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A Distance-Regular Graph with Intersection Array (5, 4, 3, 3; 1, 1, 1, 2) Does Not Exist

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Abstract. We prove that a distance-regular graph with intersection array (5, 4, 3, 3; 1, 1, 1, 2) does not exist. The proof is purely combinatorial and computer-free.

Keywords: distance-regular graph, intersection array

The problem of deciding whether a distance-regular graph with a given intersection array does exist is a very hard one, and far from solution. There are some necessary conditions ruling out unfeasible arrays, but vast numbers of arrays are left, for which there are no proofs either of existence or of nonexistence of corresponding graphs.

In a previous paper [2] we proved by a rather simple ad hoc argument that a distance-regular graph with the intersection array (5, 4, 3; 1, 1, 2) does not exist. The approach used there turned out to be applicable in yet another situation, namely, for the intersection array mentioned in the title.

The excellent monograph [1] may be recommended as an introduction to the theory of distance-regular graphs. But in this paper we shall need only the basic definition.

Definition. A distance-regular graph with the intersection array $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$ is a b_0-regular graph G of diameter d such that for every pair u, v of its vertices the following condition is satisfied:

> if $d(u, v) = i$, then exactly b_i neighbors of v are at a distance $i + 1$ from u; and exactly c_i ones are at a distance $i - 1$ from u.

THEOREM. *A distance-regular graph with intersection array* (5, 4, 3, 3; 1, 1, 1, 2) *does not exist.*

The remainder of the paper is devoted to the proof of this theorem. From now on, we suppose that G is a distance-regular graph with the specified intersection array.

We shall use the following notations:

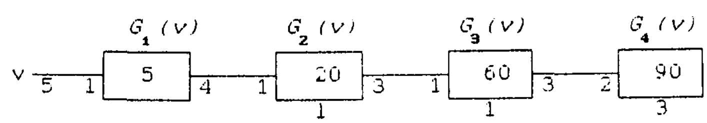
$d(x, y)$ = distance between vertices x and y in the graph G.
$G_i(v)$ = the set of vertices at a distance i from v.
$G_{\leq i}(v)$ = the set of vertices at a distance at most i from v.
An m-cycle or m-path is a cycle or path of length m.

Some of basic properties of G are collected in Lemma 1.

LEMMA 1.

(a) *For each vertex v, there are exactly* 5, 20, 60, *and* 90 *vertices at distance* 1, 2, 3, *and* 4 *from v.*
(b) *G induces matchings on $G_2(v)$ and on $G_3(v)$.*
(c) *Any two vertices at a distance* 2 *or* 3 *are connected by exactly one path of length 3.*
(d) *There are no cycles of length* 3, 4, 6, *or* 7.
(e) *Any two vertices at a distance* 4 *are connected by exactly two 4-paths.*

Proof. The diagram below represents distribution of vertices and edges of G relatively to a vertex v.



The properties (a), (b), (c), and (e) follow immediately from the diagram; the absence of 3-, 4-, and 6-cycles is also obvious. Let $vv_1 v_2 v_3 v_4 v_5 v_6$ be a cycle of length 7. It is clear that the vertices v_3 and v_4 should be in $G_3(v)$. By (c), the vertices v_1 and v_3 are connected by a unique 3-path, and, looking at the diagram, we see that it should pass through v_4 and v_5. So, $v_6 = v_1$, and there are no 7-cycles in G. □

Next we shall determine uniquely the structure of the graph induced on $G_{\leq 3}(v)$ for every vertex v.

Let $I = \{1, 2, 3, 4, 5\}$. From now on, whenever symbols i, j, k, l occur in a statement, they represent distinct elements of I.

LEMMA 2. *For every vertex $o \in G$, vertices of the graph $N = G_{\leq 3}(o)$ can be labelled by the following symbols:*

$G_1(o) = I = \{1, 2, 3, 4, 5\}$;
$G_2(o) = \{ij \mid i \neq j,\ i, j \in I\}$;
$G_3(o) = \{ijk \mid i, j, k$ *are different elements of* $I\}$;

so that all edges of N *are as follows:*

$$o - i;\ i - ij;\ ij - ji;\ ij - ijk;\ ijk - ikj.$$

Proof. Let o be any vertex of G. Label its neighbors arbitrarily by the symbols 1, 2, 3, 4, 5. Any two of them, say i and j, are at a distance 2, and so should be connected by a unique 3-path $i - x - y - j$. The vertices x and y are in $G_2(o)$. Label them by ij and ji, respectively. Each vertex of $G_2(o)$ gets at most one label, and since there are 20 vertices and 20 labels, we have labels for all vertices of $G_2(o)$. All edges inside $G_2(o)$ are also defined; they are of the form $ij - ji$. Similarly, there exists a unique 3-path $ij - x - y - ik$ for any choice of different i, j, k. Now x and y are in $G_3(o)$, and we assign to them labels ijk and ikj, respectively. Again, we have 60 labels and 60 vertices. The lemma follows. □

Now we turn to vertices in $G_4(o)$. Each one of them is adjacent to two vertices in $G_3(o)$, and is uniquely determined by them (since there are no 4-cycles). Let $abc : def$ denote the vertex in $G_4(o)$ adjacent to abc and def (if it exists). Note that $def : abc$ is another label for the same vertex.

