Journal of Applied Analysis
Vol. 3, No. 2 (1997), pp. 249–267

EXISTENCE OF EQUILIBRIA IN FINITELY ADDITIVE NONATOMIC COALITION PRODUCTION ECONOMIES

A. DE SIMONE

Received November 21, 1996 and, in revised form, May 14, 1997

Abstract. The existence of a Walras equilibrium is proved in the framework of a finitely economy with production. The theorem assumes the production set correspondence to be closed–valued and absolutely continuous with respect to the "weighting" finitely additive measure. The result is obtained by using the Kakutani's fixed point theorem and some convergence results stated for multifunctions.

1. Introduction

Coalition production economies have been introduced by Hildenbrand [9] with reference to the intuition, due to Aumann ([4] and [6]), that individuals in large economies can be represented by means of points in a continuum or, what is basically the same, in a nonatomic (countably additive) measure space. By the appearance of the two papers [1] and [2] this idea received new light: perfect competition models more properly require to be based on finite additivity of the measure space of agents (see the wide and deep discussion in [1]). Successively, the coalition production economy model has

1991 *Mathematics Subject Classification.* 90A14, 90D12, 28A12, 28B20.
Key words and phrases. Coalition production economy, Walras equilibrium, production set correspondence, Stone space.

ISSN 1425-6908

been joined with the discovery of Armstrong and Richter in the paper [7] by Basile.

However, the latter only dealt with what is known in the literature as the Core–Walras equivalence. Namely, it was only dealing with showing how it is still true that, in the finitely additive coalition production economies model, the two approaches to equilibria, the game theoretical one (due to Edgeworth) and the other based on the idea of perfect competition (due to Walras), give rise to the same set of equilibria.

As of our knowledge, the question of the existence of such equilibria (in finitely additive coalition production economies) has not been attacked yet. Armstrong and Richter in [2] prove the existence in a model which involves only exchange of goods. On the other hand the existence theorem that can be found in [9] is based on the countably additive assumption.

The aim of this paper is to prove that finitely additive coalition production economies do admit Walras equilibria under mild assumptions. Our result extends that by Armstrong and Richter [2] since the latter corresponds to the case that the production set correspondence is identically null (only exchange activities). Although the scheme of the proof repeat an idea present in [2] (i.e. via Stone space argument one moves to the countably additive setting whose equilibria can be pulled back to equilibria in the finitely additive setting), for the crucial step of finding equilibria in the countably additive setting we had to provide a suitable new existence theorem. This is done by extending to coalition production economies the existence theorem proved by Schmeidler in [12] relative to the case of exchange economies.

The paper is organized as follows: Section 2 contains the description of the model and the main theorem is stated. Then, in a quite wide Section 3, existence theorem (Theorem 2) is proved with reference to the countably additive setting where the two approaches, the coalitional and the individual, may be interchanged. Finally, Section 4 shows how to obtain a Walras equilibrium in a finitely additive coalition production economy from the one found in the countably additive setting.

2. The model and the statement of the main theorem

We use as mathematical model for a large economy a finitely additive (f.a. for short) nonatomic (= strongly continuous according to [8]) probability space $(\Omega, \mathcal{F}, \mu)$. The points of the set Ω represent the economic agents; the elements of $\mathcal{F}$ (an algebra of parts of Ω) represent the coalitions; for every coalition F in $\mathcal{F}$, $\mu(F)$ represents its "weight" in the market. Coalitions F with $\mu(F) = 0$ are called null.

Let n (≥ 2) be the number of different commodities present in the market. The generic bundle of goods is represented by a point $x = (x^1, \dots, x^n)$ of $\mathbb{R}^{\ltimes}_{+}$. We refer to the natural partial order of $\mathbb{R}^{\ltimes}$: If x and y are points of $\mathbb{R}^{\ltimes}$ then $x \leq y$ and $x < y$ have to be meant coordinatewise.

The allocations specify the way to assign a bundle of goods to every coalition. We assume that an allocation is a f.a. function from $\mathcal{F}$ to $\mathbb{R}^{\ltimes}_{+}$ and, according to Richter ([11]), we only consider allocations which are μ-absolutely continuous ($\varepsilon - \delta$ definition). In other words, by allocation we mean an element of the collection

$$\mathcal{A} = \{\alpha : \mathcal{F} \to \mathbb{R}^{\ltimes}_{+} :\ \alpha \text{ is f.a. and } \alpha \ll \mu\}.$$

Coalitions have an *initial endowment* which we represent by an element ν of $\mathcal{A}$. We suppose (it is a standard assumption) that $\int_{\Omega} \nu(t) d\mu > 0$, to ensure the presence of a positive amount of every commodity in the market.

If α is an allocation and F is in $\mathcal{F}$, we denote by α_F the restriction of α to the algebra $\mathcal{F}_F = \{E \in \mathcal{F} : E \subseteq F\}$. If β is another allocation, we write $\alpha \ll_F \beta$ to express the absolute continuity of α_F with respect to β_F.

We assume that each coalition F has a preference relation $\succ_F$ on $\mathcal{A}$.

For $\alpha, \beta \in \mathcal{A}$, the interpretation of $\alpha \succ_F \beta$ (we read that coalition F prefers allocation α to β) is that each member of F prefers what he gets from α to what he gets from β.

Our hypothesis about preferences are those of Armstrong and Richter [2], to recall them we need a few definitions more. If α and β are allocations and F is a coalition, then

$$\alpha \circ\!\!>_F \beta \quad \text{means} \quad \begin{cases} 1)\ \alpha(E) \geq \beta(E) & \forall E \in \mathcal{F}_F \\ 2)\ \beta^i \ll_F \alpha^i - \beta^i & \forall i = 1, \dots, n \end{cases}$$

$$\alpha >_F \beta \quad \text{means} \quad \begin{cases} 1)\ \alpha(E) \geq \beta(E) & \forall E \in \mathcal{F}_F \\ 2)\ \sum_{j=1}^{n} \mu^j \ll_F \alpha^i - \beta^i & \forall i = 1, \dots, n. \end{cases}$$

We also define a "weak preference" $\succeq_F$ in the following way:
$\alpha \succeq_F \beta$ if and only if there exist two sequences $\{\alpha_k\}_{k \in \mathbb{N}}$ and $\{\beta_k\}_{k \in \mathbb{N}}$ in $\mathcal{A}$ with $\alpha_k \succ_F \beta_k \quad \forall k \in \mathbb{N}$, which respectively converge (in the variation norm) to α and β.

We assume the mapping $\succ : F \in \mathcal{F} \to \succ_F \in 2^{\mathcal{A}^2} \setminus \{\emptyset\}$ to be

1a) ideal,
2a) selfish,
3a) monotone,
4a) open,
5a) positively acyclic.

