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SET THEORETIC REAL ANALYSIS

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Abstract. This article is a survey of the recent results that concern real functions (from $\mathbb{R}^n$ into $\mathbb{R}$) and whose solutions or statements involve the use of set theory. The choice of the topics follows the author's personal interest in the subject, and there are probably some important results in this area that did not make it to this survey. Most of the results presented here are left without proofs.

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1. Historical background

The study of real functions has played a fundamental role in the development of mathematics over the last three centuries. The seventeenth century discovery of calculus by Newton and Leibniz was largely due to increased understanding of the behavior of real functions. The birth of analysis is often traced to the early nineteenth century work of Cauchy, who gave precise definitions of concepts such as continuity and limits for real functions. Convergence problems while approximating real functions by Fourier series gave rise to both the Riemann and Lebesgue integrals. Cantor developed his set theory in an effort to answer uniqueness questions about Fourier series [83, 18, 151].

During this time, different techniques have been used as the theory behind them became available. For example, after Cauchy, various limiting operations such as pointwise and uniform convergence were studied, giving rise to various approximation techniques. At the turn of this century, measure theoretic techniques were exploited, leading to stochastic convergence ideas in the 1920's. Also, at about the same time topology was developed, and its applications to analysis gave rise to functional analysis.

In recent years, a new research trend has appeared which indicates the emergence of a yet another branch of inquiry that could be called *set theoretic real analysis*. This area is the study of families of real functions using modern techniques of set theory. These techniques include advanced forcing methods, special axioms of set theory such as Martin's axiom (MA) and proper forcing axiom (PFA), as well as some of their weaker consequences like additivity of measure and category. (See [97], [132], [65], [51], and [8] for examples of this work.)

Set theoretic real analysis is closely allied with descriptive set theory, but the objects studied in the two areas are different. The objects studied in descriptive set theory are various classes of (mostly nice) **sets** and their hierarchies, such as Borel sets or analytic sets. Set theoretic real analysis uses the tools of modern set theory to study **real functions** and is interested mainly in more pathological objects. Thus, the results concerning subsets of the real line (like the series of studies on "small" subsets of $\mathbb{R}$ [110], or deep studies of the duality between measure and category [121, 108, 8]) are considered only remotely related to the subject. (However, some of these duality studies spread to real analysis too. For example, see a monograph [38].)

Set theoretic real analysis already has a long history. Its roots can be traced back to the 1920's, where powerful new techniques based on the Axiom of Choice (AC) and the Continuum Hypothesis (CH) can be seen in many papers from such journals as Fundamenta Mathematicae and Studia

Mathematica. The most interesting consequences of the Continuum Hypothesis discovered in this period have been collected in 1934 monograph of Sierpiński, *Hypothèse du Continu* [140]. The influence of Sierpiński's results (and the monograph) on the set theoretic real analysis can be best seen in the next section.

The new emergence of the field was sparked by the discovery of powerful new techniques in set theory and can be compared to the parallel development of set theoretic topology during the late 1950's and 1960's. In fact, it is a bit surprising that the development of set theoretic analysis is so much behind that of set theoretic topology, since at the beginning of the century the applicability of set theory in analysis was at least as intense as in topology. This, however, can be probably attributed to the simple fact, that in the past half of a century there were many mathematicians that knew well both topology and set theory, and very few that knew well simultaneously analysis and set theory.

Our terminology is standard and follows [30].

2. New developments in classical problems

The first problem we wish to mention here is connected with the Fubini–Tonelli Theorem. The theorem says, in particular, that if a function $f\colon [0,1]^2 \to [0,1]$ is measurable then the iterated integrals $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ exist and are both equal to the double integral $\int\int f\,dm_2$, where m_2 stands for the Lebesgue measure on $\mathbb{R}^2$. But what happens when f is non-measurable? Then clearly the double integral does not exist. However, the iterated integrals might still exist. Must they be equal? The next theorem, which is a classical example of an application of the Continuum Hypothesis in real analysis, gives a negative answer to this question.

Theorem 2.1 (Sierpiński 1920, [136])**.** *If the Continuum Hypothesis holds then there exists $f\colon [0,1]^2 \to [0,1]$ for which the iterated integrals $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ exist but are not equal.*

Proof. Let $\preceq$ be a well ordering of $[0,1]$ in order type continuum $\mathfrak{c}$ and define $A = \{\langle x,y\rangle \in [0,1]^2\colon x \preceq y\}$. Let f be the characteristic function χ_A of A. Then for every fixed $y \in [0,1]$ the set $\{x \in [0,1]\colon f(x,y) \neq 0\} = \{x \in [0,1]\colon x \preceq y\}$ is an initial segment of a set ordered in type $\mathfrak{c}$. So, by CH, it is at most countable and

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 0\,dy = 0.$$

Similarly, for each $x \in [0,1]$ the set $\{y \in [0,1]\colon f(x,y) \neq 1\} = \{y \in [0,1]\colon y \prec x\}$ is at most countable and

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 1\,dy = 1.$$

Thus, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = 1 \neq 0 = \int_0^1 \int_0^1 f(x,y)\,dx\,dy$. □

Sierpiński's use of the Continuum Hypothesis in the construction of such a function begs the question whether such a function can be constructed using only the axioms of ZFC. The negative answer was given in the 1980's by Laczkovich [98], Friedman [68] and Freiling [61], who independently proved the following theorem.

Theorem 2.2. *There exists a model of set theory ZFC in which for every function $f\colon [0,1]^2 \to [0,1]$, the existence of the iterated integrals $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ implies their equality.*

It is also worthwhile to mention that the function f from the proof of Theorem 2.2 has the desired property as long as every subset of $\mathbb{R}$ of cardinality less than continuum has measure zero, i.e., when the smallest cardinality $\mathrm{non}(\mathcal{N})$ of the non-measurable subset of $\mathbb{R}$ is equal to $\mathfrak{c}$. Since the equation $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$ holds in many models of ZFC in which CH fails (for example, it is implied by MA) Theorem 2.2 is certainly not equivalent to CH. On the other hand, Laczkovich proved Theorem 2.2 by noticing that: (A) the existence of an example as in the statement of Theorem 2.1 implies the existence of such an example as in its proof, i.e., in form of χ_A; (B) there is no set $A \subset [0,1]^2$ with $f = \chi_A$ satisfying Theorem 2.2 if $\mathrm{non}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$, where $\mathrm{cov}(\mathcal{N})$ is the smallest cardinality of a covering of $\mathbb{R}$ by the sets of measure zero. (It is well known that the inequality $\mathrm{non}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$ is consistent with ZFC. See e.g. [8].)

A discussion of a similar problem for the functions $f\colon [0,1]^n \to [0,1]$ and the n-times iterated integrals can be found in a 1990 paper of Shipman [135]. The same paper contains also two easy ZFC examples of measurable functions $f\colon [0,1)^2 \to \mathbb{R}$ and $g\colon \mathbb{R}^2 \to [-1,1]$ for which the iterated integrals exist but are not equal. Thus, the restriction of the above problem to the non-negative functions is essential.

Another classical result arises from a different theorem of Sierpiński of 1928.

Theorem 2.3 (Sierpiński [137, 138])**.** *If the Continuum Hypothesis holds then there exists a set $S \subset \mathbb{R}$ of cardinality continuum such that its image $f[S] \neq [0,1]$ for any continuous $f\colon \mathbb{R} \to [0,1]$.*

The set S from the original proof of Theorem 2.3 is called *Sierpiński set* and it has the property that its intersection $S \cap N$ with any measure zero set N is at most countable.[1] Another set that satisfies the conclusion of Theorem 2.3, known as *Luzin set* (see [138] or [140, property C_5]), is defined as an uncountable subset L of $\mathbb{R}$ whose intersection $L \cap M$ with any meager set M is at most countable.[2] The existence of a Luzin set is also implied by CH. In fact, the constructions of sets S and L under the assumption of CH are almost identical: you list all G_δ measure zero sets (F_σ meager sets) as $\{Z_\xi : \xi < \mathfrak{c}\}$ and define S (L, respectively) as a set $\{x_\alpha : \alpha < \mathfrak{c}\}$ where $x_\alpha \in \mathbb{R} \setminus \left(\bigcup_{\xi<\alpha} Z_\xi\right)$. The choice is possible since, by CH, the family $\{Z_\xi : \xi < \alpha\}$ is at most countable implying that its union is not equal to $\mathbb{R}$. It is also easy to see that this construction can be carried out if $\operatorname{cov}(\mathcal{N}) = \mathfrak{c}$ (and its category analog $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ in case of construction of L). The sets constructed that way are called generalized Sierpiński and Luzin sets, respectively, and they also satisfy the conclusion of Theorem 2.3 independently of the size of $\mathfrak{c}$. Since many models of ZFC satisfy either $\operatorname{cov}(\mathcal{N}) = \mathfrak{c}$ or $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ (for example, both conditions are implied by MA) it has been a difficult task to find a model of ZFC in which the conclusion of Theorem 2.3 fails. It has been found by A. W. Miller in 1983.

Theorem 2.4 (A. W. Miller [109]). *There exists a model of set theory ZFC in which for every subset S of $\mathbb{R}$ of cardinality $\mathfrak{c}$ there exists a continuous function $f : \mathbb{R} \to [0,1]$ such that $f[S] = [0,1]$.*

In his proof of Theorem 2.4 Miller used the iterated perfect set model, which will be mentioned in this paper in several other occasions.

Some of the most recent set-theoretic results concerning classical problems in real functions are connected with a theorem of Blumberg from 1922.

Theorem 2.5 (Blumberg [10]). *For every $f : \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that the restriction $f \restriction D$ of f to D is continuous.*

The set D constructed by Blumberg is countable. In a quest whether it can be chosen any bigger Sierpiński and Zygmund proved in 1923 the following theorem.

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[1] This approach was used in the paper [137], while the Luzin set approach in the paper [138]. Since they are published in the same year, the priority is not completely clear. However in the list of Sierpiński's publications printed in [143] paper [137] precedes [138], suggesting its priority.

[2] The construction of such a set, under CH, was published by Luzin in 1914 [103]. The same construction had been also published in 1913 by Mahlo [105]. But (as is not unusual in mathematics) such a set is commonly known as a Luzin set.

Theorem 2.6 (Sierpiński, Zygmund [144]). *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ whose restriction $f \restriction X$ is discontinuous for any subset X of $\mathbb{R}$ of cardinality $\mathfrak{c}$.*

Theorem 2.6 immediately implies the following corollary, which shows that there is no hope for proving in ZFC a version of the Blumberg theorem in which the set D is uncountable.

Corollary 2.7 (Sierpiński, Zygmund [144]). *If the Continuum Hypothesis holds then there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction X$ is discontinuous for any uncountable subset X of $\mathbb{R}$.*

The proof of Theorem 2.6 is a straightforward transfinite induction diagonal argument after noticing that every continuous partial function on $\mathbb{R}$ can be extended to a continuous function on a G_δ set.

Corollary 2.7 raises the natural question about the importance of the assumption of CH in its statement. Is it consistent that the set D in Blumberg Theorem can be uncountable? Can it be of positive outer measure, or non-meager?

The cardinality part of these questions is addressed by the following theorem of Baldwin from 1990.

Theorem 2.8 (Baldwin [6]). *If Martin's Axiom holds then for every function $f\colon \mathbb{R} \to \mathbb{R}$ and every infinite cardinal number $\kappa < \mathfrak{c}$ there exists a set $D \subset \mathbb{R}$ such that $f \restriction D$ is continuous and D is κ-dense, i.e., $D \cap I$ has cardinality κ for every non-trivial interval I.*

Thus under MA the size of the set D is clear. By Theorem 2.6 it cannot be chosen of cardinality continuum (at least for some functions), but it can be always chosen of any cardinality κ less than $\mathfrak{c}$.

One might still hope to be able to prove in ZFC that for any f the set D can be found of an arbitrary cardinality $< \mathfrak{c}$. However, this is false as well, as noticed by several authors: G. Gruenhage in 1993 (see Recaw [124, Thm 4]) S. Shelah in 1995 (see [133]) and the author of this survey (unpublished).

Theorem 2.9 ([124, 133]). *There exists a model of ZFC+$\neg$CH (namely a Cohen model) in which there is a function $f\colon \mathbb{R} \to \mathbb{R}$ which is discontinuous on any uncountable subset of $\mathbb{R}$.*

The category version of a question on a size of D has been also settled in the 1995 paper of Shelah [133] mentioned above.

Theorem 2.10 (Shelah [133])**.** *There exists a model of ZFC in which for every function $f\colon \mathbb{R}\to\mathbb{R}$ there exists a set $D\subset\mathbb{R}$ such that $f\restriction D$ is continuous and D is nowhere meager, i.e., $D\cap I$ is non-meager for every non-trivial interval I.*

The measure version of the question is less clear. It has been noticed by J. Brown in 1977 that the precise measure analog of Theorem 2.10 cannot be proved. (This has been also noticed independently by K. Ciesielski, whose proof is included below.)

Theorem 2.11 (Brown [11])**.** *There exists a function $f\colon\mathbb{R}\to\mathbb{R}$ such that $f\restriction D$ is discontinuous for every set $D\subset\mathbb{R}$ which is nowhere measure zero, i.e., such that $D\cap I$ has positive outer measure for every non-trivial interval I.*

Proof. Let $\{F_n\colon n<\omega\}$ be a partition of $\mathbb{R}$ such that F_0 is a dense G_δ set of measure zero and F_n is nowhere dense for each $n>0$. Define $f\colon\mathbb{R}\to\mathbb{R}$ by putting $f(x)=n$ for $x\in F_n$. Now, $f\restriction X$ is discontinuous for any dense $X\subset\mathbb{R}$ which is nowhere measure zero.

Indeed, if $X\subset\mathbb{R}$ is dense and nowhere measure zero then there exists an $x\in X\setminus F_0$. Now, if every open set U containing x intersects $F_n\cap X$ for infinitely many n then $f\restriction X$ is discontinuous at x. Otherwise, there is an open set U containing x and intersecting only finitely many F_n's. So, we can find a non-empty open interval $I\subset U$ such that $I\cap X\subset F_0$. But this means that $I\cap X$ has measure zero, a contradiction. □

However, the following problem asked by Heinrich von Weizsäcker [67, Problem AR(a)] remains open.

Problem 1. *Is it consistent that every function $f\colon\mathbb{R}\to\mathbb{R}$ is continuous on some set $X\subset\mathbb{R}$ of positive outer measure?*

Other generalizations of Blumberg's theorem can be also found in a 1994 survey article [12]. (See also recent papers [13] and [78].)

