_0^n(\Omega)^r$, hence the approximating sequence cannot be constructed. Thus the advantage of Theorem 2 is the explicit construction of the sequence.

Remark 3. In practice it is worth transforming the equations (10) to the unit ball S if there is a $C^{2n,\nu}$ -smooth one-to-one map from $\overline{\Omega}$ onto $\overline{S}$. Then efficient methods can be used to solve the Poisson equations on S.

Example. Consider the following stationary state of a generalized reaction–diffusion type system:

$$\begin{cases} T_1(u,v)(x) := -\sum_{i=1}^{N} \partial_i \left(f_i(x, \nabla u)\right) + p_1(x,u,v) = g_1(x) \\ T_2(u,v)(x) := -\sum_{i=1}^{N} \partial_i \left(h_i(x, \nabla v)\right) + p_2(x,u,v) = g_2(x) \\ \quad u_{|\partial\Omega} = 0, \quad v_{|\partial\Omega} = 0 \end{cases}$$

with the following assumptions:

(a) $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega \in C^{2,\nu}$ (where $N \in \mathbb{N}^+$, $0 < \nu < 1$); $f_i, h_i \in C^{1,\nu}(\overline{\Omega} \times \mathbb{R}^N)$, $p_1, p_2 \in C^1(\overline{\Omega} \times \mathbb{R}^2)$, $g_1, g_2 \in C^{0,\nu}(\overline{\Omega})$;

(b) $\partial_{\xi_i} f_j(x,\xi) = \partial_{\xi_j} f_i(x,\xi) \ \left((x,\xi) \in \overline{\Omega} \times \mathbb{R}^N\right)$; $\partial_{\eta_i} h_j(x,\eta) = \partial_{\eta_j} h_i(x,\eta)$ $\left((x,\eta) \in \overline{\Omega} \times \mathbb{R}^N\right)$; $\partial_{s_2} p_1(x,s) = \partial_{s_1} p_2(x,s) \ \left((x,s) \in \overline{\Omega} \times \mathbb{R}^2\right)$;

(c) There exist constants $0 < \mu_1 \le \mu_2$ such that the matrices $\left\{\partial_{\xi_j} f_i(x,\xi)\right\}_{i,j=1}^{N}$ and $\left\{\partial_{\eta_j} h_i(x,\eta)\right\}_{i,j=1}^{N}$ have eigenvalues between μ_1 and μ_2 and the matrix $\{\partial_{s_k} p_l(x,s)\}_{k,l=1,2}$ has eigenvalues between 0 and μ_2.

Then the conditions of Theorem 2 are fulfilled.

Let $D := \left\{u \in C^{2,\nu}(\overline{\Omega}) : u_{|\partial\Omega} = 0\right\}$. Denote by G the Green function of $-\triangle$ on D, i.e. for any $\phi \in C^{0,\nu}(\overline{\Omega})$ the unique solution $u \in D$ of the equation $-\triangle u = \phi$ is given by

$$u(x) = \int_\Omega G(x,y)\phi(y)dy \quad (x \in \Omega).$$

Then Theorem 2 yields that for any $u^0, v^0 \in D^2$ the sequence

$$\begin{cases} u^{k+1}(x) := u^k(x) - \frac{2}{M+m} \int\limits_{\Omega} G(x,y) \Big(T_1(u^k, v^k)(y) - g_1(y) \Big) \, dy \qquad (x \in \Omega) \\ v^{k+1}(x) := v^k(x) - \frac{2}{M+m} \int\limits_{\Omega} G(x,y) \Big(T_2(u^k, v^k)(y) - g_2(y) \Big) \, dy \qquad (k \in \mathbb{N}) \end{cases}$$

converges to the generalized solution $(u^*, v^*) \in H_0^1(\Omega)^2$ according to the linear estimate

$$\Big(\|u^k - u^*\|^2_{H_0^1(\Omega)} + \|v^k - v^*\|^2_{H_0^1(\Omega)} \Big)^{1/2}$$
$$\leq \frac{1}{m\sqrt{p}} \Big(\|T_1(u^0, v^0) - g_1\|^2_{L^2(\Omega)} + \|T_2(u^0, v^0) - g_2\|^2_{L^2(\Omega)} \Big)^{1/2} \left(\frac{M-m}{M+m} \right)^k$$

where $p > 0$ is the smallest eigenvalue of $-\triangle$ and m, M are obtained as in Theorem 2. (It is easy to see that now $\omega_1 = 1$ and $\omega_2 = \Big(1 + \frac{1}{N} \operatorname{diam}(\Omega)^2 \Big)^{1/2}$ in (11), hence $m = \mu_1$ and $M = \Big(1 + \frac{1}{N} \operatorname{diam}(\Omega)^2 \Big) \mu_2$.)

Comparison to other methods. The most frequently used methods for (10), based on the variational principle, are the Galerkin or Ritz method and the discrete gradient method (i.e. that applied to the system of algebraic equations after discretizing (10)).

The error of the Galerkin approximations essentially arises from the projection to the corresponding finite dimensional subspaces. The dependence of this error on the dimension of the subspace possesses no geometric (i.e. linear) speed estimate. The sequence (13)–(14) in our method arises from a different, iterative approach. It yields theoretical approximation of the exact solution with linear speed, and the added numerical error only comes from the numerical solution of the Poisson equations, which are essentially simpler than the original one. (The actual performance of solving the Poisson equations is subject to choice, i.e. the Green function above is only an example. An interesting combination may arise if discretization is used for these linear problems.) That is, the main advantages of our method are the linear convergence estimate of the theoretical approximation and the reduction of computational problems to the case of easier linear equations.

Similar considerations hold for the comparison to the discrete gradient method which also approximates a kind of projected equation.

The price to be paid for this is that a new Poisson equation has to be solved in each step. A limitation on the number of iterations to achieve a prescribed accuracy is imposed by the linear convergence estimate.

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