We respectively mean that, if α and β are two allocations and F is any coalition, then:

1a) the set $\{F \in \mathcal{F} : \alpha \succ_F \beta\}$ is an ideal of $\mathcal{F}$;

2a) if α and β are identical on $\mathcal{F}_F$, then $\succ_F$ does not distinguish between them. Namely if γ is any element of $\mathcal{A}$, then the following hold

$$\alpha \succ_F \gamma \Leftrightarrow \beta \succ_F \gamma \quad \text{and} \quad \gamma \succ_F \alpha \Leftrightarrow \gamma \succ_F \beta;$$

3a) if $\alpha(E) \geq \beta(E)$ and $\alpha(E) \neq \beta(E)$ for every nonnull subcoalition E of F, then $\alpha \succ_F \beta$;

4a) if for the coalition F it is $\alpha \succ_F \beta$, then there exist two allocations, α' and β', such that $\alpha \circ\!\!>_F \alpha' \succ_F \beta' >_F \beta$;

5a) there are no cycles of the form $\alpha \circ\!\!>_F \alpha_1 \succeq_F \ldots \succeq_F \alpha_m >_F \alpha$ with $\alpha, \alpha_1, \cdots, \alpha_m \in \mathcal{A}$.

We do not restrict ourselves to consider the exchange as the only economic activity, we also suppose that every coalition F has a productive capability which we express by a subset $Y(F)$ of $\mathbb{R}^{\ltimes}$ called *production possibility set*. The points of this set are called *production plans*. If y is such a point, its positive (resp. negative) components are interpreted as outputs (resp. inputs). More precisely, $y \in Y(F)$ means that the coalition F is able to produce $y^+ = y \vee 0$ if it has at its disposal $y^- = (-y) \vee 0$.

About the **production set correspondence** $Y : F \in \mathcal{F} \to Y(F) \subseteq \mathbb{R}^{\ltimes}$ we assume

$0 \in Y(F) \qquad \forall F \in \mathcal{F}$ and

$E, F \in \mathcal{F},\ E \cap F = \emptyset \ \Rightarrow Y(E) + Y(F) = Y(E \cup F)$.

The first assumption expresses the possibility of inaction, while the second expresses the convenience for the coalitions to unit their forces forming bigger coalitions, in the sense that if two coalitions E and F can choose the productive plans y and z respectively, they can still choose them ($E \cup F$ can choose the plane $y+z$) after they join. In any case, it is $Y(E) \subseteq Y(E \cup F)$.

We denote by $\mathcal{Y}$ the set of all f.a. selections of the production set correspondence. The elements of $\mathcal{Y}$ are said to be *production allocations*.

The collection $\mathcal{E} = (\Omega, \mathcal{F}, \mu, \nu, \succ, Y)$ is called a (f.a., nonatomic) **coalition production economy** (see [7]).

Prices are represented by elements of the set

$$P = \{p \in \mathbb{R}^{\ltimes}_+ : \sum_{\beth=\nVdash}^{\ltimes} \ulcorner^{\beth} = \nVdash\}.$$

In the Walrasian idea of perfect competition the prices of goods are given, neither coalitions nor individuals can influence them.

The triple $(\alpha, \pi, p) \in \mathcal{A} \times \mathcal{Y} \times P$ is a **Walras equilibrium** if

1b) $\alpha(\Omega) = \nu(\Omega) + \pi(\Omega)$;

2b) $p \cdot \pi(F) = \max p \cdot Y(F) \qquad \forall F \in \mathcal{F}$;

3b) $p \cdot \alpha(F) \leq p \cdot \nu(F) + p \cdot \pi(F) \qquad \forall F \in \mathcal{F}$;

4b) $F \in \mathcal{F}, \quad \beta \succ_F \alpha, \quad \mu(F) > 0 \Rightarrow p \cdot \beta(F) > p \cdot \nu(F) + p \cdot \pi(F)$.

Before stating the following theorem, which is the main result of the paper, we recall that Y is said to be μ-absolutely continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\mu(F) < \delta \Rightarrow \|y\| < \varepsilon$ for all $y \in Y(F)$. We also assume that the supremum norm is used throughout the paper.

Theorem 1. *Let $\mathcal{E} = (\Omega, \mathcal{F}, \mu, \nu, \succ, Y)$ be a f.a. nonatomic coalition production economy. If the production set correspondence Y is closed-valued and μ-absolutely continuous, then there exists a Walras equilibrium.*

The outline of the proof is similar to the one used by Armstrong and Richter in [2]. It consists of three steps:

– via the Stone representation theorem the model is transferred to a σ-additive one,

– a Walras equilibrium is found in the σ-additive context,

– the latter is pulled back to the original model and it is shown to be still a Walras equilibrium.

We begin from the second step, by proving an existence theorem in σ-additive framework (Theorem 2). We complete the proof in Section 4.

3. The σ-additive case

As said in the introduction, this section is devoted to investigate the existence of Walras equilibria in σ-additive (σ-a., for short) setting. If we suppose that the space $(\Omega, \mathcal{F}, \mu)$ that represents the economy is a nonatomic σ-a. probability space (we mean that $\mathcal{F}$ is a σ-algebra and μ is a nonatomic σ-a. measure), we have one more (equivalent) way to define Walras equilibria. Because of μ-absolute continuity, every element of $\mathcal{A}$ is σ-a. and by the Radon-Nikodym theorem, there exists the μ-derivative f of α:

$$\alpha(F) = \int_F f(t)d\mu \qquad \forall F \in \mathcal{F}.$$

The same happens for the production set correspondence: it is σ-a. and it has a μ-derivative Z. The latter is an Aumann integrable multifunction on Ω such that

$$Y(F) = \int_F Z(t)d\mu \qquad \forall F \in \mathcal{F}.$$

This gives us the possibility to talk about the production set of each agent. We will denote by $\mathcal{Z}$ the set of all integrable selections of Z.

In the σ-additive case, it is also possible to consider a "density" for the preferences, in the sense that if we assume them to verify properties 1a), 2a), ..., 5a) and moreover, to be

1a') σ-ideal ($\{F \in \mathcal{F} : \alpha \succ_F \beta\}$ is a σ-ideal of $\mathcal{F}$),
6a) asimmetric ($\alpha \not\succ_F \alpha$),
7a) transitive ($\alpha \succ_F \beta, \quad \beta \succ_F \gamma \Rightarrow \alpha \succ_F \gamma$),
then it is possible to generate them (in the sense of Armstrong and Richter [2]) by individual preferences $\succ_t$ ($t \in \Omega$) which have the following properties (x, y and z are points of $\mathbb{R}^{\ltimes}_+$, f and g are in $L_1(\mu)$):

1c) $x \succ_t y, \quad y \succ_t z \Rightarrow x \succ_t z$	(transitivity);
2c) $x \not\succ_t x$	(irriflessivity);
3c) $\{x : x \succ_t y\}$ and $\{x : y \succ_t x\}$ are open	(continuity);
4c) $x \geq y, \quad x \neq y \Rightarrow x \succ_t y$	(desiderability or monotonicity);
5c) $\{t : f(t) \succ_t g(t)\}$ is measurable	(measurability).