Another problem that is related in character to the Blumberg's theorem is the following.

> *Let $\{f_n\colon\mathbb{R}\to[-\infty,\infty]\}_{n=1}^{\infty}$ be a sequence of arbitrary functions. What is the biggest size of a set $X\subset\mathbb{R}$ for which there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ convergent pointwise on X?*

Clearly such a subsequence can be found for any countable $X\subset\mathbb{R}$. Using this fact Helly [76] proved in 1921 that *any bounded sequence of monotone real functions contains a pointwise convergent subsequence.* On the other hand, answering a question of S. Saks, in 1932 Sierpiński [139] showed that the Continuum Hypothesis implies the existence of a sequence $\{f_n\colon \mathbb{R} \to [-\infty,\infty]\}_{n=1}^{\infty}$ such that $\{f_n\restriction X\}_{n=1}^{\infty}$ has no pointwise convergent subsequence for any uncountable $X \subset \mathbb{R}$. The necessity of additional set theoretical assumptions in the Sierpiński's construction was recently noticed by Fuchino and Plewik [69] who showed that the size of X having the property under consideration is characterized by the splitting number $\mathbf{s}$: *For any $X \subset \mathbb{R}$ with $|X| < \mathbf{s}$ any sequence $\{f_n\colon \mathbb{R} \to [-\infty,\infty]\}_{n=1}^{\infty}$ has a subsequence convergent pointwise on X; however for any $X \subset \mathbb{R}$ with $|X| = \mathbf{s}$ there exists a sequence $\{f_n\colon X \to [0,1]\}_{n=1}^{\infty}$ with no pointwise convergent subsequence.* (For the definition of the splitting number, see e.g. [152].)

In the past few years a lot of activity in real analysis was concentrated around symmetric properties of real functions. (See Thomson [151].) Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is symmetrically continuous at $x \in \mathbb{R}$ if

$$\lim_{h\to 0}(f(x+h) - f(x-h)) = 0,$$

and f is approximately symmetrically differentiable at x if there exists a set $S \subset \mathbb{R}$ such that x is a (Lebesgue) density point of $\mathbb{R}\setminus S$ and that the following limit exists

$$\lim_{h\to 0,\, h\notin S} \frac{f(x+h) - f(x-h)}{2h}.$$

This limit, which does not depend on the choice of a set S, is called *the approximate symmetric derivative of f at x* and is denoted by $D^s_{ap}f(x)$. We will say that f has a *co-countable symmetric derivative at x* and denote it by $D^s_c f(x)$ if the set S in the above definition can be chosen to be countable.

One of the long standing conjectures (with several incorrect proofs given earlier, some even published) was settled by Freiling and Rinne in 1988 by proving the following theorem.

Theorem 2.12 (Freiling, Rinne [64]). *If $f\colon \mathbb{R} \to \mathbb{R}$ is measurable and such that $D^s_{ap}f(x) = 0$ for all $x \in \mathbb{R}$ then f is constant almost everywhere.*

The importance of the measurability assumption in Theorem 2.12 was long known from the following theorem of Sierpiński of 1936.

Theorem 2.13 (Sierpiński [141]). *If the Continuum Hypothesis holds then there exists a non-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ (which is a characteristic function χ_A of some set A) for which $D^s_c f(x) = 0$ for all $x \in \mathbb{R}$.*

In fact, in [141] Theorem 2.13 is stated in a bit stronger form[3] from which it follows immediately that the theorem remains true under MA, if the co-countable symmetric derivatives $D^s_c f(x)$ are replaced by the approximate symmetric derivatives $D^s_{ap}(x)$. However, neither Theorem 2.13 nor its version with $D^s_{ap}(x)$ can be proved in ZFC. This follows from the following two theorems of Freiling from 1990.

Theorem 2.14 (Freiling [62])**.** *If the Continuum Hypothesis fails then for every function $f\colon \mathbb{R} \to \mathbb{R}$ with $D^s_c f(x) = 0$ for all $x \in \mathbb{R}$ there exists a countable set S such that f is constant on $\mathbb{R} \setminus S$.*

Thus the existence of a function as in Theorem 2.13 is in fact equivalent to the Continuum Hypothesis.

Theorem 2.15 (Freiling [62])**.** *It is consistent with ZFC that for every function $f\colon \mathbb{R} \to \mathbb{R}$ with $D^s_{ap} f(x) = 0$ for all $x \in \mathbb{R}$ there exists a measure zero set S such that f is constant on $\mathbb{R} \setminus S$.*

More precisely, Freiling proves that the conclusion of Theorem 2.15 follows the property that is just a bit stronger than the inequality $\mathrm{non}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$. (Compare comment following Theorem 2.2.)

Another direction in which the symmetric continuity research went was the study of how far symmetric continuity can be destroyed. First note that clearly every continuous function is symmetrically continuous, but not vice versa, since the characteristic function of a singleton is symmetrically continuous. However, it is not difficult to find functions which are nowhere symmetrically continuous. For example, the characteristic function of any dense Hamel basis is such a function.[4] How much more can we destroy symmetric continuity?

In the non-symmetric case probably the weakest (bilateral) version of continuity that can be defined is the following. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly continuous* at x if there are sequences $a_n \nearrow 0$ and $b_n \searrow 0$ such that

$$\lim_{n\to\infty} f(x+a_n) = f(x) = \lim_{n\to\infty} f(x+b_n).$$

This notion is so weak that it is impossible to find a function $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere weakly continuous. This follows from the following easy, but a little surprising theorem.

[3]Co-countable symmetric derivatives are replaced by co-$<\mathfrak{c}$ symmetric derivatives and the theorem is proved in ZFC.

[4]In fact, a Hamel basis $\mathcal{B}$ can be chosen to be both first category and measure zero. Thus $\chi_{\mathcal{B}}$ can be measurable and have the Baire property.

Theorem 2.16 ([48, p. 82]). *Every function $f\colon \mathbb{R} \to \mathbb{R}$ is weakly continuous everywhere on the complement of a countable set.*

A natural symmetric counterpart of weak continuity is defined as follows. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *weakly symmetrically continuous* at x if there is a sequence $h_n \to 0$ such that

$$\lim_{n\to\infty} (f(x+h_n) - f(x-h_n)) = 0.$$

However, the symmetric version of Theorem 2.16 badly fails: there exist nowhere weakly symmetrically continuous functions (which are also called *uniformly antisymmetric functions*). Their existence follows immediately from the following theorem of Ciesielski and Larson from 1993.

Theorem 2.17 (Ciesielski, Larson [37]). *There exists a function $f\colon \mathbb{R} \to \mathbb{N}$ such that the set*

$$S_x = \{h > 0\colon f(x+h) = f(x-h)\}$$

is finite for every $x \in \mathbb{R}$.

The function f from Theorem 2.17 raises the questions in two directions. Can the range of f be any smaller? Can the size of all sets S_x be uniformly bounded? The first of this questions leads to the following open problem from [37]. (See also problems listed in [151].)

Problem 2. *Does there exist a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with range $f[\mathbb{R}]$ being (a) finite? (b) bounded?* [5]

Concerning part (a) of this problem it has been proved in 1993 by Ciesielski [26] that the range of a uniformly antisymmetric function must have at least 4 elements. (Compare also [28].)

The estimation of sizes of sets S_x from Theorem 2.17 has been examined by Komjáth and Shelah in 1993, leading to the following two theorems.

Theorem 2.18 (Komjáth, Shelah [94]). *The Continuum Hypothesis is equivalent to the existence of a function $f\colon \mathbb{R} \to \mathbb{N}$ such that the set*

$$S_x = \{h > 0\colon f(x+h) = f(x-h)\}$$

has at most 1 element for every $x \in \mathbb{R}$.

[5] A uniformly antisymmetric function $f\colon \mathbb{R} \to [0,1]$ has been recently constructed by S. Shelah.

Theorem 2.19 (Komjáth, Shelah [94]). *If $\mathfrak{c} > \omega_{k+1}$, $k = 0, 1, 2, \ldots$, then there is no function $f : \mathbb{R} \to \mathbb{N}$ such that the set*

$$S_x = \{h > 0 : f(x+h) = f(x-h)\}$$

has at most 2^k elements for every $x \in \mathbb{R}$.

Theorem 2.18 suggests that the converse of Theorem 2.19 should also be true. However, this is still unknown, leading to another open problem.

Problem 3. *Does the assumption that $\mathfrak{c} \leq \omega_{k+1}$ imply that there exists a function $f : \mathbb{R} \to \mathbb{N}$ such that the set*

$$S_x = \{h > 0 : f(x+h) = f(x-h)\}$$

has at most 2^k elements for every $x \in \mathbb{R}$?

For $k = 0$ the positive answer is implied by Theorem 2.18. Also, it is consistent that $\mathfrak{c} = \omega_{k+1}$ and there exists $f : \mathbb{R} \to \mathbb{N}$ such that each S_x has at most 2^k elements. This follows from another theorem of Komjáth and Shelah [95, Thm 1]. (See also a paper [29] of Ciesielski related to this subject.)

In fact, the proof of Theorem 2.17 gives also the following version for functions on $\mathbb{R}^n$:

- There exists a function $f : \mathbb{R}^n \to \mathbb{N}$ such that the set

$$\{h \in \mathbb{R}^n : f(x+h) = f(x-h)\}$$

  is finite for every $x \in \mathbb{R}^n$.

This statement is related to the following recent theorem of J. Schmerl, which solves a long standing problem of Erdős [111, Problem 15.9]. (See also a survey article [93] for more on this problem.)

Theorem 2.20 (Schmerl [131]). *There exists a function $f : \mathbb{R}^n \to \mathbb{N}$ such that for any distinct $a, b, x \in \mathbb{R}^n$ with $\|a - x\| = \|x - b\|$ all the values $f(a)$, $f(x)$ and $f(b)$ are not equal.*

Thus, this theorem says, that there exists (in ZFC) a countable partition of $\mathbb{R}^n$ such that no three vertices a, b, x spanning isosceles triangle belong to the same element of the partition.

3. New classic-like results

Consider a function $F = \langle f_1, f_2 \rangle$ from $\mathbb{R}$ onto $\mathbb{R}^2$. By a well known theorem of Peano from 1890 (see e.g. [129]) such an F can be continuous. However, it is not difficult to see that it cannot be differentiable. It follows easily from the fact that every differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the Banach condition T_2, i.e., the set $\{y\colon f^{-1}(y) \text{ is uncountable}\}$ has Lebesgue measure zero. (See e.g. [130, Chap. VII, p. 221].) Thus, Morayne in 1987 considered the following question: can function $F = \langle f_1, f_2 \rangle$ be chosen in such a way that at every point $x \in \mathbb{R}$ either f_1 or f_2 is differentiable? The surprising answer is given below.

Theorem 3.1 (Morayne [112]). *The Continuum Hypothesis is equivalent to the existence of a function $F = \langle f_1, f_2 \rangle$ from $\mathbb{R}$ onto $\mathbb{R}^2$ such that at every point $x \in \mathbb{R}$ either f_1 or f_2 is differentiable.*

The proof of this theorem is based on a well known theorem of Sierpiński [140, Property P_1] from 1919 that CH is equivalent to the existence of a decomposition of $\mathbb{R}^2$ into two sets A and B such that all horizontal sections of A and all vertical sections of B are at most countable. It is also worthwhile to point out that the function F from Theorem 3.1 is not a Peano curve, since it is not continuous. In fact Morayne proves in the same paper that for such an $F = \langle f_1, f_2 \rangle$ it is impossible that even one of f_1 or f_2 is measurable.

Next, recall that if two continuous functions $f, g\colon \mathbb{R} \to \mathbb{R}$ agree on some dense set $M \subset \mathbb{R}$ then they are equal. Does the statement remain true if the clause "agree on M" is replaced by "$f[M] = g[M]$?" Clearly not, as shown by $M = \mathbb{Q}$ and any two different rational translations of the identity function. What about finding some more complicated set $M \subset \mathbb{R}$ for which the implication

$$\text{if } f[M] = g[M] \text{ then } f = g$$

holds for any continuous f and g? Even this is too much to ask, as recently noted by Burke and Ciesielski [19, Remark 6.6]. On the other hand, the following theorem of Berarducci and Dikranjan from 1993 gives a positive (consistent) answer to this question in the class of continuous nowhere constant functions. (A function is nowhere constant if it is not constant on any non-empty open set.)

Theorem 3.2 (Berarducci, Dikranjan [9]). *If the Continuum Hypothesis holds then there exists a set $M \subset \mathbb{R}$ (called* magic*) such that for every continuous nowhere constant functions $f, g\colon \mathbb{R} \to \mathbb{R}$,*

$$\textit{if } f[M] \subset g[M] \textit{ then } f = g.$$

The construction of a magic set given in [9] is done by an easy diagonal transfinite induction argument and uses only the assumption that less than continuum many meager sets do not cover $\mathbb{R}$. In particular, CH can be replaced by MA in Theorem 3.2.

Examining the problem of existence of a magic set in ZFC Burke and Ciesielski noticed the following properties of a magic set.

Theorem 3.3 (Burke, Ciesielski [19]). *If $M \subset \mathbb{R}$ is a magic set then*
(a) *M is dense and nowhere meager;*
(b) *$f[M] \not\supset [0,1]$ for every continuous $f\colon \mathbb{R} \to \mathbb{R}$.*

In fact part (b) of Theorem 3.3 is just a remark: if there were a continuous $f\colon \mathbb{R} \to \mathbb{R}$ with $f[M] \supset [0,1]$ then it could be easily modified to a nowhere constant function such that $f[M] = \mathbb{R}$, and the functions f and $g = 1 + f$ would give a contradiction. But (b) shows that there is no magic set of cardinality continuum in the model from Theorem 2.4, the iterated perfect set model. Although it was noticed in [19] that in this model there exists a magic set (clearly of cardinality less than $\mathfrak{c}$), Theorem 3.3 was used by Ciesielski and Shelah as a base in proving that magic set cannot be constructed in ZFC.

Theorem 3.4 (Ciesielski, Shelah [45]). *There is a model of ZFC in which*
(a) *every subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$ is meager;*
(b) *for every set $M \subset \mathbb{R}$ of cardinality continuum there exists a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[M] = [0,1]$.*

In particular, there is no magic set in this model.

The magic sets for different classes of functions have also been considered. Burke and Ciesielski [19] studied such sets (which they call *sets of range uniqueness*) for the classes of measurable functions with respect to abstract measurable spaces with negligibles. In particular, they proved the following theorem concerning the Lebesgue measurable functions.

Theorem 3.5 (Burke, Ciesielski [19]).