By a pattern we shall mean a label in which some digits are replaced by the symbol $\star$. A pattern defines the set of vertices, labels of which are obtained by replacing all stars by some digits.

LEMMA 3. *Of* 90 *vertices in* $G_4(o)$, *there are* 60 *having labels of the pattern* $ij\star : kl\star$, *where* i, j, k, l *are all different; and 30 with labels of the pattern* $ij\star : ji\star$.
Each of the patterns $ijk : ji\star$, $ijk : l\star\star$, $ij\star : kl\star$ *defines exactly one vertex.*

Proof. Let us consider the vertices ijk and ji. We have $d(ijk, ji) = 2$, so they are joined by a unique 3-path. It should look like $ji - ji\star - v - ijk$ for some vertex $v \in G_4(o)$. This vertex will be the only one defined by the pattern $ijk : ji\star$.

Similarly, we have $d(ijk, l) = 4$. One of the two 4-paths between them is $ijk - ij - i - o - l$; the other should be

$$ijk - v - l\star\star - l\star - l$$

for some $v \in G_4(o)$; and v is the only vertex defined by the pattern $ijk : l\star\star$.

Considering, at last, vertices ij and kl, we find in the same way a unique vertex corresponding to $ij\star : kl\star$.

So, we have $(5 \cdot 4 \cdot 3 \cdot 2)/2 = 60$ vertices with labels of the pattern $ij\star : kl\star$, and $(5 \cdot 4 \cdot 3)/2 = 30$ vertices with labels of the pattern $ij\star : ji\star$. Since $60 + 30 = 90$, we have no other vertices, and the lemma is proved. □

We will call vertices $ij\star : kl\star$ *good.* (The cause for this will be seen later). Every vertex in $G_3(o)$ has exactly two good neighbors.

LEMMA 4. *If G contains a vertex $axy : bzt$, then it cannot contain any of the vertices (a) $ayx : btz$; (b) $ayx : bzt'$; and (c) $axy : zbt'$.*

Proof. Let $v = axy : bzt$, and w be one of the vertices (a), (b), (c). Then the following cycles contradict Lemma 1(d):

(a) $v - axy - ayx - w - btz - bzt - v$;
(b) $v - axy - ayx - w - bzt' - bz - bzt - v$;
(c) $v - axy - w - zbt' - zb - bz - bzt - v$. □

LEMMA 5. *The pattern $i\star\star : j\star\star$ defines exactly six good vertices. They have labels*

$$iab : jba,\ iac : jca,\ ibc : jcb,$$
$$iba : jac,\ ica : jbc,\ icb : jab$$

for some uniquely determined permutation a, b, c of the set $I' = I \backslash \{i, j\}$.

Proof. By Lemma 3, there are six vertices determined by patterns $ixy : j\star\star$ for $x, y \in I'$. Since there are 60 good vertices, and 10 unordered pairs $\{i, j\}$, each pattern $i\star\star : j\star\star$ defines six good vertices with labels $ix_sy_s : jz_st_s$, $1 \leq s \leq 6$, and any ordered pair of elements of I' is met exactly once among x_sy_s, and once among z_st_s. We define an auxiliary graph with the vertex set $S = \{1, 2, 3, 4, 5, 6\}$ and edges of four sorts: *blue (simple or double)* and *red (simple or double)*. The rules for drawing edges are as follows.

s and s' are joined by:

A simple blue edge, iff $x_s = y_{s'}$ and $x_{s'} = y_s$.
A double blue edge, iff $x_s = x_s$.
A simple red edge, iff $z_s = t_{s'}$ and $z_{s'} = t_s$.
A double red edge, iff $z_s = z_s$.

We claim that each pair of vertices is joined by at most one edge. Indeed, if s and s' are joined by simple blue and simple red edges, then the corresponding vertices contradict Lemma 4(a); if they are joined by one simple edge and one double edge, then they contradict Lemma 4(b); and if they are joined by two double edges, then there are two vertices of the form $ix\star : jz\star$ contradicting Lemma 3.

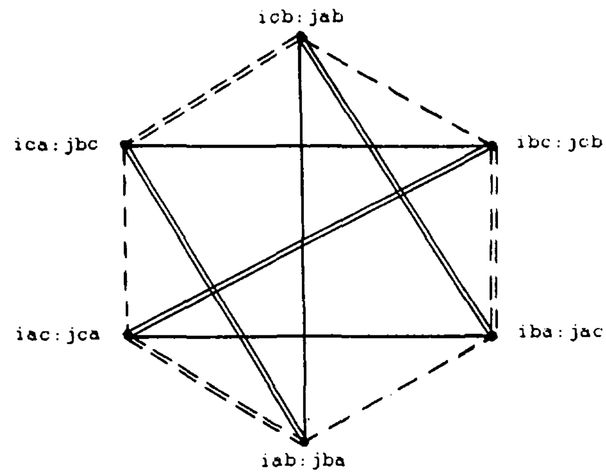


Figure 1. Vertices $i{\star}{\star} : j{\star}{\star}$.