The converse is also true: individual preferences verifying 1c), ..., 5c) above, generate coalitional preferences for which 1a'), 2a), ..., 7a) hold.

If $(\Omega, \mathcal{F}, \mu)$ is a σ-a. probability space and preferences satisfy 1a'), 2a), ..., 7a), we will talk about **σ-additive economies**.

In this section we prove a theorem which estabilish the existence of a Walras equilibrium for a nonatomic σ-additive economy. To obtain this result, we need preferences to satisfy the additional hypothesis 1a'), 6a) and 7a). This is not a real restriction for our goal. In fact, when in Section 4 we will use the Stone representation theorem to transfer the original economy to a σ-additive one, we will obtain a model in which preferences will automatically verify hypothesis 1a'), 2a), ..., 7a).

To state the theorem we need, we have to suppose that

1d) $Z(t)$ is closed and it contains zero $\quad \forall t \in \Omega$;

2d) $\exists \gamma \in \mathbb{R}^+$ such that $\quad z \leq (\gamma) \quad \forall z \in Z(t) \quad \forall t \in \Omega$ where $(\gamma) = (\gamma, \dots, \gamma)$.

To give an upper bound to the production sets (hypothesis 2d) expresses the fact that the production cannot be infinite even if we assume free disposal of goods. The uniformity of the bond (γ does not depend on t) is justified by the fact that when we pass to coalitional point of view, it is still impossible to have infinite production and these production sets $Z(t)$ are the basis to construct the production set of the whole economy.

Let us introduce some other notations:

$N(p,t) = \max\{p \cdot z : z \in Z(t)\}$,
$E(p,t) = \{z \in Z(t) : p \cdot z = N(p,t)\}$,
$A(p,t) = \{x \in \mathbb{R}^{\ltimes}_+ : \imath \cdot \curvearrowleft \leq \imath \cdot \nu(\approx) + \mathbb{N}(\imath, \approx)\}$.

Note that $N(p,t)$ exists because $p \in \mathbb{R}^{\ltimes}_+$ and Z is closed and upper bounded. For short, we write $\int f$ (resp. $\int_F f$) rather than $\int_\Omega f(t)d\mu$ (resp. $\int_F f(t)d\mu$), we still indicate by ν the μ-derivative of the initial endowment and we neglect null sets in the definitions (in the sense that when we write $\forall t \in \Omega$ we mean $\forall t \in \Omega$ μ-a.e.).

At this point it is immediate to verify that if $f \in L_1(\mu)$ and $z \in \mathcal{Z}$ are respectively the μ-derivative of the allocation α and the μ-derivative of $\pi \in \mathcal{Y}$, then the triple (α, π, p) is a Walras equilibrium if and only if (f, z, p) is such that

1e) $\int f = \int \nu + \int z$;

2e) $z(t) \in E(p,t) \qquad \forall t \in \Omega$;

3e) $p \cdot f(t) \leq p \cdot \nu(t) + N(p,t) \qquad \forall t \in \Omega$;

4e) $z \geq 0, \quad z \succ_t f(t) \Rightarrow p \cdot z > p \cdot \nu(t) + N(p,t)$.

Of course an equivalent formulation for 3e) and 4e) is

5e) $\forall t \in \Omega \quad f(t)$ is a maximal element in $A(p,t)$ with respect to the preference relation $\succ_t$.

To sum up, we can say that in the σ-additive setting we can replace the original (coalitional) point of view (expressed by definition 1b), ..., 4b) and assumptions 1a'), 2a), ..., 7a)) with an individual one (expressed by definition 1e), ..., 4e) and assumptions 1c), ..., 5c)). The latter will be used along the proof of the next, special and preliminary, case of Theorem 1.

Theorem 2. *Let $\mathcal{E}$ be as in Theorem 1. Assume further that it is σ-additive and that the individual production sets $Z(t)$ verify 1d) and 2d), then there exists a Walras equilibrium.*

The idea of the proof is the following (we improve the method used by Schmeidler in [12] for the case of pure exchange economy): we define some equilibria, called k-equilibria, whose definition is similar to that of Walras equilibria, except for their allocations and production plans, that are confined to belong to some ball of $\mathbb{R}^\ell$ whose ray depends on k. Using a fixed point argument (Proposition 1), we find a k-equilibrium for every $k > 1$ (Proposition 2) and we show that, when k goes to infinity, from these equilibria we can obtain a Walras equilibrium (Section 3.3).

With respect to a fixed $k > 1$ and to the (integrable) function $\lambda(t) = \sum_{i=1}^n \nu^i(t)$, we define the following subsets of $\mathbb{R}^\ell$ which will be useful in the sequel:

$B(k) = \{x \in \mathbb{R}^\ell : \|x\| \leq k \int \lambda\}$;

$B_+(k) = B(k) \cap \mathbb{R}^\ell_+$;

$B(k,t) = \{x \in \mathbb{R}^\ell : \|x\| \leq k\lambda(t)\} \quad (t \in \Omega)$;

$B_+(k,t) = B(k,t) \cap \mathbb{R}^\ell_+ \quad (t \in \Omega)$.

For each t in Ω, the set $B(k,t) \cap Z(t)$ is compact because of the closedness of $Z(t)$, so it is possible to consider

$$M(k,p,t) = \max\{p \cdot x : x \in B(k,t) \cap Z(t)\}$$

which represents the (nonnegative) maximum income that agent t can obtain by using his productive capability, if we admit that he must choose his production plan in the ball $B(k,t)$.

The set of production plans that permit the agent t to obtain the income $M(k,p,t)$ is denoted by $E(k,p,t)$ so

$$E(k,p,t) = \{x \in B(k,t) \cap Z(t) : p \cdot x = M(k,p,t)\}.$$

Since agent t is able to obtain $M(k,p,t)$ with a suitable choice of a production plan in his production set, his budget set is

$$S(k,p,t) = \{x \in \mathbb{R}_+^\ell : p \cdot x \leq p \cdot \nu(t) + M(k,p,t)\}.$$

As already said, we do not consider all the bundles in this budget set. We restrict ourselves to consider

$$C(k,p,t) = S(k,p,t) \cap B_+(2k,t).$$

Finally, we denote by $C'(k,p,t)$ the set of the maximal elements of $C(k,p,t)$ with respect to the preference relation $\succ_t$, so

$$C'(k,p,t) = \{x \in C(k,p,t) : y \succ_t x \Rightarrow y \notin C(k,p,t)\}$$

and we define

$$D(k,p,t) = \begin{cases} C(k,p,t) & \text{if} \quad p \cdot \nu(t) + M(k,p,t) = 0 \\ C'(k,p,t) & \text{if} \quad p \cdot \nu(t) + M(k,p,t) > 0. \end{cases}$$

3.1. **A fixed point argument.**

For a number $k > 1$, let us consider the following correspondence ϕ from $B_+(2k) \times B(k) \times P$ into itself:

$\phi(x,y,p) = \phi^1(p) \times \phi^2(p) \times \phi^3(x,y)$

where

$\phi^1(p) = \int D(k,p,t)$,

$\phi^2(p) = \int E(k,p,t)$,

$\phi^3(x,y) = \{p \in P : \forall q \in P \quad p \cdot (x - y - \int \nu) \geq q \cdot (x - y - \int \nu)\}$.