(a) *If* $\mathrm{cov}(\mathcal{N})=\mathfrak{c}$ *(thus under CH or MA) then there exists a set* $M\subset\mathbb{R}$ *with the property that for every measurable functions* $f,g\colon\mathbb{R}\to\mathbb{R}$ *which are not constant on any set of positive measure*

$$\text{if } f[M]\subset g[M] \text{ then } f=g \text{ almost everywhere.}$$

(b) *There is a model of ZFC in which a set from part (a) does not exist.*

The model satisfying Theorem 3.5(b) is a modification of the iterated perfect set model and was constructed by Corazza [49] in 1989. Once again it satisfies property (b) of Theorem 3.4, while part (a) is replaced by $\mathrm{cov}(\mathcal{N})=\mathfrak{c}$. It has also been proved by Burke, Ciesielski, and Larson that for the class D^1 of differentiable functions the existence of a magic set can be proved in ZFC.

Theorem 3.6 (Burke, Ciesielski [20]). *There exists a set* $M\subset\mathbb{R}$ *such that for every* D^1 *nowhere constant functions* $f,g\colon\mathbb{R}\to\mathbb{R}$

$$\text{if } f[M]\subset g[M] \text{ then } f=g.$$

Note also that the existence of a countable magic set (a convergent sequence) for the class of analytic functions has been proved already in 1981 by Diamond, Pomerance, and Rubel [57]. However, not all convergent sequences form a magic set for this class.

For the following consideration recall that a function $f\colon\mathbb{R}^n\to\mathbb{R}$ is *Darboux* (or *has the Darboux property*) if $f[C]$ is connected for every connected subset C of $\mathbb{R}^n$. Thus, in case of $n=1$ Darboux functions are precisely the functions for which the Intermediate Value Theorem holds. The class of Darboux functions will be denoted here by $\mathcal{D}$ (with n clear from the context, usually $n=1$).

The class of Darboux functions has been studied for a long time as one of possible generalizations of the class of continuous functions. (Clearly every continuous function is Darboux.) However, it has some peculiar properties. For example, it is not closed under addition. In fact, in 1927 Lindenbaum [102] noticed (without a proof) that every function $f\colon\mathbb{R}\to\mathbb{R}$ can be written as a sum of two Darboux functions. (For proofs, see [142, 106].) This theorem has been improved in several ways. Erdős [59] showed that if f is measurable, both of the summands can be chosen to be measurable. Another improvement was done by Fast [60] in 1959 who proved that for every family $\mathcal{F}$ of real functions that has cardinality continuum there is just one Darboux function g such that the sum of g with any function in $\mathcal{F}$ has the Darboux property. The natural question of whether such a "universal" summand exists also for families of larger cardinality has been studied by Natkaniec [114] and lead to the development described in Section 4.

A problem that is in some sense opposite to the existence of a "universal" summand is for which families $\mathcal{F}$ of functions there is a "universally bad" Darboux function g, in the sense that the sum of g with any function in $\mathcal{F}$ does not have the Darboux property. In 1990 Kirchheim and Natkaniec addressed this problem for the class $\mathcal{F}$ of continuous nowhere constant functions.

Theorem 3.7 (Kirchheim, Natkaniec [91])**.** *If union of less than $\mathfrak{c}$ many meager subsets of $\mathbb{R}$ is meager (thus under CH or MA) then there exists a Darboux function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is not Darboux for every continuous nowhere constant function $f\colon \mathbb{R} \to \mathbb{R}$.*

The problem whether the additional set-theoretic assumptions are necessary in this theorem was investigated in 1992 by Komjáth [92] and was settled in 1995 by Steprāns.

Theorem 3.8 (Steprāns [148])**.** *It is consistent with ZFC that for every Darboux function $g\colon \mathbb{R} \to \mathbb{R}$ there exists a continuous nowhere constant function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is Darboux.*

A model having this property is the iterated perfect set model. Note also that in Theorem 3.7 the restriction to the nowhere constant functions is important. This has been proved independently by T. Natkaniec (in his 1992/93 paper [116]) and by J. Steprāns (in the 1995 paper mentioned above).

Theorem 3.9 (Natkaniec [116], Steprāns [148])**.** *For every Darboux function $g\colon \mathbb{R} \to \mathbb{R}$ there exists a continuous non-constant function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f + g$ is Darboux.*

To state further results recall the following generalizations of continuity. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *almost continuous* (in the sense of Stallings) if each open subset of $\mathbb{R}^n \times \mathbb{R}$ containing the graph of f contains also a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$ [146]. Function $f\colon \mathbb{R} \to \mathbb{R}$ *has a perfect road* at $x \in \mathbb{R}$ if there exists a perfect set C such that x is a bilateral limit point of C and $f \restriction C$ is continuous at x [107]. The classes of all almost continuous functions and all functions having a perfect road at each point are denoted by AC and PR, respectively. It is easy to see that $\mathcal{C} \subset \mathrm{AC} \subset \mathcal{D}$ (for functions on $\mathbb{R}$) and that the inclusions are strict (see e.g. [14]), where $\mathcal{C}$ stands for the class of all continuous functions. We will also consider the class SZ of *Sierpiński-Zygmund (SZ-) functions*, i.e., functions $f\colon \mathbb{R} \to \mathbb{R}$ whose restrictions $f \restriction X$

are discontinuous for all subsets X of $\mathbb{R}$ of cardinality continuum. (That is, functions from Theorem 2.6.)

The classes SZ and PR recently appeared in a 1993 paper of Darji [52], who constructed in ZFC a function $f \in \mathrm{SZ} \cap \mathrm{PR}$. Answering a question posed by Darji the following theorem has been proved recently by Balcerzak, Ciesielski, and Natkaniec.

Theorem 3.10 (Balcerzak, Ciesielski, Natkaniec [4]).

(a) *If $\mathbb{R}$ is not a union of less than continuum many of its meager subsets (thus under CH or MA) then there exists an $f \in \mathrm{SZ} \cap \mathrm{PR} \cap \mathrm{AC}$.*

(b) *There is a model of ZFC in which every Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on some set of cardinality continuum. In particular, in this model we have $\mathrm{SZ} \cap \mathrm{AC} = \mathrm{SZ} \cap \mathcal{D} = \emptyset$.*

The model satisfying Theorem 3.10(b) is, once again, the iterated perfect set model.

Another generalization of continuity is that of countable continuity: a function $f\colon \mathbb{R} \to \mathbb{R}$ is *is countably continuous* if there exists a countable partition $\{X_n\}_{n=1}^{\infty}$ of $\mathbb{R}$ such that the restriction of f to any X_n is continuous. (See also Section 4.) In 1995 Darji gave the following combinatorial characterization of this notion.

Theorem 3.11 (Darji [53, 54]). *If the Continuum Hypothesis holds then*

($\star$) *$f\colon \mathbb{R} \to \mathbb{R}$ is countably continuous if and only if for every uncountable set $U \subseteq \mathbb{R}$ there is an uncountable set $V \subseteq U$ such that the restriction $f \restriction V$ is continuous.*

The characterization ($\star$) cannot be proved in ZFC. This follows from a result of Cichoń and Morayne [21] from 1988 which implies that in some models of ZFC (actually, when $\mathfrak{c} = \omega_2$ and $d = \omega_1$, where d is the dominating number) ($\star$) is false. However, it is not known, whether the equivalence ($\star$) can be proved in absence of CH, leading to the following open problem.

Problem 4. Is ($\star$) from Theorem 3.11 equivalent to the Continuum Hypothesis?

Another recent theorem concerning countable and symmetric continuities is the following theorem of Ciesielski and Szyszkowski, answering a question of L. Larson.

Theorem 3.12 (Ciesielski, Szyszkowski [46]). *There exists a symmetrically continuous function $f: \mathbb{R} \to \mathbb{R}$ such that for some set $Z \subset \mathbb{R}$ of cardinality continuum $f \restriction Z$ is of Sierpiński-Zygmund type, i.e., $f \restriction X$ is discontinuous for any subset X of Z of cardinality continuum.*

In particular, f is not countably continuous.

We will finish this section with the following two interesting results. The first one was proved independently in 1978 by Grande and Lipiński and in 1979 by Kharazishvili.

Theorem 3.13 (Grande, Lipiński [71], Kharazishvili [90]). *If the Continuum Hypothesis holds then there exists a non-measurable function $F: \mathbb{R}^2 \to \mathbb{R}$ such that for every measurable $f: \mathbb{R} \to \mathbb{R}$, the composition $F(x, f(x))$ is measurable.*

This theorem has important consequences concerning the existence of solutions of the differential equation $y' = F(x, y)$ in the class of absolutely continuous functions. In 1992 Balcerzak [3] showed that in Theorem 3.13 the CH assumption can be weakened to $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$. However, the following problem remains open.

Problem 5. Can Theorem 3.13 be proved in ZFC?

In fact, all functions F satysfying Theorem 3.13 are of the form χ_h, where h is a (partial) function from $\mathbb{R}$ to $\mathbb{R}$. It is worth to mention here that, by [3, Prop. 1.5], the property considered in Problem 1 implies that no $F = \chi_h$ with $h: \mathbb{R} \to \mathbb{R}$ can satisfy Theorem 3.13. Similarly, the property considered in the following stronger version of Problem 1 implies that F from Theorem 3.13 cannot be of the form χ_h for a function h from $Y \subset \mathbb{R}$ into $\mathbb{R}$.

Problem 6. *Is it consistent that for every subset Y of $\mathbb{R}$ of positive outer measure and every function $f: Y \to \mathbb{R}$ there exists a set $X \subset Y$ of positive outer measure such that $f \restriction Y$ is continuous?*

The second result is the following 1974 theorem of R. O. Davies.

Theorem 3.14 (Davies [56]). *If the Continuum Hypothesis holds then for every $f: \mathbb{R}^2 \to \mathbb{R}$ there exist functions $g_n, h_n: \mathbb{R} \to \mathbb{R}$, $n < \omega$, such that*

$$f(x, y) = \sum_{n=0}^{\infty} g_n(x) \cdot h_n(y).$$

Problem 7 ([111, Problem 15.11])**.** Is the conclusion of Theorem 3.14 equivalent to CH?

Note that Theorem 3.14 is related to Hilbert's Problem 13 (from his famous Paris lecture of 1900) and a 1957 theorem of Kolmogorov, in which he proves that every continuous function $f\colon [0,1]^n \to \mathbb{R}$ can be represented in a certain form (similar to the above) by continuous functions of one variable. An interesting account on this and related results can be found in a 1984 paper of Sprecher [145].

4. Cardinal functions in analysis

The important recent developments in set theoretical analysis concern the cardinal functions that are defined for different classes of real functions. These investigations seem to be analogous to those concerning of cardinal functions in topology from the 1970's and 1980's. (See [81, 77, 82, 152].) They are also related to the deep studies of cardinal invariants associated with different small subsets of the real line. (For a summary of the results concerning cardinals related to the measure and category see [65] or [8]. For a survey concerning cardinals associated with the thin sets derived from harmonic analysis see [18].)

The first group of functions is motivated by the notion of countable continuity and was introduced in 1991 by J. Cichoń, M. Morayne, J. Pawlikowski, and S. Solecki in [22]. More precisely, they define the *decomposition* function $\mathrm{dec}(\mathcal{F},\mathcal{G})$ for arbitrary families $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and $\mathcal{G} \subset \bigcup\{\mathbb{R}^X\colon X \subset \mathbb{R}\}$, where Y^X stands for the set of all functions from X to Y.

$$\mathrm{dec}(\mathcal{F},\mathcal{G}) = \min\{\kappa \le \mathfrak{c}\colon (\forall f \in \mathcal{F})(\exists \mathcal{X} \in \Pi_\kappa)(\forall X \in \mathcal{X})(f \restriction X \in \mathcal{G})\} \cup \{\mathfrak{c}^+\},$$

where Π_κ denotes the family of all coverings of $\mathbb{R}$ with at most κ many sets. In particular, if $\mathcal{C}$ stands for the family of all continuous functions (from subsets of $\mathbb{R}$ into $\mathbb{R}$) then

$$f\colon \mathbb{R} \to \mathbb{R} \text{ is countably continuous if and only if } \mathrm{dec}(\{f\},\mathcal{C}) \le \omega.$$

In [22] the authors considered the values of $\mathrm{dec}(\mathcal{B}_\beta,\mathcal{B}_\alpha)$ for $\alpha < \beta < \omega_1$, where $\mathcal{B}_\alpha$ stands for the functions of α-th Baire class.

The motivation for this definition comes from a question of N. N. Luzin whether every Borel function is countable continuous. This question was answered negatively by P. S. Novikov (see Keldyš [84]) and was subsequently generalized by Keldyš [84] (in 1934), and S. I. Adian and P. S. Novikov [1] (in 1958). The most general result in this direction was obtained in late 1980's by M. Laczkovich (see Cichoń, Morayne [21]) who proved, in particular, that $\mathrm{dec}(\mathcal{B}_\beta,\mathcal{B}_\alpha) > \omega$ for every $\alpha < \beta < \omega_1$.

One of the most interesting results from the paper [22] is the following theorem.

Theorem 4.1 (Cichoń, Morayne, Pawlikowski, Solecki [22]).

$$\mathrm{cov}(\mathcal{M}) \le \mathrm{dec}(\mathcal{B}_1, \mathcal{C}) \le d,$$

where $\mathrm{cov}(\mathcal{M})$ *is the smallest cardinality of a covering of* $\mathbb{R}$ *by meager sets, and* d*, the dominating number, is the smallest cardinality of a dominating family* $D \subset \omega^\omega$*, i.e., such that for every* $f \in \omega^\omega$ *there exists* $g \in D$ *with* $f \le^* g$.[6]

It has been also shown by J. Steprāns and S. Shelah that none of these inequalities can be replaced by the equation.

Theorem 4.2 (Steprāns [147]). *It is consistent with ZFC that*

$$\mathrm{cov}(\mathcal{M}) < \mathrm{dec}(\mathcal{B}_1, \mathcal{C}).$$

Theorem 4.3 (Shelah, Steprāns [134]). *It is consistent with ZFC that*

$$\mathrm{dec}(\mathcal{B}_1, \mathcal{C}) < d.$$

There are also some interesting results concerning the value of $\mathrm{dec}(\mathcal{C}, \mathcal{D}^1)$, where $\mathcal{D}^1$ is the class of all (partial) differentiable functions. It has been proved by Morayne (see Steprāns [149, Thm 6.1]) that

Theorem 4.4 (Morayne [149, Thm 6.1]). $\mathrm{cov}(\mathcal{M}) \le \mathrm{dec}(\mathcal{C}, \mathcal{D}) \le \mathfrak{c}$.