Proposition 1. *The correspondence ϕ has a fixed point.*

Proof. The sets $B_+(2k), B(k)$ and P are non–empty, convex and compact. If we show that for any (x,y,p) in $B_+(2k) \times B(k) \times P$ its image $\phi(x,y,p)$ is non–empty and convex, and that ϕ is closed, the existence of a fixed point for ϕ will follow from Kakutani's fixed point theorem. It is enough to refer to the components ϕ^1 and ϕ^2 of ϕ, since for ϕ^3 the result is trivial.

CLAIM. *For any price p , the sets $\phi^1(p)$ and $\phi^2(p)$ are non–empty.*

Proof. For any t in Ω we have

$$D(k,p,t) \subseteq B_+(2k,t),$$

hence the correspondences $t \to D(k,p,t)$ and $t \to E(k,p,t)$ are integrably bounded (uniformly in p). They are also measurable for any price p. The first one because of measurability of $\succ_t$, the second one because of the equality

$$\{(t,x) : t \in \Omega \quad x \in E(k,p,t)\} =$$
$$= [\Omega \times (B(k,t) \cap Z(t))] \cap \{(t,x) : t \in \Omega \quad p \cdot x = M(k,p,t)\}.$$

Corollary 5.2 in [5] ensures that for any p in P the sets $\phi^1(p)$ and $\phi^2(p)$ are non–empty. □

CLAIM. *For any t in Ω the correspondence $\theta_1 : p \in P \to D(k,p,t) \subseteq \mathbb{R}_+^{\ltimes}$ is closed.*

Proof. Let us fix an element t in Ω. Suppose that $\lim_h p_h = p, \quad \lim_h x_h = x$ and that, for each h in $\mathbb{N}$, the point x_h belongs to $D(k, p_h, t)$. From the relation $x_h \in D(k,p_h,t) \subseteq C(k,p_h,t) = B_+(2k,t) \cap S(k,p_h,t)$ it follows that $\|x_h\| \leq 2k\lambda(t)$ and $p_h \cdot x_h \leq p_h \cdot \nu(t) + M(k,p_h,t)$ for each $h \in \mathbb{N}$.

Taking the limits of these relations when h tends to infinity and observing that $\lim_h M(k,p_h,t) = M(k,p,t)$, we have $\|x\| \leq 2k\lambda(t)$ and $p \cdot x \leq p \cdot \nu(t) + M(k,p,t)$. This ensures that x belongs to the set $C(k,p,t)$. Let us show that x is in fact in $D(k,p,t)$.

Let $p \cdot \nu(t) + M(k,p,t)$ be positive and let us assume that there exists an element w in $C(k,p,t)$ such that $w \succ_t x$. We can suppose that for a point z of $C(k,p,t)$ we have $z \succ_t x$ and $p \cdot z < p \cdot \nu(t) + M(k,p,t)$. In fact, if this is not true for w, then $p \cdot w = p \cdot \nu(t) + M(k,p,t)$; the continuity of $\succ_t$ (hypothesis 3c)) ensures that there exists $z < w$ such that (z is still preferred to x by t) $z \succ_t x$, this is the wanted point.

When h is large enough ($h > h_1$), we have

$$p_h \cdot z < p_h \cdot \nu(t) + M(k,p_h,t) \qquad z \succ_t x_h.$$

This means that x_h is eventually not maximal.

On the other hand since $p \cdot \nu(t) + M(k,p,t)$ is positive, for h large enough ($h > h_2$) it results

$$p_h \cdot \nu(t) + M(k,p_h,t) > 0.$$

By the definition of $D(k,p_h,t)$, from this it follows that x_h is eventually maximal. A contradiction. □

CLAIM. *For any t in Ω the correspondence $\theta_2 : p \in P \to E(k,p,t) \subseteq \mathbb{R}^{\ltimes}$ is closed.*

Proof. If $\lim_h p_h = p, \quad \lim_h y_h = y$ and, for each h in $\mathbb{N}$, $\curvearrowright_{\eqsim}$ belongs to $E(k,p_h,t)$, then, for each h in $\mathbb{N}$ we have

$y_h \in B(k,t) \cap Z(t)$ and

$p_h \cdot y_h = M(k,p_h,t)$.

From these relations, when h goes to infinity, we respectively have

$y \in B(k,t) \cap Z(t)$ (since this set is closed),
$p \cdot y = M(k,p,t)$ (since $\lim_h M(k,p_h,t) = M(k,p,t)$).
So y belongs to the set $E(k,p,t)$ and the proof is complete. □

Now we terminate the proof of Proposition 1 by appealing to Theorem 1 in [9] and nonatomicity of μ. They ensure that the sets $\int D(k,p,t)$ and $\int E(k,p,t)$ are convex. By applying Kakutani's theorem to ϕ we get the desired conclusion.

3.2. **Existence of k-equilibria.**

We call **k-equilibrium** a triple $(f,z,p) \in L_1^+(\mu) \times \mathcal{Z} \times P$ such that
1f) $\int f = \int \nu + \int z$;
2f) $z(t) \in E(k,p,t) \quad \forall t \in \Omega$;
3f) $p \cdot f(t) \le p \cdot \nu(t) + M(k,p,t) \quad$ and $\quad f(t) \in B_+(2k,t) \quad \forall t \in \Omega$;
4f) $p \cdot \nu(t) + M(k,p,t) > 0 \Rightarrow f(t) \in C'(k,p,t)$.
We observe that 3f) and 4f) together are equivalent to
5f) $f(t) \in D(k,p,t) \quad \forall t \in \Omega$.

Proposition 2. *For any k greater than 1 there exists a k-equilibrium.*

Proof. Fix $k > 1$ and let $(\overline{x},\overline{y},\overline{p})$ be a fixed point of ϕ, namely $\overline{x} \in \int D(k,\overline{p},t)$, $\overline{y} \in \int E(k,\overline{p},t)$ and $\overline{p} \in \{p \in P : \forall q \in P \quad p \cdot (\overline{x} - \overline{y} - \int \nu) \ge q \cdot (\overline{x} - \overline{y} - \int \nu)\}$. There exist elements w in $L_1^+(\mu)$ and r in $L_1(\mu)$ such that
$\overline{x} = \int w$ and $w(t) \in D(k,\overline{p},t) \quad \forall t \in \Omega$,
$\overline{y} = \int r$ and $r(t) \in E(k,\overline{p},t) \quad \forall t \in \Omega$.
Note that $r(t)$ is an element of $E(k,\overline{p},t)$, so we have $\overline{p} \cdot r(t) = M(k,\overline{p},t)$. Since $w(t)$ belongs to $D(k,\overline{p},t)$, which is contained in $B_+(2k,t)$, all the components of $w(t)$ are less than or equal to $2k\lambda(t)$. Let us show that it is impossible that they are all simultaneously equal to $2k\lambda(t)$. □

CLAIM. *At least for an index i, we have $w^i(t) < 2k\lambda(t)$.*

Proof. If x is an element of $B(k,t)$, then, by definition, we have $x^i \le k\lambda(t) = k\sum_{j=1}^n \nu^j(t)$ for every i. So, for x in $B(k,t) \cap Z(t)$, we have $\overline{p} \cdot x \le k\lambda(t) \sum_{s=1}^n \overline{p}_s$ and therefore, $M(k,\overline{p},t) \le k\lambda(t) \sum_{s=1}^n \overline{p}_s$. If we assume that $w^i(t) = 2k\lambda(t)$ for all i, then we have

$$\overline{p} \cdot w(t) > \lambda(t) \sum_{s=1}^n \overline{p}_s + k\lambda(t) \sum_{s=1}^n \overline{p}_s \ge \overline{p} \cdot \nu(t) + M(k,\overline{p},t).$$

This is a contradiction since $w(t)$ belongs to $D(k,\overline{p},t)$ which is a subset of $S(k,\overline{p},t)$. □

CLAIM. *For any t in Ω it is $\overline{p} \cdot w(t) = \overline{p} \cdot \nu(t) + \overline{p} \cdot r(t)$.*

Proof. We can suppose that $\overline{p} \cdot \nu(t) + \overline{p} \cdot r(t)$ is positive. From the previous claim, we know that we can increase one of the components of $w(t)$ remaining in $B(2k, t)$. Namely there exists an index i and a positive ε such that

$$\|w(t) + \varepsilon e^i\| \leq 2k\lambda(t),$$

where by e^i we denote the i-th vector of the natural base of $\mathbb{R}^{\ltimes}$.

If we assume that

$$\overline{p} \cdot w(t) < \overline{p} \cdot \nu(t) + \overline{p} \cdot r(t) = \overline{p} \cdot \nu(t) + M(k, \overline{p}, t),$$

we can find ε' in $]0, \varepsilon[$ such that

$$\overline{p} \cdot w(t) + \varepsilon' \leq \overline{p} \cdot \nu(t) + M(k, \overline{p}, t).$$

Since prices are normalized we have

$$\overline{p} \cdot (w(t) + \varepsilon' e^i) \leq \overline{p} \cdot w(t) + \varepsilon' \leq \overline{p} \cdot \nu(t) + M(k, \overline{p}, t).$$

The desiderability of $\succ_t$ (hypothesis 4c) ensures that

$$w(t) + \varepsilon' e^i \succ_t w(t).$$

We have found an element in $B_+(2k, t)$ which is preferred to $w(t)$, this is impossible because it contradicts the maximality of $w(t)$ in $C(k, \overline{p}, t)$. □

Let us set $b = \int (w - \nu - r)$ for short. Evidently $\overline{p} \cdot b = 0$. Since $\overline{p}$ is an element of $\phi^3(\overline{x}, \overline{y})$, then $q \cdot b \leq 0 \quad \forall q \in P$; consequently $b^i \leq 0 \quad \forall i \in \{1, \ldots, n\}$. Therefore $\overline{p}^i = 0$ for any i such that $b^i < 0$.

If $b = 0$, the wanted equilibrium is $(w, r, \overline{p})$. Othervise, for any i such that b^i is negative, we define

$A^i = \{t \in \Omega : w^i(t) < \nu^i(t) + r^i(t)\}$.

It comes out that $\mu(A^i) > 0$ (otherwise $b^i = \int_{\Omega \setminus A^i} (w^i - \nu^i - r^i) \geq 0$).

Let us set $C^i = \int_{A^i} (\nu^i + r^i - w^i)$ and observe that from $\mu(A^i) > 0$ it follows that $C^i > 0$. Moreover $|b^i| \leq C^i$ since $|b^i| = -b^i = \int_\Omega (\nu^i + r^i - w^i) \leq \int_{A^i} (\nu^i + r^i - w^i) = C^i$.

At this point define

$$f^i(t) = \begin{cases} w^i(t) + (|b^i|/C^i)(\nu^i(t) + r^i(t) - w^i(t)) & \text{if } b^i < 0 \text{ and } t \in A^i \\ w^i(t) & \text{otherwise} \end{cases}.$$

CLAIM. *The triple $(f, r, \overline{p})$ is a k-equilibrium.*

Proof. 1f) and 2f) are promptly verified once it is observed that $\int_\Omega f^i = \int_{A^i} f^i + \int_{\Omega \setminus A^i} f^i = \int_{A^i} w^i + (|b^i|/C^i) \int_{A^i} (\nu^i + r^i - w^i) + \int_{\Omega \setminus A^i} w^i = \int_\Omega w^i + (|b^i|/C^i) C^i = \int_\Omega w^i + \int_\Omega (\nu^i + r^i - w^i) = \int_\Omega (\nu^i + r^i)$. Then

$$\int_\Omega z = \int_\Omega (\nu + r) \qquad [*]$$

(for i such that $b^i = 0$ the equality trivially holds).

Let us move to prove 3f). Note that $f(t)$ and $w(t)$ can only differ in some component i such that b^i is negative, but in that case we observed that $\overline{p}^i = 0$, hence this difference does not influence the scalar product of $f(t)$ or $w(t)$ with $\overline{p}$. From this and from the previous claim we have

$$\overline{p} \cdot f(t) = \overline{p} \cdot w(t) = \overline{p} \cdot \nu(t) + M(k, \overline{p}, t).$$

Moreover, since $|b^i| \le C^i$, then

$$\frac{|b^i|}{C^i}(\nu^i + r^i - w^i) \le \nu^i + r^i - w^i.$$

This implies that, $f^i(t) \le \nu^i(t) + r^i(t) \le 2k\lambda(t)$ if $f^i(t)$ differs from $w^i(t)$. So, for any t in Ω, the vector $f(t)$ belongs to $B_+(2k, t)$ and 3f) is proved. Finally, assume that there exists t in Ω and z' in $C(k, \overline{p}, t)$ such that $z' \succ_t f(t)$; by desiderability of $\succ_t$ it follows that $f(t) \succ_t w(t)$ and by transitivity one gets $z' \succ_t w(t)$.

The vector $w(t)$ is in $D(k, \overline{p}, t)$, so t must be such that

$$\overline{p} \cdot \nu(t) + M(k, \overline{p}, t) = 0,$$

this completes the proof. ☐

Remark. We have found, for any $k > 1$, a k-equilibrium for which the equality holds in statement 3f) of the definition. In addition equality [$*$] (see the proof of claim above) holds.