Also, Steprāns proved that

Theorem 4.5 (Steprāns [149]). *It is consistent with ZFC that*

$$\mathrm{dec}(\mathcal{C}, \mathcal{D}) < \mathfrak{c}.$$

However, the relation between numbers $\mathrm{dec}(\mathcal{C}, \mathcal{D})$, $\mathrm{dec}(\mathcal{B}_1, \mathcal{C})$ and $\mathrm{dec}(\mathcal{B}_\beta, \mathcal{B}_\alpha)$ for $0 < \alpha < \beta < \omega_1$ is unclear.

In the same direction, K. Ciesielski recently noticed that (obviously)

$$\mathrm{cf}(\mathfrak{c}) \le \mathrm{dec}(\mathrm{SZ}, \mathcal{C}) \le \mathfrak{c}$$

and that it is the best that can be said in ZFC.

[6] Recall that for $f, g \in \omega^\omega$ we write $f \le^* g$ if there exists an $n < \omega$ such that $f(m) \le g(m)$ for every $m > n$.

Theorem 4.6 (Ciesielski [32])**.**

(1) *For every κ with* $\mathrm{cf}(\kappa) > \omega$ *there exists a model of ZFC in which* $\mathfrak{c} = \kappa$ *and* $\mathrm{dec}(\mathrm{SZ}, \mathcal{C}) = \mathfrak{c}$*.*

(2) *For every κ with* $\mathrm{cf}(\kappa) > \omega$ *there exists a model of ZFC in which* $\mathfrak{c} = \kappa$ *and* $\mathrm{dec}(\mathrm{SZ}, \mathcal{C}) = \mathrm{cf}(\kappa) = \mathrm{cf}(\mathfrak{c})$*.*

In fact, (1) happens in a model obtained by extending a ground model with GCH by adding κ many Cohen reals. The equation $\mathrm{dec}(\mathrm{SZ}, \mathcal{C}) = \mathfrak{c}$ follows immediately from Theorem 2.9.

The model for (2) is obtained as follows. You start with a model with GCH, assume that $\lambda = \mathrm{cf}(\kappa) < \kappa$ and take an increasing sequence $\{\lambda_\xi \colon \xi < \lambda\}$ cofinal with λ and such that each λ_ξ is a cardinal successor. The desired model is obtained by a generic extension via forcing P which a finite support iteration of forcings M_ξ, where each M_ξ is a standard ccc forcing adding the Martin's Axiom over the previous model and making $\mathfrak{c} = \lambda_\xi$.

The second group of cardinal functions is defined in terms of algebraic operations on functions. Their definition was motivated by the following property of Darboux functions (from $\mathbb{R}$ to $\mathbb{R}$) due to Fast and mentioned in the previous section:

- for every family $\mathcal{H} \subset \mathbb{R}^\mathbb{R}$ with $|\mathcal{H}| \leq \mathfrak{c}$

$$\text{there exists } g \in \mathbb{R}^\mathbb{R} \text{ such that } g + h \in \mathcal{D} \text{ for every } h \in \mathcal{H}, \tag{1}$$

where $|Z|$ denotes the cardinality of Z. In 1974 Kellum [85] proved the similar result for the class AC of almost continuous functions and in 1991 Natkaniec [114] defined the following cardinal functions for every $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ to study these phenomena more closely.

$$\begin{aligned}
\mathrm{A}(\mathcal{F}) &= \min\Big\{|\mathcal{H}| : \mathcal{H} \subset \mathbb{R}^\mathbb{R} \ \&\ \neg\exists g \in \mathbb{R}^\mathbb{R}\ \forall h \in \mathcal{H}\ \} + \langle \in \mathcal{F}\Big\} \cup \{(2^\mathfrak{c})^+\} \\
&= \min\Big\{|\mathcal{H}| : \mathcal{H} \subset \mathbb{R}^\mathbb{R} \ \&\ \forall \} \in \mathbb{R}^\mathbb{R}\ \exists \langle \in \mathcal{H}\ \} + \langle \notin \mathcal{F}\Big\} \cup \{(2^\mathfrak{c})^+\} \\
\mathrm{M}(\mathcal{F}) &= \min\Big\{|\mathcal{H}| : \mathcal{H} \subset \mathbb{R}^\mathbb{R} \ \&\ \neg\exists \} \in \mathbb{R}^\mathbb{R} \setminus \{\chi_\emptyset\}\ \forall \langle \in \mathcal{H}\ \} \cdot \langle \in \mathcal{F}\Big\} \cup \{(2^\mathfrak{c})^+\} \\
&= \min\Big\{|\mathcal{H}| : \mathcal{H} \subset \mathbb{R}^\mathbb{R} \ \&\ \forall \} \in \mathbb{R}^\mathbb{R} \setminus \{\chi_\emptyset\}\ \exists \langle \in \mathcal{H}\ \} \cdot \langle \notin \mathcal{F}\Big\} \cup \{(2^\mathfrak{c})^+\}.
\end{aligned}$$

The extra assumption that $g \neq \chi_\emptyset$ is added in the definition of M since otherwise for every family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ containing constant zero function $\chi_\emptyset$ we would have $\mathrm{M}(\mathcal{F}) = (2^\mathfrak{c})^+$.

Its easy to see that the functions A and M are monotone in a sense that $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$ and $\mathrm{M}(\mathcal{F}) \leq \mathrm{M}(\mathcal{G})$ for every $\mathcal{F} \subset \mathcal{G} \subset \mathbb{R}^\mathbb{R}$. Also clearly (1) is false for $\mathcal{H} = \mathbb{R}^\mathbb{R}$. Thus, in language of the function A the results of Fast and Kellum can be expressed as follows:

$$\mathfrak{c} < \mathrm{A}(\mathrm{AC}) \leq \mathrm{A}(\mathcal{D}) \leq 2^\mathfrak{c}.$$

If $2^{\mathfrak{c}} = \mathfrak{c}^+$ (so, under the Generalized Continuum Hypothesis GCH) the values of A(AC) and A($\mathcal{D}$) are clear: $\mathrm{A(AC)} = \mathrm{A}(\mathcal{D}) = 2^{\mathfrak{c}} = \mathfrak{c}^+$. Thus, Natkaniec asked [114, p. 495] (see also [72, Problem 1]) whether the equation $\mathrm{A(AC)} = 2^{\mathfrak{c}}$ can be proved in ZFC.

This question was investigated by Ciesielski and Miller in 1994. They proved that $\mathrm{A(AC)} = \mathrm{A}(\mathcal{D})$, that the cofinality $\mathrm{cf}(\mathrm{A}(\mathcal{D}))$ of $\mathrm{A}(\mathcal{D})$ is greater than $\mathfrak{c}$, and that this together with the inequalities $\mathfrak{c} < \mathrm{A}(\mathcal{D}) \leq 2^{\mathfrak{c}}$ is essentially all that can be proved in ZFC.

Theorem 4.7 (Ciesielski, Miller [40])**.**
(a) $\mathrm{A(AC)} = \mathrm{A}(\mathcal{D}) = \mathfrak{e}_{\mathfrak{c}}$, *where*
$$\mathfrak{e}_\kappa = \min\{|F| \colon F \subseteq \kappa^\kappa \ \&\ \forall g \in \kappa^\kappa\ \exists f \in F\ |f \cap g| < \kappa\}.$$
(b) $\mathrm{cf}(\mathrm{A}(\mathcal{D})) > \mathfrak{c}$.
(c) *Let* $\lambda \geq \kappa \geq \omega_2$ *be cardinals such that* $\mathrm{cf}(\lambda) > \omega_1$ *and* κ *is regular. Then it is relatively consistent with ZFC that the Continuum Hypothesis is true,* $2^{\mathfrak{c}} = \lambda$, *and* $\mathrm{A}(\mathcal{D}) = \kappa$.
(d) *Let* λ *be a cardinal such that* $\mathrm{cf}(\lambda) > \omega_1$. *Then it is relatively consistent with ZFC that the Continuum Hypothesis holds and* $\mathrm{A}(\mathcal{D}) = \lambda = 2^{\mathfrak{c}}$.

In particular Theorem 4.7 says that $\mathrm{A}(\mathcal{D})$ does not have to be a regular cardinal (part (d)) and that $\mathrm{A}(\mathcal{D})$ can be any regular cardinal number between c^+ and 2^c, with 2^c being "arbitrarily large" (part (c)).

At the same time Natkaniec and Recaw established the values of M(AC) and M($\mathcal{D}$) proving

Theorem 4.8 (Natkaniec, Recaw [120])**.** $\mathrm{M(AC)} = \mathrm{M}(\mathcal{D}) = \mathrm{cf}(\mathfrak{c})$.

The first systematic study of functions A and M was done by Ciesielski and Recaw in the later part of 1995. They collected basic properties of operators A and M, which are stated below, and found the values of A and M for some other classes of functions.

Proposition 4.9 ([44])**.** *Let* $\emptyset \neq \mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R}^{\mathbb{R}}$. *Then*
(0) $\mathrm{A}(\emptyset) = \mathrm{M}(\emptyset) = 1$;
(1) $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$ *and* $\mathrm{M}(\mathcal{F}) \leq \mathrm{M}(\mathcal{G})$;
(2) $\mathrm{A}(\mathcal{F}) \geq 2$;
(2′) $\mathrm{M}(\mathcal{F}) \geq 2$ *if* $\chi_\emptyset, \chi_{\mathbb{R}} \in \mathcal{F}$;
(3) $\mathrm{A}(\mathcal{F}) = (2^{\mathfrak{c}})^+$ *if and only if* $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$;
(3′) $\mathrm{M}(\mathcal{F}) \leq \mathfrak{c}$ *if* $r\chi_{\{x\}} \notin \mathcal{F}$ *for every* $r, x \in \mathbb{R}$, $r \neq 0$;
(4) $\mathrm{A}(\mathcal{F}) = 2$ *if and only if* $\mathcal{F} - \mathcal{F} = \{f_1 - f_2 \colon f_1, f_2 \in \mathcal{F}\} \neq \mathbb{R}^{\mathbb{R}}$.[7]

[7] In [44] this was proved with the additional assumption that $\chi_\emptyset \in \mathcal{F}$. This extra assumption was removed by F. Jordan in [79].

In particular, (4) from Proposition 4.9 shows that every function is a difference of two functions from a class $\mathcal{F}$ if and only if $\mathrm{A}(\mathcal{F}) > 2$.

To state the other results from [44] recall the definitions the following classes of functions, where X is an arbitrary topological space.

- $\mathrm{Conn}(X)$ of *connectivity functions* $f\colon X \to \mathbb{R}$, i.e., such that the graph of f restricted to C (that is $f \cap [C \times \mathbb{R}]$) is connected in $X \times \mathbb{R}$ for every connected subset C of X.
- $\mathrm{Ext}(X)$ of *extendable functions* $f\colon X \to \mathbb{R}$, i.e., such that there exists a connectivity function $g\colon X \times [0,1] \to \mathbb{R}$ with $f(x) = g(x,0)$ for every $x \in X$.
- $\mathrm{PC}(X)$ of *peripherally continuous functions* $f\colon X \to \mathbb{R}$, i.e., such that for every $x \in X$ and any pair $U \subseteq X$ and $V \in \mathbb{R}$ of open neighborhoods of x and $f(x)$, respectively, there exists an open neighborhood W of x with $\mathrm{cl}(W) \subseteq U$ and $f[\mathrm{bd}(W)] \subseteq V$, where $\mathrm{cl}(W)$ and $\mathrm{bd}(W)$ stand for the closure and the boundary of W, respectively.

We will write Conn, Ext and PC in place of $\mathrm{Conn}(X)$, $\mathrm{Ext}(X)$, and $\mathrm{PC}(X)$ if $X = \mathbb{R}$. Notice also, that $f \in \mathrm{PC}$ if and only if f is weakly continuous, as defined on page 151.

For the generalized continuity classes of functions (from $\mathbb{R}$ into $\mathbb{R}$) defined so far we have the following proper inclusions $\subset$, marked by arrows $\longrightarrow$. (See [14].)

$$\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathcal{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[drr] & & & & \mathrm{PC} \\
 & & \mathrm{PR} \arrow[urr] & &
\end{tikzcd}$$

Chart 1.

In particular, inclusions $\mathrm{AC} \subset \mathrm{Conn} \subset \mathcal{D}$, monotonicity of A and Theorem 4.7(a) imply that $\mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathcal{D})$. Similarly, Theorem 4.8 implies that $\mathrm{M}(\mathrm{Conn}) = \mathrm{M}(\mathrm{AC}) = \mathrm{M}(\mathcal{D})$. The values of A and M for the remaining classes are as follows.

Theorem 4.10 (Ciesielski, Recaw [44])**.**

(1) $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$.
(2) $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$.
(3) $\mathrm{M}(\mathrm{Ext}) = \mathrm{M}(\mathrm{PR}) = 2$.
(4) $\mathrm{M}(\mathrm{PC}) = \mathfrak{c}$.

Notice also that $\mathrm{Ext} \subset \mathrm{AC} \cap \mathrm{PR} \subset \mathrm{Conn} \cap \mathrm{PR} \subset \mathcal{D} \cap \mathrm{PR} \subset \mathrm{PR}$. Thus, by monotonicity of A and the above theorem we obtain the following corollary.

Corollary 4.11.

(1) $\mathrm{A}(\mathrm{AC} \cap \mathrm{PR}) = \mathrm{A}(\mathrm{Conn} \cap \mathrm{PR}) = \mathrm{A}(\mathcal{D} \cap \mathrm{PR}) = \mathfrak{c}^+$*; and,*
(2) $\mathrm{M}(\mathrm{AC} \cap \mathrm{PR}) = \mathrm{M}(\mathrm{Conn} \cap \mathrm{PR}) = \mathrm{M}(\mathcal{D} \cap \mathrm{PR}) = \in$.

The values of functions A and M for the class SZ has been studied by Ciesielski and Natkaniec. First they noticed that if the definition of M from page 162 is used then trivially $\mathrm{M}(\mathrm{SZ}) = 1$, since for any function $h \in \mathbb{R}^{\mathbb{R}}$ with $|h^{-1}(0)| = \mathfrak{c}$ we have $g \cdot h \notin \mathrm{SZ}$ for every $g \in \mathbb{R}^{\mathbb{R}}$. Thus, they modified the definition of M(SZ) to

$$\mathrm{M}(\mathrm{SZ}) = \min\left\{|\mathcal{H}| : \mathcal{H} \subseteq \mathcal{R}_{\prime} \ \& \ \neg\exists \} \in \mathbb{R}^{\mathbb{R}} \ \forall \langle \in \mathcal{H} \ \} \cdot \langle \in \mathrm{SZ}\right\} \cup \{(2^{\mathfrak{c}})^+\},$$

where

$$\mathcal{R}_{\prime} = \left\{\{ \in \mathbb{R}^{\mathbb{R}} \colon \ |\{^{-\infty}(\prime)| < \mathfrak{c}\right\}.$$

With this agreement in place they proved the following result.