3.3. **Existence of Walras equilibria (proof).**

The last step consists of a limit argument. We will obtain a Walras equilibrium from the given k-equilibria, by letting k run to infinity.

For any $k > 1$ we have found a k-equilibrium (x_k, y_k, p_k). Since P is compact, we can suppose that there exists p in P such that the sequence $(p_k)_{k \in \mathbb{N}}$ converges to p.

CLAIM 1. *If there exists an index i such that p^i is zero and if t is such that the scalar product $p \cdot \nu(t)$ is positive, then the sequence $\{x_k(t)\}_{k \in \mathbb{N}}$ does not have limit points.*

Proof. Suppose x to be such a point, then there exists a subsequence of $\{x_k(t)\}_k$ (we still indicate it by $\{x_k(t)\}_k$) whose limit is x. Since

$$0 \le M(k, p_k, t) \le \gamma \qquad \forall k \in \mathbb{N},$$

then, by passing to a subsequence if necessary, we have that $\{M(k, p_k, t)\}_{k \in \mathbb{N}}$ has a limit, let us denote it by $L(t)$.

For any k in $\mathbb{N}$ it is

$$p_k \cdot x_k(t) = p_k \cdot \nu(t) + M(k, p_k, t) \ge p_k \cdot \nu(t).$$

taking the limit of this relation for k that goes to infinity we have

$$p \cdot x(t) = p \cdot \nu(t) + L(t) \geq p \cdot \nu(t) > 0,$$

so there exists an index j in $\{1, \dots , n\}$ such that $p^j > 0$ and $x^j(t) > 0$ (obviously $j \neq i$).

Monotonicity of preferences ensures that $x + e^i \succ_t x$ and for their continuity there exists a positive δ such that if we define $z = x + e^i - \delta e^j$ it is $z \in \mathbb{R}^{\ltimes}_{+}$ and $z \succ_t x$.

If we take k large enough ($k > k_1$), we have $\|z\| \leq 2k\lambda(t)$. Since $x_k(t)$ goes to x and the preferences are continuous, then there exists a k_2 in $\mathbb{N}$ such that

$$z \succ_t x_k \qquad \forall k > k_2.$$

If we take $k > \max\{k_1, k_2\}$, then the maximality of x_k in $C(k, p_k, t)$ implies that

$$p_k \cdot z > p_k \cdot \nu(t) + M(k, p_k, t)$$

which gives, for k that goes to infinity,

$$p \cdot z \geq p \cdot \nu(t) + L(t) = p \cdot x.$$

This is impossible because

$$p \cdot z = p \cdot x + p^i - \delta p^j < p \cdot x.$$

□

CLAIM 2. $p > 0$.

Proof. Suppose that there exists an index i in $\{1, \dots, n\}$ for which p^i is zero. Let us define $T = \{t \in \Omega : p \cdot \nu(t) > 0\}$. If we observe that $\int p \cdot \nu = p \cdot \int \nu$ and that the latter integral is positive because, by hypothesis, all the components of $\int \nu$ are positive, we conclude that $\mu(T)$ is positive. Let us define

$$\eta = \frac{2}{\mu(T)} \left[\int \sum_{j=1}^{n} \nu^j + \limsup_k \int \sum_{j=1}^{n} y_k^j \right]$$

and $F = \{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x^j \leq \eta\}$.

Let t be in T, from Claim 1 it follows that $x_k(t)$ belongs to F for at most a finite number of k. So for any t in T there exists k_t in $\mathbb{N}$ such that $\sum_{j=1}^{n} x_k^j(t) > \eta$ for $k > k_t$. Then

$$\liminf_k \sum_{j=1}^{n} x_k^j(t) \geq \eta \qquad \forall t \in T;$$

this ensures that

$$\int_T \liminf_k \sum_{j=1}^{n} x_k^j \geq \eta \mu(T),$$

so we have
$\eta\mu(T) \leq \int_T \liminf_k \sum_{j=1}^{n} x_k^j \leq \liminf_k \int_T \sum_{j=1}^{n} x_k^j \leq \liminf_k \int_\Omega \sum_{j=1}^{n} x_k^j =$ (since $\int x_k = \int(\nu + y_k)$) $\liminf_k \int \sum_{j=1}^{n}(\nu^j + y_k^j) \leq \limsup_k \int \sum_{j=1}^{n} \nu^j + \limsup_k \int \sum_{j=1}^{n} y_k^j = \int \sum_{j=1}^{n} \nu^j + \limsup_k \int \sum_{j=1}^{n} y_k^j = \mu(T)\eta/2$,
a contradiction. □

From the equality $p = \lim_k p_k$ and from Claim 2 it follows that there exists a positive δ such that when k is sufficiently large ($k > K$) all the components p_k^i of p_k are greater than δ.

CLAIM 3. *For $k > K$ it is $y_k^i(t) \geq (1 - n)\gamma/\delta \quad \forall i = 1, \dots, n$ and $\forall t \in \Omega$.*

Proof. Let k be greater than K and let t be an element of Ω. If for a particular index i it is $y_k^i(t) < [(1-n)\gamma]/\delta$, then

$$p_k \cdot y_k = \sum_{j=1}^{n} p_k^j y_k^j = \sum_{j \neq i} p_k^j y_k^j + p_k^i y_k^i \leq \sum_{j \neq i} p_k^j \gamma + p_k^i y_k^i.$$

From $p_k^j \leq 1$ we obtain $\sum_{j \neq i} p_k^j \gamma \leq (n-1)\gamma$. On the other hand, $p_k^i > \delta$ and $y_k^i < (1-n)\gamma/\delta < 0$ imply that $p_k^i y_k^i < [\delta(1-n)\gamma]/\delta$. Therefore we have that the scalar product $p_k \cdot y_k$ is negative. The latter is impossible because (x_k, y_k, p_k) is a k-equilibrium. □

Let us define, $G_k(t) = \{y_k(t)\}$ for each t in Ω and denote by $G(t)$ the set of the limit points of $G_k(t)$. We have (see [5])

$$\int G \supseteq \limsup_k \int G_k(t).$$

Let y be such that $y(t) \in G(t)\ \ \forall t \in \Omega$ and $\int y = \lim_k \int y_k$.

Such an y exists. In fact let us observe that

$$-\int \nu \leq \int y_k \leq (\gamma) \qquad \forall k \in \mathbb{N}, \qquad [**]$$

where first inequality follows from the observation that $\int \nu + \int y_k = \int x_k \geq 0$ implies that $\int y_k \geq -\int \nu$ and the second one follows from the observation that $y_k(t) \in Z(t) \quad \forall t \in \Omega$ and $y_k(t) \leq (\gamma) \quad \forall t \in \Omega$.