Theorem 4.12 (Ciesielski, Natkaniec [41])**.**

(a) $\mathrm{M}(\mathrm{SZ}) = \mathrm{A}(\mathrm{SZ}) = \mathrm{d}_{\mathfrak{c}}$*, where*

$$\mathrm{d}_\kappa = \min\{|F| \colon F \subseteq \kappa^\kappa \& \ \forall g \in \kappa^\kappa \ \exists f \in F \ |f \cap g| = \kappa\}.$$

(b) *Let* $\lambda \geq \kappa \geq \omega_2$ *be cardinals such that* $\mathrm{cf}(\lambda) > \omega_1$ *and* κ *is regular. Then it is relatively consistent with ZFC that the Continuum Hypothesis is true,* $2^{\mathfrak{c}} = \lambda$*, and* $\mathrm{A}(\mathrm{SZ}) = \mathrm{A}(\mathcal{D}) = \kappa$.
(c) *Let* $\lambda > \omega_2$ *be a cardinal such that* $\mathrm{cf}(\lambda) > \omega_1$*. Then it is relatively consistent with ZFC that the Continuum Hypothesis holds,* $2^{\mathfrak{c}} = \lambda$*, and* $\mathrm{A}(\mathrm{SZ}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} = \mathrm{A}(\mathcal{D})$.

However, the following problems remain open.

Problem 8 ([41, Problems 2.13 and 2.17])**.**

(a) *Is it consistent that* $\mathrm{A}(\mathrm{SZ}) > \mathrm{A}(\mathcal{D})$*?*
(b) *Can* A(SZ) *be a singular cardinal?*

Another systematic study of the operator A was done by F. Jordan in 1996. In his study he examined the values of $\mathrm{A}(\neg\mathcal{F})$ where $\neg\mathcal{F} = \mathbb{R}^{\mathbb{R}} \setminus \mathcal{F}$ and classes $\mathcal{F}$ are chosen from those discussed above. Notice that $\mathrm{A}(\neg\mathcal{F})$ has the following very nice interpretation:

$$\mathrm{A}(\neg\mathcal{F}) \text{ is the smallest cardinality of an } \mathcal{H} \subset \mathbb{R}^{\mathbb{R}} \text{ such that } \mathcal{F} - \mathcal{H} = \mathbb{R}^{\mathbb{R}},$$

where $\mathcal{F} - \mathcal{H} = \{f - h \colon f \in \mathcal{F} \ \&\ h \in \mathcal{H}\}$. To make this study non-trivial Jordan notes first that the value of $\mathrm{A}(\mathcal{F})$ does not determine the value of $\mathrm{A}(\neg\mathcal{F})$:

Theorem 4.13 (Jordan [79]). *For every cardinal number $2 \le \lambda \le 2^{\mathfrak{c}}$ there exists $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{A}(\mathcal{F}) = 2$ and $\mathrm{A}(\neg\mathcal{F}) = \lambda$.*

In particular, there exist families $\mathcal{G}, \mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{A}(\mathcal{F}) = \mathrm{A}(\mathcal{G})$ and $\mathrm{A}(\neg\mathcal{F}) \neq \mathrm{A}(\neg\mathcal{G})$.

This paper [79] contains also the following results, where for a cardinal number κ and functions $f, g \colon X \to Y$ we define $[X]^{\kappa} = \{Y \subset X \colon |Y| = \kappa\}$ and $[f = g] = \{x \in X \colon f(x) = g(x)\}$.

Theorem 4.14 (Ciesielski [79, Thm. 7]). $\mathrm{A}(\neg\,\mathrm{PC}) = \omega_1$.

Theorem 4.15 (Jordan [79]).

(1) $\mathrm{A}(\neg\,\mathrm{PR}) = \mathrm{A}(\neg\mathrm{Ext}) = 2^{\mathfrak{c}}$.

(2) $\mathrm{A}(\mathrm{SZ}) = d_{\mathfrak{c}} \le \mathrm{A}(\neg\mathcal{D}) \le \mathrm{A}(\neg\mathrm{Conn}) \le \mathrm{A}(\neg\mathrm{AC}) \le d^*_{\mathfrak{c}}$, *where*

$$d^*_{\kappa} = \min\{|F| \colon F \subseteq \kappa^{\kappa} \ \&\ (\forall G \in [\kappa^{\kappa}]^{\kappa})(\exists f \in F)(\forall g \in G)\\ (|[f = g]| = \kappa)\}.$$

(3) $\mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathcal{D}) = \mathfrak{e}_{\mathfrak{c}} \le \mathrm{A}(\neg\,\mathrm{SZ}) \le \mathfrak{e}^*_{\mathfrak{c}}$, *where*

$$\mathfrak{e}^*_{\kappa} = \min\{|F| \colon F \subseteq \kappa^{\kappa} \ \&\ (\forall G \in [\kappa^{\kappa}]^{\kappa})(\exists f \in F)(\forall g \in G)\\ (|[f = g]| < \kappa)\}.$$

(4) *If $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ then $d_{\mathfrak{c}} = d^*_{\mathfrak{c}}$*[8] *and $\mathfrak{e}_{\mathfrak{c}} = \mathfrak{e}^*_{\mathfrak{c}}$. In particular*

$$\mathrm{A}(\mathrm{SZ}) = d_{\mathfrak{c}} = \mathrm{A}(\neg\mathcal{D}) = \mathrm{A}(\neg\mathrm{Conn}) = \mathrm{A}(\neg\mathrm{AC}) = d^*_{\mathfrak{c}}$$

and

$$\mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathrm{AC}) = \mathfrak{e}_{\mathfrak{c}} = \mathrm{A}(\neg\,\mathrm{SZ}) = \mathfrak{e}^*_{\mathfrak{c}}.$$

(5) *If $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ and $\mathfrak{c} = \lambda^+$ for some cardinal λ then $d_{\mathfrak{c}} \le \mathfrak{e}_{\mathfrak{c}}$. In particular*

$$\mathrm{A}(\neg\mathcal{D}) = \mathrm{A}(\neg\mathrm{AC}) = \mathrm{A}(\mathrm{SZ}) = d_{\mathfrak{c}} \le \mathfrak{e}_{\mathfrak{c}} = \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\neg\,\mathrm{SZ}).$$

The importance of the extra assumptions in (4) and (5) of Theorem 4.15 is not clear. In particular, the following problem is still open.

Problem 9. *When does either $d_{\mathfrak{c}} = d^*_{\mathfrak{c}}$ or $\mathfrak{e}_{\mathfrak{c}} = \mathfrak{e}^*_{\mathfrak{c}}$ hold?*

[8] This part was proved by K. Ciesielski.

Note also that (4) and (5) of Theorem 4.15, and Theorem 4.12 imply immediately the following corollary.

Corollary 4.16 (Jordan [79]).

(1) *Let $\lambda \geq \kappa \geq \omega_2$ be cardinals such that $\mathrm{cf}(\lambda) > \omega_1$ and κ is regular. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$ and*

$$\mathrm{A}(\neg\mathcal{D}) = \mathrm{A}(\neg\mathrm{AC}) = \mathrm{A}(\mathrm{SZ}) = \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\neg\,\mathrm{SZ}) = \kappa.$$

(2) *Let $\lambda > \omega_2$ be a cardinal such that $\mathrm{cf}(\lambda) > \omega_1$. Then it is relatively consistent with ZFC+CH that $2^{\mathfrak{c}} = \lambda$, and*

$$\mathrm{A}(\neg\mathcal{D}) = \mathrm{A}(\neg\mathrm{AC}) = \mathrm{A}(\mathrm{SZ}) = \mathfrak{c}^+ < 2^{\mathfrak{c}} = \mathrm{A}(\mathcal{D}) = \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\neg\,\mathrm{SZ}).$$

Finally, the following three classes of functions have been brought to this picture.

CIVP of functions $f\colon \mathbb{R} \to \mathbb{R}$ having the *Cantor Intermediate Value Property*, i.e., such that for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$;

SCIVP of functions $f\colon \mathbb{R} \to \mathbb{R}$ having the *Strong Cantor Intermediate Value Property*, i.e., such that for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous;

WCIVP of functions $f\colon \mathbb{R} \to \mathbb{R}$ having the *Weak Cantor Intermediate Value Property*, that is, such that for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there is a Cantor set C between x and y such that $f[C] \subset (f(x), f(y))$.

They fit Chart 1 in the following way. (See Gibson, Natkaniec [70].)

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathcal{D} \arrow[dr] & \\
\mathcal{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] \arrow[dr] & \mathrm{PR} \arrow[ur] & \\
 & & & & \mathrm{WCIVP} &
\end{tikzcd}$$

Chart 2: "Darboux like" functions.

Clearly the above inclusions, monotonicity of A and M, and Theorem 4.10 imply immediately:

$$\mathrm{A}(\mathrm{SCIVP}) = \mathrm{A}(\mathrm{CIVP}) = \mathfrak{c}^+ \quad \text{and} \quad \mathrm{M}(\mathrm{SCIVP}) = \mathrm{M}(\mathrm{CIVP}) = 2.$$

The values of functions A and M for the class WCIVP, and for the classes formed by the intersections of SZ with each of the remaining classes mentioned above were not studied too carefully so far. However, obviously

SZ $\cap$SCIVP $= \emptyset$ implying

$$\mathrm{A}(\mathrm{SZ}\cap\mathrm{SCIVP}) = \mathrm{M}(\mathrm{SZ}\cap\mathrm{SCIVP}) = 1.$$

Also, it follows from Theorem 3.10 that

$$\text{it is consistent that } \mathrm{A}(\mathrm{SZ}\cap\mathcal{D}) = \mathrm{M}(\mathrm{SZ}\cap\mathcal{D}) = 1,$$

while also

$$\text{it is consistent that } \mathrm{A}(\mathrm{SZ}\cap\mathrm{PR}\cap\mathrm{AC}) \geq 2.$$

A stronger version of this last inequality follows also from the following recent theorem of K. Banaszewski and Natkaniec.

Theorem 4.17 (Banaszewski, Natkaniec [7]).
(1) SZ $\cap$CIVP $\neq \emptyset$. *In particular* SCIVP $\neq$ CIVP.
(2) *If union of less than* $\mathfrak{c}$ *many meager subsets of* $\mathbb{R}$ *is meager (thus under CH and MA) then* AC $\cap$CIVP $\cap$ SZ $\neq \emptyset$.

In particular,

$$\text{it is consistent that } \mathrm{A}(\mathrm{SZ}\cap\mathrm{CIVP}\cap\mathrm{AC}) \geq 2$$

and

$$\mathrm{A}(\mathrm{SZ}\cap\mathrm{CIVP}) \geq 2.$$

This last inequality has been recently improved by F. Jordan, who proved the following.

Theorem 4.18 (Jordan [80]). A(SZ $\cap\neg\mathcal{D}\cap$ CIVP) $> \mathfrak{c}$. *In particular*

$$\mathrm{A}(\mathrm{SZ}\cap\neg\mathcal{D}\cap\mathrm{CIVP}) = \mathrm{A}(\mathcal{F}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$$

for every $\mathcal{F}\subset\mathbb{R}^{\mathbb{R}}$ *such that* SZ $\cap\neg\mathcal{D}\cap$ CIVP $\subset\mathcal{F}\subset$ PR.

This theorem gives the value of A for many classes that can be obtained intersecting classes from Chart 2 and SZ.

Some of the difficulties of studying operators A and M for the intersections AC $\cap$SCIVP and AC $\cap$CIVP is that there is relatively little known about these classes. For example, although Theorem 4.17(2) implies that consistently these classes are different, a ZFC example was unknown until the following very recent theorem of Ciesielski.

Theorem 4.19 (Ciesielski [31]). *There exists an* $f\in$ AC $\cap$CIVP *which is discontinuous on every perfect set. In particular,* $f\notin$ SCIVP.

Also the only example of an $f\in$ AC $\cap$SCIVP$\setminus$Ext results from the following recent theorem of Rosen. (See also [31] for a generalization.)

Theorem 4.20 (Rosen [127])**.** *If the Continuum Hypothesis holds then there exists an* $f \in \mathrm{AC} \cap \mathrm{SCIVP} \setminus \mathrm{Ext}$.

In fact, the conclusion of Theorem 4.20 remains true under the assumption that union of less than continuum many meager sets is meager. However, the problem of existence of such a function in ZFC remains open.

Several other operators similar to A and M have also been studied. Thus, in 1995 Natkaniec [117] introduced the following operators connected to the composition of functions, where Const stands for the family of all constant functions.

$$\mathrm{C}_{out}(\mathcal{F})=\min\Big\{|\mathcal{H}|\colon \mathcal{H}\subset\mathbb{R}^{\mathbb{R}}\ \&\ \neg\exists\} \in \mathbb{R}^{\mathbb{R}}\setminus\mathrm{Const}\ \forall\langle\in\mathcal{H}\ \}\circ\langle\in\mathcal{F}\Big\}\cup\{(2^{\mathfrak{c}})^+\}$$

$$\mathrm{C}_{in}(\mathcal{F})=\min\Big\{|\mathcal{H}|\colon \mathcal{H}\subset\mathbb{R}^{\mathbb{R}}\ \&\ \neg\exists\} \in \mathbb{R}^{\mathbb{R}}\setminus\mathrm{Const}\ \forall\langle\in\mathcal{H}\ \langle\circ\}\in\mathcal{F}\Big\}\cup\{(2^{\mathfrak{c}})^+\}$$

He proved also the following.

Theorem 4.21 (Natkaniec [117])**.**

(1) $\mathrm{C}_{out}(\mathrm{Ext}) = \mathrm{C}_{out}(\mathrm{CIVP}) = \mathrm{C}_{out}(\mathrm{PR}) = 1$.
(2) $\mathrm{C}_{out}(\mathrm{AC}) = \mathrm{C}_{out}(\mathrm{Conn}) = \mathrm{C}_{out}(\mathcal{D}) = \mathrm{cf}(\mathfrak{c})$.
(3) $\mathrm{C}_{out}(\mathrm{PC}) = \mathfrak{c}$.
(4) $\mathrm{C}_{in}(\mathrm{Ext}) = \mathrm{C}_{in}(\mathrm{AC}) = \mathrm{C}_{in}(\mathrm{Conn}) = \mathrm{C}_{in}(\mathcal{D}) = 1$.
(5) $\mathrm{C}_{in}(\mathrm{CIVP}) = \mathrm{C}_{in}(\mathrm{PR}) = \mathrm{C}_{in}(\mathrm{PC}) = \mathfrak{c}^+$.