Thus, by [**], the sequence $\{\int y_k\}_k$ admits a convergent subsequence (we still indicate it by $\{\int y_k\}_k$) for which we have $\lim_k \int y_k \in \limsup_k \int G_k \subseteq \int G$.

We note that the sequence $(\int x_k)_{k\in\mathbb{N}}$ converges because we have

$$\int (\nu + y) = \lim_k \int (\nu + y_k) = \lim_k \int x_k.$$

The nonnegativity of p_k and of $x_k(t)$, jointly with

$$p_k \cdot x_k(t) = p_k \cdot \nu(t) + M(k, p_k, t) \qquad \forall k \in \mathbb{N},\ \forall \approx \in \Omega$$

(that follows from the remark at the end of Section 3.2), gives

$$p_k^i x_k^i(t) \leq p_k \cdot \nu(t) + M(k, p_k, t)$$
$$\forall k \in N,\ \forall t \in \Omega,\ \forall i \in \{1, \ldots, n\}. \quad [***]$$

Since $Z(t)$ is bounded by γ, using $[***]$ we have

$$p_k^i x_k^i(t) \leq \sum_{i=1}^{n} \nu^i(t) + \gamma \qquad \forall k \in \mathbb{N},\ \forall \approx \in \Omega,\ \forall \sqsupset \in \{\nVdash, \ldots, \ltimes\}.$$

Therefore for $k > K$ we have

$$\delta \|x_k(t)\| \leq \sum_{i=1}^{n} \nu^i(t) + \gamma \qquad \forall t \in \Omega.$$

Let us now define $X_k(t) = \{x_k(t)\}$ and denote by $X(t)$ the set of the limit points of $X_k(t)$.

From the latter inequality it follows by Fatou's lemma that

$$\lim_k \int x_k \in \limsup_k \int X_k \subseteq \int X.$$

This allows us to choose x such that $x(t) \in X(t)\ \ \forall t \in \Omega$ and $\int x = \int(\nu + y)$.

CLAIM 4. *The triple* (x, y, p) *is a Walras equilibrium.*

Proof. 1e) is obviously verified.

Since we know, from Claim 3, that there exists a positive integer K such that $\|y_k\| \leq (n-1)\gamma/\delta$ for any $k > K$, then for $k > \max\{K, (n-1)\gamma/\delta\}$ the production plan $y_k(t)$ permits to obtain, at prevalent price p_k, the maximum income not only in $Z(t) \cap B(k,t)$ but in all $Z(t)$. We mean that for k sufficiently large

$$p_k \cdot y_k = \max\{p \cdot z : z \in Z(t)\}.$$

Let us denote this maximum by $N(p,t)$. This ensures that the sequence $\{M(k, p_k, t)\}_{k\in\mathbb{N}}$ is eventually constant (equal to $N(p,t)$) and it gives 3e).

Suppose that t and z can be found such that $t \in \Omega$, $z \in A(p,t)$ and $z \succ_t x(t)$. Because of desiderability of preferences, we have $z \neq 0$. To fix our ideas let us assume that z^1 is positive.

Let us denote by β a positive number such that $z_\beta = z - (\beta, 0, \dots, 0) \in \mathbb{R}^{\ltimes}_+$ and $z_\beta \succ_t x(t)$ (continuity of preferences). Since $\lim_k p_k \cdot z_\beta = p \cdot z - \beta p^1 < p \cdot z \leq p \cdot \nu(t) + N(p,t) = \lim_k (p_k \cdot \nu(t) + M(k, p_k, t))$ we can affirm that there exists k_1 in $\mathbb{N}$ such that

$$p_k \cdot z_\beta < p_k \cdot \nu(t) + M(k, p_k, t) \qquad \forall k > k_1.$$

Moreover there exists k_2 in $\mathbb{N}$ such that

$$\|z_\beta\| \leq 2k \sum_{i=1}^{n} \nu^i(t) \qquad \forall k > k_2.$$

Since $x(t) \in X(t)$ then, using if necessary a subsequence, we have $x(t) = \lim_k x_k(t)$ and there exists k_3 in $\mathbb{N}$ such that $z_\beta \succ_t x_k(t)$ $\forall k > k_3$. For $k > \max\{k_1, k_2, k_3\}$, the vector $x_k(t)$ is not maximal in $C(k, p_k, t)$. This contradiction proves 4e).

Finally, to show 2e), let us prove that the set $R = \{t \in \Omega : p \cdot y(t) < \max p \cdot Z(t)\}$ has measure zero.

Since $E(p,t)$ is measurable and integrably bounded, we have $\int E(p,t) \neq \emptyset$ (see [10] or [9]). Let e be an integrable selection of $E(p,t)$, then, for any t in R,

$$p \cdot e(t) = \max p \cdot Z(t) > p \cdot y(t)$$

holds and moreover

$$0 \geq \int_R (p \cdot y - p \cdot e) = \lim_k \int_R (p_k \cdot y_k - p_k \cdot e) \geq 0.$$

This implies that $\mu(R)$ is zero as wanted. □

The proof of Proposition 2 is now complete.

4. Proof of the main theorem

We can turn now our attention to the proof of Theorem 1.

Let $(\Omega', \mathcal{F}', \mu')$ be the Stone space (canonically) associated to $(\Omega, \mathcal{F}, \mu)$; in this space μ' is not only nonatomic but also σ-additive.

There is a natural way to define $Y'(F')$, $\succ'_{F'}$ (for $F' \in \mathcal{F}'$) and the initial allocation ν'. They keep all the properties we have seen in $(\Omega, \mathcal{F}, \mu)$ and, in addition, Y' and ν' are σ-additive.

We will show the existence of a Walras equilibrium in $(\Omega', \mathcal{F}', \mu')$ (this is enough for us). For brevity we will avoid to use apices.

Let us call $\mathcal{B}$ the σ-algebra generated by $\mathcal{F}$, it is in fact the Baire algebra on Ω ([8]), let us extend μ and ν to $\mathcal{B}$ and call these extensions μ^* and ν^*. The triple $(\Omega, \mathcal{B}, \mu^*)$ is a nonatomic σ-additive probability space. As Armstrong and Richter show ([2]), in this hypothesis it is possible to extend preferences from $\mathcal{F}$ to $\mathcal{B}$ and to generate them by individual preferences $(\succ_t, t \in \Omega)$ which verify 1c), ..., 5c). Let us indicate by $\mathcal{S}$ the set of all the compact convex, non–empty subsets of $\mathbb{R}^{\ltimes}$, this is a complete, uniform, separated semigroup if we define the addition in usual pointwise way and we use the Hausdorff metric.

Since a f.a., nonatomic, closed-valued set correspondence is compact-valued (see [7]), we can interpret Y as a function from $\mathcal{F}$ to $\mathcal{S}$ and we can extend it to $\mathcal{B}$ obtaining a σ-additive function $Y^* : \mathcal{B} \to \mathcal{S}$. Since this extension is obtained by "continuity", then it is μ^*-absolutely continuous.