Similar functions have been also studied by Ciesielski and Natkaniec [41]:

$$\mathrm{C}_{out}(\mathrm{SZ})=\min(\{|\mathcal{H}| : \mathcal{H}\subseteq\mathcal{R}_{\wr\sqcap\sqcup}\ \&\ \neg\exists\}\in\mathbb{R}^{\mathbb{R}}\ \forall\langle\in\mathcal{H}\ \}\circ\langle\in\mathrm{SZ}\}\cup\{(\in^{\mathfrak{c}})^+\})$$

$$\mathrm{C}_{in}(\mathrm{SZ})=\min(\{|\mathcal{H}| : \mathcal{H}\subseteq\mathcal{R}_{\rangle\backslash}\ \&\ \neg\exists\}\in\mathbb{R}^{\mathbb{R}}\ \forall\langle\in\mathcal{H}\ \langle\circ\}\in\mathrm{SZ}\}\cup\{(\in^{\mathfrak{c}})^+\})$$

where $\mathcal{R}_{\wr\sqcap\sqcup}$ ($\mathcal{R}_{\rangle\backslash}$) is the set of all $h\in\mathbb{R}^{\mathbb{R}}$ for which there exists $g\in\mathbb{R}^{\mathbb{R}}$ such that $g\circ h\in\mathrm{SZ}$ ($h\circ g\in\mathrm{SZ}$, respectively). In fact, the classes $\mathcal{R}_{\wr\sqcap\sqcup}$ and $\mathcal{R}_{\rangle\backslash}$ have the following nice characterizations:

$$\mathcal{R}_{\wr\sqcap\sqcup}=\Big\{\{\in\mathbb{R}^{\mathbb{R}}\colon\ |\{^{-\infty}(\dagger)|<\mathfrak{c}\ \text{ for every } \dagger\in\mathbb{R}\Big\},$$

and, when $\mathfrak{c}$ is a regular cardinal,

$$\mathcal{R}_{\rangle\backslash}=\Big\{\{\in\mathbb{R}^{\mathbb{R}}\colon\ |\{[\mathbb{R}]|=\mathfrak{c}\Big\}.$$

In [41] the authors proved that

Theorem 4.22 (Ciesielski, Natkaniec [41]).
(1) $\mathrm{C}_{in}(\mathrm{SZ}) = 2$.
(2) $\mathrm{C}_{out}(\mathrm{SZ}) = \mathrm{A}(\mathrm{SZ})$ *if* $\mathfrak{c} = \lambda^+$ *for some cardinal* λ.
(3) $\mathfrak{c} < \mathrm{C}_{out}(\mathrm{SZ}) \leq 2^{\mathfrak{c}}$, *if* $\mathfrak{c}$ *is regular.*
(4) $\mathrm{cf}(\mathfrak{c}) \leq \mathrm{C}_{out}(\mathrm{SZ}) \leq 2^{\mathrm{cf}(\mathfrak{c})}$, *if* $\mathfrak{c}$ *is singular.*

Also, in a recent short survey paper [119] Natkaniec evaluated the values of operators A, M, C_{in} and C_{out} for the class HAC of *almost continuous functions in sense of Husain*, i.e., such $f\colon \mathbb{R} \to \mathbb{R}$ that $f^{-1}(U) \subset \mathrm{int}(\mathrm{cl}(f^{-1}(U)))$ for every non-empty open set $U \subset \mathbb{R}$.

Theorem 4.23 (Natkaniec [119]).
(1) $\mathrm{A}(\mathrm{HAC}) = 2^{\mathfrak{c}}$.
(2) $\mathrm{M}(\mathrm{HAC}) = \mathrm{cov}(\mathcal{M})$ *, where* $\mathrm{cov}(\mathcal{M})$ *is the smallest cardinality of a family of meager sets that covers* $\mathbb{R}$.
(3) $\mathrm{C}_{in}(\mathrm{HAC}) = (2^{\mathfrak{c}})^+$ *and* $\mathrm{C}_{out}(\mathrm{HAC}) = \mathfrak{c}$.

Some other cardinal operators connected with composition and concerning some kind of coding were also studied by Ciesielski and Recaw [44], Ciesielski and Natkaniec [41], and Natkaniec [119].

Another variant of function A is connected to the families of bounded functions. To define it properly the following notation is necessary. For a Let UB stand for all uniformly bounded families $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$, and let $\mathcal{B}$ be the class of all bounded functions $f\colon \mathbb{R} \to \mathbb{R}$. Then we define

$$\mathrm{A}_b(\mathcal{F}) = \min\{|\mathcal{H}| : \mathcal{H} \in \mathrm{UB} \ \& \ \neg\,(\exists\} \in \mathcal{B})\,(\forall\langle \in \mathcal{H})\,(\} + \langle \in \mathcal{F})\}\,.$$

In 1994 Maliszewski [104] proved that

$$\mathrm{A}_b(\mathcal{D}) = \mathrm{cf}(\mathfrak{c})$$

so that $\mathrm{A}_b(\mathcal{D}) < \mathrm{A}(\mathcal{D})$. Moreover, he proved that if $\mathcal{F} \in \mathrm{UB}$, all functions in $\mathcal{F}$ are measurable (have Baire property), and the size of $\mathcal{F}$ is less than the additivity of measure (category) then there exists a "universal summand" bounded function for $\mathcal{F}$ with the same property. Similar results were also proved for countable families of Borel measurable functions of α class when $\alpha > 1$ and for finite families of Baire one functions.

The values of A_b for the other classes of functions from Chart 1 has been investigated by Ciesielski and Maliszewski [39]. In particular, they proved

Theorem 4.24 (Ciesielski, Maliszewski [39])**.**

(1) $\mathrm{A}_b(\mathrm{Ext}) = \mathrm{A}_b(\mathrm{PR}) = 2$.
(2) $\mathrm{A}_b(\mathrm{AC}) = \mathrm{A}_b(\mathrm{Conn}) = \mathrm{A}_b(\mathcal{D}) = \mathrm{cf}(\mathfrak{c})$.
(3) $\mathrm{A}_b(\mathrm{PC}) = \mathfrak{c}$.

Notice also that Theorem 4.24 implies immediately the following corollary.

Corollary 4.25 (Ciesielski, Maliszewski [39])**.**

(1) *Every bounded function $f\colon \mathbb{R} \to \mathbb{R}$ is the sum of two bounded almost continuous functions.*
(2) *There exists a bounded function $f\colon \mathbb{R} \to \mathbb{R}$ which is not the sum of two bounded functions with perfect road.*

In particular, Corollary 4.25(1) generalizes a result of Darji and Humke [55] that every bounded function can be expressed a sum of three bounded almost continuous functions. On the other hand Corollary 4.25(2) shows that the following result of Natkaniec is sharp.

Theorem 4.26 (Natkaniec [118])**.** *Every bounded function can be expressed as a sum of three bounded extendable functions.*

It might be also interesting to examine a bounded version of M, defined as

$$\mathrm{M}_b(\mathcal{F}) = \min\{|\mathcal{H}| : \mathcal{H} \in \mathrm{UB}\ \ \&\ \neg\,(\exists\} \in \mathcal{B},\, \} \neq \prime)\,(\forall\langle \in \mathcal{H})\,(\} \cdot \langle \in \mathcal{F})\}\,.$$

However this function has not been studied so far.

One might also consider the study of the operator A (and M) for the functions from $\mathbb{R}^n$ into $\mathbb{R}$ with $n > 1$. This has indeed been done by Ciesielski and Wojciechowski in [47]. The study concerned only the classes $\mathrm{Ext}(\mathbb{R}^n)$, $\mathrm{AC}(\mathbb{R}^n)$, $\mathrm{Conn}(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, and $\mathrm{PC}(\mathbb{R}^n)$ since other classes from Chart 2 do not have natural generalizations into functions of more than one variable. First, one should recall that for $n > 1$ Chart 1 is not valid any more. The new inclusions (for $n > 1$) are as follows:

$$\mathrm{Ext}(\mathbb{R}^n) \subset \mathrm{PC}(\mathbb{R}^n) = \mathrm{Conn}(\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n),$$

$$\mathcal{D}(\mathbb{R}^n) \not\subset \mathrm{AC}(\mathbb{R}^n), \quad \mathrm{AC}(\mathbb{R}^n) \not\subset \mathcal{D}(\mathbb{R}^n), \quad \mathcal{D}(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \not\subset \mathrm{Conn}(\mathbb{R}^n).$$

(The inclusion "$\mathrm{PC}(\mathbb{R}^n) \subset \mathrm{Conn}(\mathbb{R}^n)$" was proved by Hamilton [75] and by Stallings [146], and the inclusion "$\mathrm{Conn}(\mathbb{R}^n) \subset \mathrm{PC}(\mathbb{R}^n)$" by Hagan [74]. The proof of the inclusion "$\mathrm{Conn}(\mathbb{R}^n) \subset \mathrm{AC}(\mathbb{R}^n)$" is presented in [146]. The examples showing that $\mathcal{D}(\mathbb{R}^n) \not\subset \mathrm{AC}(\mathbb{R}^n)$ and $\mathrm{AC}(\mathbb{R}^n) \not\subset \mathcal{D}(\mathbb{R}^n)$ can be found in [115, Examples 1.1.9 and 1.1.10] or [114, Examples 1.7 and 1.6], while a simple Baire class 1 function in $\mathcal{D}(\mathbb{R}^n) \cap \mathrm{AC}(\mathbb{R}^n) \setminus \mathrm{Conn}(\mathbb{R}^n)$

was described in [128, Example 1].) We do not know whether the inclusion $\mathrm{Ext}(\mathbb{R}^n) \subset \mathrm{PC}(\mathbb{R}^n)$ is proper.

The problem with studying the value of the operator A for all these classes (except for $\mathrm{AC}(\mathbb{R}^n)$) is that there exists a function $f\colon \mathbb{R}^n \to \mathbb{R}$ which is not a sum of n Darboux functions, implying that

$$\mathrm{A}(\mathrm{Ext}(\mathbb{R}^n)) = \mathrm{A}(\mathrm{PC}(\mathbb{R}^n)) = \mathrm{A}(\mathrm{Conn}(\mathbb{R}^n)) = \mathrm{A}(\mathcal{D}(\mathbb{R}^n)) = 2.$$

However, every function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a sum of $n+1$ extendable functions. To express these results nicely, define for $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}^n}$ the *repeatability* $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$ as the smallest integer k such that any function $f\colon \mathbb{R}^n \to \mathbb{R}$ can be expressed as a sum of k functions from $\mathcal{F}$. (We put $\mathcal{R}(\mathcal{F}) = \infty$ if such a number does not exist.) In this language the results of Ciesielski and Wojciechowski can be stated as follows.

Theorem 4.27 (Ciesielski, Wojciechowski [47]).

$$\mathcal{R}(\mathrm{Ext}(\mathbb{R}^\backslash)) = \mathcal{R}(\mathrm{Conn}(\mathbb{R}^\backslash)) = \mathcal{R}(\mathrm{PC}(\mathbb{R}^\backslash)) = \backslash + \infty.$$

Clearly Theorem 4.27 implies that $\mathcal{R}(\mathcal{D}(\mathbb{R}^\backslash)) \leq \backslash + \infty$. The problem (stated in [47]) whether this equation can be replaced by the equality has been recently solved by F. Jordan.

Theorem 4.28 (F. Jordan [80]). *For every n there exists a Baire 1 class function $f\colon \mathbb{R}^n \to \mathbb{R}$ which is not a sum of n Darboux functions. In particular,*

$$\mathcal{R}(\mathcal{D}(\mathbb{R}^\backslash)) = \backslash + \infty.$$

The value of $\mathcal{R}(\mathrm{AC}(\mathbb{R}^\backslash))$ is clearly equal to 2, since Natkaniec [114] proved that $\mathrm{A}(\mathrm{AC}(\mathbb{R}^n)) > \mathfrak{c}$. This fact has been recently improved by F. Jordan, who proved

Theorem 4.29 (F. Jordan [80]). *For every $n \geq 1$*

$$\mathrm{A}(\mathrm{AC}(\mathbb{R}^n)) = \mathrm{A}(\mathrm{AC}) = \mathfrak{e}_{\mathfrak{c}}.$$

In [80] F. Jordan considers also the following version of additivity function for the classes $\mathcal{F}$ of functions from $\mathbb{R}^n$ into $\mathbb{R}$

$$\mathrm{GA}_{n,k}(\mathcal{F}) = \min\left(\left\{|G|\colon G \subseteq \mathbb{R}^{\mathbb{R}^n} \,\&\, \Psi_{n,k}(G) \text{ holds}\right\} \cup \left\{(2^{\mathfrak{c}})^+\right\}\right)$$

where $k\mathcal{F} = \{f_0 + \cdots + f_{k-1}\colon f_i \in \mathcal{F}\}$ and $\Psi_{n,k}(G)$ denotes the statement

$$(\forall f \in (n-k)\mathcal{F})(\exists g \in G)(g - f \notin (k+1)\mathcal{F}).$$

This function makes a good generalization of both functions A and $\mathcal{R}$ since

$$\mathcal{R}(\mathcal{F}) = \backslash + \infty \text{ if and only if } \infty < \mathrm{GA}_{\backslash,\|}(\mathcal{F}) < (\in^{\mathfrak{c}})^{+} \text{ for some } \| < \backslash,$$

and

$$\mathrm{GA}_{1,0}(\mathcal{F}) + 1 = \mathrm{A}(\mathcal{F}) \quad \text{for any } \mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}} \text{ such that } \mathcal{F} = \{-f \colon f \in \mathcal{F}\}.$$

Thus, the following theorem generalizes Theorems 4.7(a), 4.10(1), 4.27, and 4.28.

Theorem 4.30 (F. Jordan [80]). *For every $n \geq 1$*

(1) $\mathrm{GA}_{n,n-1}(\mathrm{Ext}(\mathbb{R}^n)) = \mathrm{GA}_{n,n-1}(\mathrm{Conn}(\mathbb{R}^n)) = \mathrm{GA}_{n,n-1}(\mathrm{PC}(\mathbb{R}^n)) = \mathfrak{c}^+$;
(2) $\mathrm{GA}_{n,n-1}(\mathcal{D}(\mathbb{R}^n)) = \mathfrak{e}_{\mathfrak{c}}$;
(3) $\mathrm{GA}_{n,j}(\mathcal{D}(\mathbb{R}^n)) = \mathrm{A}((j+1)\mathcal{D}(\mathbb{R}^n)) = \mathfrak{c}^+$ *for any $j < n-1$ such that $2j \geq n-1$.*

Notice also, that in the language of $\mathcal{R}$ operator the results from Theorem 4.26 and Corollary 4.25(2) can be expressed by the equation

$$\mathcal{R}_{\lfloor}(\mathrm{Ext}) = \mathcal{R}_{\lfloor}(\mathrm{SCIVP}) = \mathcal{R}_{\lfloor}(\mathrm{CIVP}) = \mathcal{R}_{\lfloor}(\mathrm{PR}) = \ni,$$

where $\mathcal{R}_{\lfloor}$ is the natural generalization of $\mathcal{R}$ for the class of bounded functions.

5. Some elements of topology

Let X and Y be arbitrary sets. For arbitrary families $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$, where $\mathcal{P}(Z)$ stands for the collection of all subsets of a set Z, define

$$\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \{f \in Y^X \colon f^{-1}(B) \in \mathcal{A} \text{ for every } B \in \mathcal{B}\}$$

and

$$\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in Y^X \colon f[A] \in \mathcal{B} \text{ for every } A \in \mathcal{A}\}.$$

If families $\mathcal{A}$ and $\mathcal{B}$ are the topologies on X and Y, respectively, then $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ is a well known object: the class of all continuous functions from $\langle X, \mathcal{A}\rangle$ to $\langle Y, \mathcal{B}\rangle$. Similarly a class of measurable functions with respect to an algebra $\mathcal{A}$ of subsets of X is equal to $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, where $\mathcal{B}$ is an appropriate topology on Y.

In both these approaches one starts with families of sets $\mathcal{A}$ and $\mathcal{B}$ and obtain, in return, a family of functions. But what if a class of functions $\mathcal{F} \subset Y^X$ is given to begin with? When can we find families $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$ such that $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ or $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$? And how nice can these families be, if they exist?

These questions have been studied recently by several authors. To talk about their results, let us fix the following terminologies. We say that a family $\mathcal{F} \subset Y^X$ can be

- *characterized by images of sets* if $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ for some families $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$;
- *characterized by preimages of sets* if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some families $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{P}(Y)$;
- *topologized* if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some topologies $\mathcal{A}$ on X and $\mathcal{B}$ on Y;
- *characterized by associated sets* if $Y = \mathbb{R}$ and $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some family $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B} = \{(a,\infty)\colon a \in \mathbb{R}\} \cup \{(-\infty,b)\colon b \in \mathbb{R}\}$.

From all these notions only the problem of characterizing by associated sets has been extensively studied. Clearly, all classes of continuous function $\mathcal{C}(X)$ from a topological space X into $\mathbb{R}$ (considered with the natural topology) can be characterized by associated sets. So can be the family of $\mathcal{B}$-measurable functions from X into $\mathbb{R}$, for any σ-algebra $\mathcal{B}$ of subsets of X. However, there are also many examples of classes of functions that do not admit such a characterization. In fact, the real interest in the characterizations of functions by associated sets has been initiated by the 1950 paper of Zahorski [155], in which he tried to characterize derivatives (from $\mathbb{R}$ to $\mathbb{R}$) in that way.[9] Today we know that derivatives cannot be characterized by associated sets: any class $\mathcal{F}$ that can be characterized that way has the property that $h \circ f \in \mathcal{F}$ for every $f \in \mathcal{F}$ and every homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$; however derivatives do not have this property. (See Bruckner's book [16] on this subject. Compare also [17].) This negative result has been followed by several others, in which the authors prove that the following classes of functions (from $\mathbb{R}$ to $\mathbb{R}$) cannot be characterized by associated sets: $\mathcal{D}$ (Bruckner [15, 1967]), Conn (B. Cristian, I. Tevy [50, 1980]), AC (Kellum [86, 1982]), Ext (Rosen [126, 1996]) and the remaining classes from Chart 2 (Ciesielski, Natkaniec [42, 1997]).

The question about topologizing different classes of real functions has been first systematically studied in early 1990's by Ciesielski in [27].[10] He starts with the following theorem listing basic properties of classes that can be topologized. In the theorem $\mathbb{C}$ stands for the set of complex numbers, $\mathcal{L}$ for the class of linear functions $f(x) = ax + b$, $\mathcal{T}_{\mathcal{O}}$ for the natural topology on $\mathbb{R}$, and id_X for the identity function from X to X.

[9] See also 1969 paper of Mrówka [113] on characterizing functions by associated sets.

[10] According to [123] already in a 1988 manuscript [150] Tartaglia proved that the class of all derivatives cannot be topologized.

Theorem 5.1 (Ciesielski [27]). *Let τ_1 and τ_2 be the topologies on sets X and Y, respectively, and let $\mathcal{F} = \mathcal{C}^{-1}_{\tau_1,\tau_2} \neq Y^X$. If τ is the weak topology on X generated by $\mathcal{F}$, that means, generated by the family $\{f^{-1}(U)\colon U \in \tau_2, f \in \mathcal{F}\}$, then*

(i) Const $\subset \mathcal{F}$, $\tau \subset \tau_1$, $\tau_1 \neq \mathcal{P}(X)$, $\tau_2 \neq \{\emptyset, Y\}$ *and* $\mathcal{F} = \mathcal{C}^{-1}_{\tau,\tau_2}$;

(ii) *if* $X = Y$ *and* $\mathrm{id}_X \in \mathcal{F}$ *then* $\tau_2 \subset \tau \subset \tau_1$;

(iii) *if* $Y = \mathbb{R}$ *and* $\mathcal{T}_{\mathcal{O}} \subset \tau_2$ *then* $\mathcal{F}$ *is closed under the maximum and minimum operations;*

(iv) *if* $\mathcal{G} \subset Y^Y$ *is such that* $\mathrm{id}_Y \in \mathcal{G}$ *and* $g \circ f \in \mathcal{F}$ *for all* $f \in \mathcal{F}$ *and* $g \in \mathcal{G}$ *then* $\mathcal{F} = \mathcal{C}^{-1}_{\tau,\tau'}$, *where* τ' *is a topology generated by* $\{g^{-1}(U)\colon U \in \tau_2, g \in \mathcal{G}\}$;
in particular, if $\mathcal{G} = \mathcal{L}$ *than we may assume that* τ_2 *is a homothetically closed* T_1 *topology;*

(v) *if* $X = Y$, $\mathrm{id}_X \in \mathcal{F}$ *and* $\mathcal{F}$ *is closed under the composition, then* $\mathcal{F} = \mathcal{C}^{-1}_{\tau,\tau}$;

(vi) *if* $X = Y \in \{\mathbb{R}, \mathbb{C}\}$ *and* $\mathcal{L} \subset \mathcal{F}$ *then* τ_1 *is a* T_1 *topology;*

(vii) *if* $X = Y \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{L} \subset \mathcal{F}$ *and* τ_2 *contains two nonempty disjoint sets, then* τ_1 *is Hausdorff;*

(viii) *if* $X = Y = \mathbb{R}$ *and every* $f \in \mathcal{F}$ *is Darboux then* τ_1 *is connected;*

(ix) *if* $X = Y = \mathbb{R}$, $\mathcal{L} \subset \mathcal{F}$ *and* τ_2 *contains a non-empty set which has either upper or lower bound, then* $\mathcal{T}_{\mathcal{O}} \subset \tau_1$.

Of all these properties only (iii) needs a little longer (but still easy) argument. Note also, that (i) shows, that in order to topologize some family, only the search for the range topology is essential. Condition (v) shows that the question when topologies τ_1 and τ_2 can be chosen equal is answered by the following corollary.

Corollary 5.2 (Ciesielski [27]). *Let $\mathcal{F} \subset X^X$. If $\mathcal{F}$ can be topologized then $\mathcal{F} = \mathcal{C}_{\tau,\tau}$ for some topology τ on X if and only if $\mathrm{id}_X \in \mathcal{F}$ and $\mathcal{F}$ if closed under the composition operation.*

Next, from Theorem 5.1 (conditions (iii), (vi) and (ix)) Ciesielski concludes the following fact

Theorem 5.3 (Ciesielski [27]). *Let $\mathcal{F}$ be a family of real functions closed under composition and such that $\mathcal{C}^\infty \subset \mathcal{F}$. If $\mathcal{F}$ can be topologized then $\mathcal{F}$ is closed under the maximum and minimum operations.*

which easily leads to the following corollary:

Corollary 5.4 (Ciesielski [27]). *The classes: $\mathcal{C}^\infty$ of infinitely many times differentiable functions, $\mathcal{D}^n$ of n-times differentiable functions, and $\mathcal{C}^n$ of functions with continuous n-th derivative cannot be topologized. The same* *is true, when in the above we replace differentiability with symmetric differentiability, approximate differentiability, symmetric approximate differentiability, $\mathcal{I}$-approximate differentiability or symmetric $\mathcal{I}$-approximate differentiability.*

(The definitions of all classes of functions from this, and the next corollary can be found in [16] and in [38].)

With a little more effort he also concludes

Corollary 5.5 (Ciesielski [27]). *The following classes cannot be topologized: the class Δ of all derivatives,*[11] *the Zahorski's classes $\mathcal{M}_i$ for $i = 0, 1, 2, 3, 4$, the class of all symmetrically (symmetrically approximately or symmetrically $\mathcal{I}$-approximately) continuous functions, the class of all Darboux functions, the class of all measurable functions and the class of all functions having the Baire property.*

From the positive side, paper [27] contains the following deeper result.

Theorem 5.6 (Ciesielski [27]). *Let $|X| = |Y| = \mathfrak{c}$, $\mathcal{R} \subset \mathcal{Y}^{\mathcal{X}}$ be of cardinality $\leq \mathfrak{c}$ and let $\mathcal{I}$ be a proper σ-ideal on X containing all singletons.*

(A) *If GCH holds then there is a Hausdorff, connected and locally connected topology τ_2 on Y with the property that for every family $\mathcal{F} \subset \mathrm{Const} \cup \mathcal{R}$ such that* Const $\subset \mathcal{F}$ *and*

$$\{x \in X \colon f(x) = g(x)\} \in \mathcal{I} \quad \textit{for every distinct} \quad f, g \in \mathcal{F}$$

we have

$$\mathcal{F} = \mathcal{C}^{-1}_{\tau,\tau_2},$$

where τ is generated by the family $\{f^{-1}(U) \colon U \in \tau_2 \ \&\ f \in \mathcal{F}\}$. Topology τ is connected and locally connected. It is also Hausdorff, provided $\mathcal{F}$ separates points.

(B) *Moreover, it is consistent with the set theory ZFC+GCH that the topologies τ and τ_2 are completely regular and Baire.*

Applying Theorem 5.6 to the σ-ideal $\mathcal{M}$ of the first category subsets of $\mathbb{R}^n$, and using the fact that for any different harmonic functions $f, g \colon \mathbb{R}^n \to \mathbb{R}^m$ we have $\mathrm{int}_{\tau_{\mathcal{O}}}(\{x \in X \colon f(x) = g(x)\}) = \emptyset$ we can conclude that the class of all harmonic functions $f \colon \mathbb{R}^n \to \mathbb{R}^m$ can be topologized.

Another σ-ideal that can be used with Theorem 5.6 is the ideal $\mathcal{I}_\omega$ of at most countable sets. Since for any two different analytic functions $f, g \in \mathcal{A}$ we have $\{x \colon f(x) = g(x)\} \in \mathcal{I}_\omega$, we can also conclude the following corollary.

[11] See also [150].

Corollary 5.7 (Ciesielski [27]). *If GCH holds then there is a Hausdorff, connected and locally connected topology $\tau_{\mathcal{A}}$ (on $\mathbb{R}$ or $\mathbb{C}$) such that for any family $\mathcal{F} \supset \mathrm{Const}$ of analytic functions we have*

$$\mathcal{F} = \mathcal{C}^{-1}_{\tau_{\mathcal{F}},\tau_{\mathcal{A}}},$$

where $\tau_{\mathcal{F}}$ is generated by the family $\{f^{-1}(U)\colon U \in \tau_{\mathcal{A}} \ \&\ f \in \mathcal{F}\}$. Moreover, $\tau_{\mathcal{F}}$ is connected and locally connected, and it is Hausdorff provided $\mathcal{F}$ separates points.

It is also consistent with ZFC+GCH that all these topologies are completely regular and Baire.

Notice also, that if the family $\mathcal{F}$ in Corollary 5.7 is closed under the composition and $\mathrm{id} \in \mathcal{F}$, then, by Theorem 5.1(v), $\mathcal{F} = \mathcal{C}^{-1}_{\tau_{\mathcal{F}},\tau_{\mathcal{F}}}$. We can write this in the form of next corollary, where $\mathcal{A}$ stands for the family of all analytic functions and $\mathcal{P}$ for the family of all polynomials.

Corollary 5.8 (Ciesielski [27]). *If GCH holds and $\mathcal{F}$ is a family of real functions which is closed under the composition and such that $\{\mathrm{id}\}\cup\mathrm{Const} \subset \mathcal{F} \subset \mathcal{A}$, then there exists a Hausdorff, connected and locally connected topology $\mathcal{T}_{\mathcal{F}}$ (on $\mathbb{R}$ or $\mathbb{C}$) such that $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{T}_{\mathcal{F}},\mathcal{T}_{\mathcal{F}}}$. In particular, there exist a "linear topology" $\mathcal{T}_{\mathcal{L}}$, a "polynomial topology" $\mathcal{T}_{\mathcal{P}}$ and an "analytic topology" $\mathcal{T}_{\mathcal{A}}$ which are Hausdorff, connected and locally connected such that $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{\mathcal{P}} \subset \mathcal{T}_{\mathcal{A}}$ and for which $\mathcal{L} = \mathcal{C}^{-1}_{\mathcal{T}_{\mathcal{L}},\mathcal{T}_{\mathcal{L}}}$, $\mathcal{P} = \mathcal{C}^{-1}_{\mathcal{T}_{\mathcal{P}},\mathcal{T}_{\mathcal{P}}}$, and $\mathcal{A} = \mathcal{C}^{-1}_{\mathcal{T}_{\mathcal{A}},\mathcal{T}_{\mathcal{A}}}$.*

Moreover, it is consistent with ZFC+GCH that all these topologies are completely regular and Baire.

The following questions in these subject are open.

Problem 10 (Ciesielski [27]).