Using the extension of Radon-Nikodym theorem to multifunctions ([10] or [3]), we obtain the existence of a multifunction $Z_1 : \Omega \to \mathbb{R}^{\ltimes}$ such that

$$Y^*(F^*) = \int_{F^*} Z_1(t) d\mu^* \qquad \forall F^* \in \mathcal{B}.$$

Let us define the closed-valued correspondence $Z : t \in \Omega \to Z_1(t) \cup \{0\} \subseteq \mathbb{R}^{\ltimes}$.

CLAIM. *For any element F^* in $\mathcal{B}$, it is $Y^*(F^*) = \int_{F^*} Z(t) d\mu^*$.*

Proof. We have to show that

$$\int_{F^*} Z(t) d\mu^* = \int_{F^*} Z_1(t) d\mu^*$$

holds for any F^* in $\mathcal{B}$. This means that

$$\{\int_{F^*} f : f \in I\} = \{\int_{F^*} f : f \in I_1\} \qquad \forall F^* \in \mathcal{B},$$

where I is the set of the integrable selections of Z and I_1 is analogously defined using Z_1 instead of Z.

Since $I_1 \subseteq I$, one of the two inclusions is trivial. Let us show the other one. Take f in I, since f is integrable then the set $X = \{t \in \Omega^* : f(t) = 0\}$ is measurable, so we can write

$$\int_{F^*} f(t)d\mu^* = \int_X f(t)d\mu^* + \int_{F^*\setminus X} f(t)d\mu^* = \int_{F^*\setminus X} f(t)d\mu^* \qquad \forall F^* \in \mathcal{B}.$$

If $t \in F^* \setminus X$, then $f(t)$ belongs to $Z_1(t)$, so we have

$$\int_{F^*} f(t)d\mu^* \in \int_{F^*\setminus X} Z_1(t)d\mu^* = Y^*(F^* \setminus X) \subseteq Y^*(F^*).$$

From this, the desired inclusion follows. □

The function ν^*, is σ-additive and μ^*-absolutely continuous, so it admits a density that we indicate by $n(t)$.

At this point the *coalition production economy* verifies all the hypothesis of Theorem 2. From this theorem we obtain the existence in $(\Omega, \mathcal{B}, \mu^*)$ of a Walras equilibrium which we indicate by (x, y, p).

Let us define for any F in $\mathcal{F}$

$$\alpha(F) = \int_F x(t)d\mu^* \qquad \pi(F) = \int_F y(t)d\mu^*$$

and let us prove that (α, π, p) is a Walras equilibrium for $(\Omega, \mathcal{F}, \mu)$.

For proving 1b), 2b) and 3b) only some computation is necessary. Here it follows.

1b) $\alpha(\Omega) = \int_\Omega x(t)d\mu^* = \int_\Omega n(t)d\mu^* + \int_\Omega Z(t)d\mu^* = \nu^*(\Omega) + Y^*(\Omega) = \nu(\Omega) + Y(\Omega)$;

2b) $p \cdot \pi(F) = p \cdot \int_F y(t)d\mu^* = \int_F p \cdot y(t)d\mu^* \geq \int_F p \cdot Z(t)d\mu^* = p \cdot \int_F Z(t)d\mu^* = p \cdot Y^*(F) = p \cdot Y(F)$;

3b) $p{\cdot}\alpha(F) = p{\cdot}\int_F x(t)d\mu^* = \int_F p{\cdot}x(t)d\mu^* \leq \int_F p{\cdot}n(t)d\mu^* + \int_F p{\cdot}y(t)d\mu^* = p \cdot \nu^*(F) + p \cdot \pi(F) = p \cdot \nu(F) + p \cdot \pi(F)$;

To get 4b) suppose that there exist an element F of $\mathcal{F}$ with $\mu(F) > 0$ and an allocation β such that $\beta \succ_F \alpha$ and $p \cdot \beta(F) \leq p \cdot \nu(F) + p \cdot \pi(F)$.

By passing to the Baire algebra we have: $\beta^* \succ_{F^*} \alpha^*$, $\quad p \cdot \beta^*(F^*) \leq p \cdot \nu^*(F^*) + p \cdot \pi^*(F^*)$ and $\mu^*(F^*) = \mu(F) > 0$. Recall that ([2])

$$\beta^* \succ_{F^*} \alpha^* \overset{\text{def}}{\Leftrightarrow} (\frac{d\beta^*}{d\mu^*})(t) \succ_t (\frac{d\alpha^*}{d\mu^*})(t) \qquad \text{a.e. on} \quad F^*$$

so we have

$$p \cdot (\frac{d\beta^*}{d\mu^*})(t) \leq p \cdot (\frac{d\nu^*}{d\mu^*})(t) + p \cdot (\frac{d\pi^*}{d\mu^*})(t) \qquad \text{a.e. on} \quad F^*$$

and this is a contradiction because of the definition of Walras equilibrium. This completes the proof. □

Acknowledgements. The author wishes to express her gratitude to the Referee for the accurancy in reading the paper and for some useful suggestions to improve it.

References

[1] Armstrong T.E., Richter M.K., *The core-Walras equivalence.* J. Econom. Theory **33** (1984), 116–151.

[2] Armstrong T.E., Richter M.K., *Existence of nonatomic core-Walras allocations.* J. Econom. Theory **38** (1986), 137–159.

[3] Artstein Z., *Set–valued measures.* Trans. Amer. Math. Soc. **165** (1972), 103–125.

[4] Aumann R.J., *Markets with a continuum of traders.* Econometrica **32** (1964), 39–50.

[5] Aumann R.J., *Integrals of set–valued functions.* J. Math. Anal. Appl. **12** (1965), 1–12.

[6] Aumann R.J., *Existence of competitive equilibria in markets with a continuum of traders.* Econometrica **34** (1966), 1–17.

[7] Basile A., *Finitely additive nonatomic coalition production economies: Core–Walras equivalence.* Internat. Econom. Rev. **34** (1993), 983–995.

[8] Bhaskara Rao K.P.S., Bhaskara Rao M., *Theory of Charges.* Academic Press, London, New York, 1983.

[9] Hildenbrand W., *Core and Equilibria of a Large Economy.* Princeton Univ. Press, Princeton, New Jersey, 1974.

[10] Klein E., Thompson A.C., *Theory of Correspondences.* Wiley Interscience, New York, 1984.

[11] Richter M.K., *Coalitions, core and competition.* J. Econom. Theory **3** (1971), 323–334.

[12] Schmeidler D., *Competitive equilibria in markets with a continuum of traders and incomplete preferences.* Econometrica **37** (1969), 578–585.

Anna De Simone
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Universitá degli Studi di Napoli "Federico II"
Complesso Universitario Monte S. Angelo — Via Cintia
80126 Napoli, ITALIA
email: annades@matna2.dma.unina.it