(1) Can we prove Theorem 5.6 or any of the Corollaries 5.7, or 5.8 without any additional set-theoretical assumptions?
(2) Can topologies from Theorem 5.6 or any of the Corollaries 5.7, or 5.8 be normal? Lindelöf? hereditarily Lindelöf? compact? metrizable?

The general problem of characterizing classes of functions by preimages of sets (in a sense defined above) has been studied only in two papers: [33] and [42]. In paper [33] Ciesielski proves the following theorem, which generalizes a similar result of Preiss and Tartaglia [123].

Theorem 5.9 (Ciesielski [33]). *Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a family of cardinality less than or equal to $\mathfrak{c}^+$ and let $\mathcal{G} \subset \mathbb{R}^{\mathbb{R}}$ be such that*

(1) *$\mathcal{G}$ contains all constant functions;*
(2) *$|\mathcal{G}| \le \mathfrak{c}$; and,*
(3) *$|f[\mathbb{R}]| = \mathfrak{c}$ for any non-constant $f \in \mathbb{R}^{\mathbb{R}}$ which is a difference of two functions from $\mathcal{G}$.*

Then there exists a family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ of cardinality less than or equal to $|\mathcal{F}|$ such that

$$\mathcal{G} \cap \mathcal{F} = \mathcal{C}_{\mathcal{D},\mathcal{A}} \cap \mathcal{F}$$

where $\mathcal{D} = \{f^{-1}(A)\colon A \in \mathcal{A} \ \&\ f \in \mathcal{G}\}$.

Clearly the family $\mathcal{G} = \Delta$ of all derivatives satisfies the above conditions (1)–(3). Thus, using the theorem with $\mathcal{G} = \Delta$ and $\mathcal{F}$ equal to the family $\mathcal{B}\mathrm{or}$ of Borel functions we obtain the following corollary.

Corollary 5.10. *There exists a family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ such that $|\mathcal{A}| \le \mathfrak{c}$ and*

$$\Delta = \mathcal{B}\mathrm{or} \cap \mathcal{C}(\mathcal{D}, \mathcal{A}),$$

where $\mathcal{D} = \{f^{-1}(A)\colon f \in \Delta \ \&\ A \in \mathcal{A}\}$.

However, the following stronger characterization of Δ is also described in [33], where $\mathcal{DB}_1$ stands for the class of Darboux Baire one functions.

Theorem 5.11. *There exists a Bernstein set $B \subset \mathbb{R}$ such that*

$$\Delta = \mathcal{DB}_1 \cap \mathcal{C}(\mathcal{D}_0, \{B + c\colon c \in \mathbb{R}\}) = \mathcal{C}(\mathcal{D}, \mathcal{A}),$$

where $\mathcal{A} = \bigcup_{c\in\mathbb{R}}\{(-\infty, c), (c, \infty), B + c\}$, $\mathcal{D} = \{f^{-1}(A)\colon f \in \Delta \ \&\ A \in \mathcal{A}\}$, and $\mathcal{D}_0 = \{f^{-1}(B + c)\colon f \in \Delta \ \&\ c \in \mathbb{R}\}$.

Note that by Corollary 5.5 the families $\mathcal{D}$ and $\mathcal{A}$ in Theorem 5.11 cannot be topologies. Also, they cannot be algebras:

Theorem 5.12 (Ciesielski [33]). *If $\Delta = \mathcal{C}^{-1}_{\mathcal{D},\mathcal{A}}$ for some families $\mathcal{D}$ and $\mathcal{A}$ of subsets of $\mathbb{R}$ then neither $\mathcal{A}$ nor $\mathcal{D}$ contain simultaneously a non-empty proper subset S of $\mathbb{R}$ and its complement $\mathbb{R} \setminus S$.*

In particular, neither $\mathcal{A}$ nor $\mathcal{D}$ is an algebra.

The following problem remains open.

Problem 11 (Ciesielski [33]). Can the family $\mathcal{A}$ in Corollary 5.10 or Theorem 5.11 consist of any kind of regular sets like Lebesgue measurable, Borel, or sets with Baire property?

An interesting discussion concerning characterizations of the derivatives can be also found in a recent article of Freiling [63].

The problem of characterizing by preimages of sets families from Chart 2 has been recently addressed by Ciesielski and Natkaniec.

Theorem 5.13 (Ciesielski, Natkaniec [42]).

(1) *The classes:* SZ, Ext, AC, Conn, $\mathcal{D}$, SCIVP, CIVP, *and* WCIVP *cannot be characterized by preimages of sets.*
(2) *The classes:* PR *and* PC *can be characterized by preimages of sets as* $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ *with* $\mathcal{B}$ *being the natural topology on* $\mathbb{R}$. *However, they can neither be topologized nor be characterized by associated sets.*

The problem of characterizing a family of functions by images of sets was first studied by Velleman for the class $\mathcal{C}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 5.14 (Velleman [153]).

(1) $\mathcal{C} = \mathcal{C}_{\mathcal{A},\mathcal{A}} \cap \mathcal{C}_{\mathcal{B},\mathcal{B}}$, *where* $\mathcal{A}$ *is the family of all connected subsets of* $\mathbb{R}$ *and* $\mathcal{B}$ *the family of all compact subsets of* $\mathbb{R}$.
(2) $\mathcal{C}$ *cannot be characterized by images of sets.*

Note that a family $\mathcal{C}_{\mathcal{A},\mathcal{A}}$ from Theorem 5.14 is just the family $\mathcal{D}$ of Darboux functions.

Theorem 5.14(1) has been recently generalized by Arenas and Puertas [2]. Theorem 5.14(2) has been essentially generalized by Ciesielski, Dikranjan and Watson in [34]. In this paper the authors list a basic properties of classes that can be characterized by images of sets, which is similar in flavor to Theorem 5.1. Then, they prove the following generalization of Theorem 5.14.

Theorem 5.15 (Ciesielski, Dikranjan, Watson [34]). *For a Tychonoff topological space* X *the class* $\mathcal{F} = \mathcal{C}(X, \mathbb{R})$ *of all continuous functions from* X *to* $\mathbb{R}$ *can be characterized by images of sets if and only if* X *is a discrete space.*

They also remarked that there is a compact subset $K \subset \mathbb{R}^2$, a Cook continuum, for which $\mathcal{C}(K, K) = \mathrm{Const} \cup \{\mathrm{id}_K\}$, and so, it can be characterized by images of sets.

For the classes of functions from $\mathbb{R}$ to $\mathbb{R}$, their generalization of Theorem 5.14 appears as follows.

Theorem 5.16 (Ciesielski, Dikranjan, Watson [34]). *If* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ *are such that* $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *then there is a non-measurable function* $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

This, in particular, implies the following corollary.

Corollary 5.17 (Ciesielski, Dikranjan, Watson [34]). *Neither of the following classes of functions from $\mathbb{R}$ to $\mathbb{R}$ can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$:*

- *the class of upper or lower semicontinuous functions;*
- *the class Δ of derivatives;*
- *the class of approximately continuous functions;*
- *the class of Baire class 1 functions;*
- *the class of Borel functions;*
- *the class of measurable functions.*

They also noticed that the class $\mathcal{D}$ of Darboux functions can be characterized by images of sets. (It is defined that way.)

It has been also recently noticed by Ciesielski and Natkaniec [42] that in Theorem 5.16 the clause "non-measurable" cannot be replaced by "without the Baire property." More precisely, they proved

Theorem 5.18 (Ciesielski, Natkaniec [42]). *Let $f \in \mathcal{C}_{\mathcal{A},\mathcal{A}}$, where*

$$\mathcal{A} = \{D \cap I \colon D \text{ is dense in } \mathbb{R} \text{ and } I \subset \mathbb{R} \text{ is an interval } \}.$$

Then f is continuous on a dense set.

In particular f has the Baire property and

$$\mathcal{C} \subset \mathcal{C}_{\mathcal{A},\mathcal{A}} \subset \mathcal{B}aire,$$

where $\mathcal{B}aire$ stands for the class of functions $g \colon \mathbb{R} \to \mathbb{R}$ with the Baire property.

Finally, Ciesielski and Natkaniec [42] proved that it is impossible to characterize by images of sets the classes SZ, and $\mathcal{B}aire$ of functions (from $\mathbb{R}$ to $\mathbb{R}$) with the Baire property. They also proved the following theorem.

Theorem 5.19 (Ciesielski, Natkaniec [42]). *The classes:* CIVP, WCIVP, *and $\mathcal{D}$ can be characterized by images of sets. However, the remaining classes from Chart 2 cannot be represented that way.*

The following problem in this area remains open.

Problem 12 (Ciesielski, Dikranjan, Watson [34]). *Can any of the following classes of real functions be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$?*

- *The class of all linear functions $f(x) = ax + b$.*
- *The class of all polynomials.*
- *The class of all real analytic functions.*
- *The class C^∞ of infinitely many times differentiable functions.*
- *The class D^n of n-times differentiable functions, with $1 \le n < \omega$.*

Another interesting problem (loosely related to real functions, but having the same flavor that the topologizing question has) concerns the existence of a topology on a given set X, often the real line, satisfying the best possible separation axioms, for which a given ideal (σ-ideal) of subsets of X consists precisely of sets that are nowhere dense (or first category) in X. Ciesielski and Jasinski [35, 1995] obtained several positive results in this direction under some additional set-theoretic assumptions. The problem was also investigated in the papers [125] by Rogowska and [5] by Balcerzak and Rogowska.

There are also many interesting theorems concerning different classes of functions $f\colon \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is equipped with some abstract topology refining of the natural topology. A survey of some recent results in this direction can be found in the last issue of the Real Analysis Exchange [73]. The topologies on $\mathbb{R}$ that were most studied in this aspect in recent years are the $\mathcal{I}$-density topology (defined in 1982 by Wilczyński [154]) and the deep $\mathcal{I}$-density topologies (defined in 1986 independently by azarow [101], and by Poreda and Wagner-Bojakowska [122]). These are category analogues of the density topology. The survey of the results in this direction can be found in a monograph of Ciesielski, Larson and Ostaszewski [38]. (In particular, see [36] or [38, Sec. 1.5] for some set theoretic results and open problems concerning these topologies.)

6. Elements of measure theory

The Lebesgue measure, being a function from family of subsets $\mathcal{L}$ of $\mathbb{R}^n$ into $[0,\infty]$, is not of the form $f\colon \mathbb{R}^n \to \mathbb{R}$, so it does not lie directly in a scope of this article. However, it is certainly one of the main tools of real analysis and many results concerning its generalizations have a deep set theoretical context. Therefore, a short section concerning the newest developments in this area has been added to this paper.

An accessible survey concerning different extensions of Lebesgue measure can be found in the 1989 Mathematical Intelligencer article [23] of K. Ciesielski. The best survey concerning universal (i.e., defined on $\mathcal{P}(\mathbb{R})$)

countable additive extensions of Lebesgue measure can be found in the 1993 survey article of D. H. Fremlin [66]. Thus, we will concentrate here only on the newest developments, that concern isometrically invariant extensions of Lebesgue measure. (See also M. Laczkovich survey article [100] on this subject.)

Recall here, that by the 1923 theorem of Banach there is a finitely additive isometrically invariant measure $\mu\colon \mathcal{P}(\mathbb{R}^2) \to [0,\infty]$ extending Lebesgue measure, while such a measure on $\mathbb{R}^3$ does not exist by a famous Banach-Tarski Paradox (1924):

- the ball $B \subset \mathbb{R}^3$ and the cube $Q \subset \mathbb{R}^3$ (of arbitrary volumes) are isometrically equivalent, i.e., there is a finite partition $\{B_k\}_{k=1}^n$ of B and isometries $\{i_k\}_{k=1}^n$ of $\mathbb{R}^3$ such that $\{i_k[B_k]\}_{k=1}^n$ forms a partition of Q.

There were two famous problems around this subject. The first one, due to Marczewski, was whether the pieces $\{B_k\}_{k=1}^n$ in the Banach-Tarski Paradox can have the Baire property. The answer to this question, surprisingly positive, was obtained by Dougherty and Foreman in 1994.

Theorem 6.1 (Dougherty, Foreman [58])**.** *For any $n \geq 3$ any two bounded non-meager sets $B, Q \subset \mathbb{R}^n$ with the Baire property are isometrically equivalent with pieces having the Baire property.*

The second famous question was the Tarski's circle-squaring problem: *is a circle $C \subset \mathbb{R}^2$ of the unit area equivalent to a square $S \subset \mathbb{R}^2$ of the unit area?* Note that if the areas of C and D were different, then Banach's theorem of 1923 would immediately imply the negative answer. However, the answer to Tarski's circle-squaring problem is positive, as proved by Laczkovich in 1990.

Theorem 6.2 (Laczkovich [99])**.** *Any two sets $B, Q \subset \mathbb{R}^n$ having the same area and being bounded by Jordan curves are isometrically equivalent.*

The other class of isometrically invariant extensions of Lebesgue measures on $\mathbb{R}^n$ concerns countably additive extensions. In 1936 Sierpiński asked, whether such an extension can be maximal. The negative answer to this question was given in 1977 by A. B. Kharazishvili [87] (for $n = 1$) and in 1985 by Ciesielski and Pelc [43] (for arbitrary n). (Compare also [24, 25, 96, 156].)

Theorem 6.3 (Ciesielski, Pelc [43])**.** *For every $n \geq 1$ and every countably additive isomatrically invariant extension $\mu\colon \mathcal{M} \to [0,+\infty]$ of the Lebesgue measure on $\mathbb{R}^n$ there exists a proper countably additive isomatrically invariant extension $\mu'\colon \mathcal{M}' \to [0,+\infty]$ of μ.*

A weak side of this theorem was that the extension μ' of μ was only by (new) measure zero sets so, in a way, trivial. This has been recently improved by Zakrzewski, who showed

Theorem 6.4 (Zakrzewski [157]). *For every $n \geq 1$ and every countably additive isomatrically invariant extension $\mu\colon \mathcal{M} \to [0,+\infty]$ of the Lebesgue measure on $\mathbb{R}^n$ there exists a countably additive isomatrically invariant extension $\mu'\colon \mathcal{M}' \to [0,+\infty]$ of μ such that the canonical embedding $\mathrm{e}\colon \mathcal{M}/\mu \to \mathcal{M}'/\mu'$ of measure algebras defined by $\mathrm{e}([A]_\mu) = [A]_{\mu'}$ is not surjective.*

Zakrzewski's proof is based on a construction of Kharazishvili [88] from 1997, which was known earlier to imply Theorem 6.4 for $n = 1$.

Many interesting results concerning invariant extensions of Lebesgue measure can be also found in 1983 book of Kharazishvili [89].

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