Edges of each sort form a matching; edges of each color form a 6-cycle. It's an easy exercise to verify that, up to isomorphism, there is only one possible way to draw such a configuration; it is presented on the Figure 1. Reading out from the picture labels of the vertices (this can be done in a unique way), we get the desired result. □

We will use the shorthand notation $ij \rightarrow abc$ meaning that G contains the six vertices mentioned in Lemma 5.

To determine labels of all good vertices in $G_4(o)$ we need to define 20 relations of the form $ij \rightarrow abc$. It turns out that we can do this uniquely up to isomorphism.

From now on, a, b, c, d, e is an arbitrary permutation of elements of I.

LEMMA 6.

(a) *If* $ab \rightarrow cde$, *then* $ba \rightarrow dec$.
(b) *If* $ab \rightarrow cde$, *then* $ab' \rightarrow c'd'e$ *cannot hold for* $b' \neq b$.

Proof.

(a) These two relations define the same set of vertices.
(b) If this were the case, then G would contain vertices $aex : bb'{\star}$ and $aex : b'b{\star}$, contrary to Lemma 4(c). □

LEMMA 7. *The graph* G *induces a matching on good vertices of the pattern* $ab{\star} : {\star}{\star}{\star}$.

Proof. Consider the set $G_2(ab)$. It consists of 20 vertices: o, b, ax, axb, bax, $bax : ab\star$ and six good vertices $ab\star : \star\star\star$ (here x is an arbitrary element of $I\backslash\{a, b\}$). By Lemma 1(b) a matching is induced on them. But the first 14 of them already form a matching; it follows that so do the rest. □

LEMMA 8. *Let* $ab \to cde$. *Then*

(a) *The matching induced on the vertices* $ae\star : \star\star\star$ *is the following:*
$ae\star : cb\star \ - \ ae\star : bd\star$,
$ae\star : cd\star \ - \ ae\star : db\star$,
$ae\star : dc\star \ - \ ae\star : bc\star$.
(b) *Either* $ad \to ecb$, *or* $ad \to ebc$.

Proof. Consider the set X of all good vertices of the form $ae\star : \star\star\star$. Its elements are

$$v_1 = aex_1 : cby_1,\ v_2 = aex_2 : cdy_2,$$
$$v_3 = aex_3 : dby_3,\ v_4 = aex_4 : dcy_4,$$
$$v_5 = aed : bcd, \quad v_6 = aec : bde.$$

The labels of last two vertices are given by $ab \to cde$.

Let's define values of x_i. By Lemma 3, $\{x_1, x_2\} = \{b, d\}$ and $\{x_3, x_4\} = \{b, c\}$. By Lemma 4(c), $x_1 \neq d$ and $x_3 \neq c$. So, $x_1 = b$, $x_2 = d$, $x_3 = b$, $x_4 = c$.

By Lemma 7, G induces on X a matching. We can determine it uniquely. Indeed, the vertex v_5 can be adjacent only to v_4; other choices lead to cycles contradicting Lemma 1(d). For instance.

$$v_5 - v_3 - dby_3 - db - bd - bdc - bcd - v_5;\ \text{or}$$
$$v_5 - v_2 - aed - v_5.$$

Then we find similarly that v_3 is adjacent to v_2 and v_6 to v_1, and the assertion (a) is proved.

The value of y_3 is c or e. If $y_3 = c$, then we have the 7-cycle

$$v_3 - v_2 - cdy_2 - cd - dc - dcb - dbc - v_3.$$

Hence, $y_3 = e$.

If $y_4 = b$, then there are three 4-paths from v_4 to bd:

$$v_4 - dcb - dbc - db - bd,$$
$$v_4 - v_5 - bcd - bdc - bd,$$
$$v_4 - aec - v_6 - bde - bd,$$

contradicting Lemma 1(e). Hence, $y_4 = e$.

So, there are vertices $aeb : dbe$ and $aec : dce$. Now the assertion (b) follows from Lemma 5. □

LEMMA 9. *If* $ab \to cde$, *then all relations* $xy \to pqr$ *are determined uniquely. They are shown in Table 1.*

Table 1. Relations $xy \to pqr$.

	a	b	c	d	e
a	⊠⊠⊠	cde	bed	ecb	dbc
b	edc	⊠⊠⊠	dae	cea	acd
c	deb	ead	⊠⊠⊠	abe	bda
d	bce	aec	eba	⊠⊠⊠	cab
e	cbd	dca	adb	bac	⊠⊠⊠

Proof. Lemma 6(b) implies that in each row of the table all last symbols of triples are different. Moreover, all second symbols in each row are also different. Indeed, if $xy_1 \to p_1qr_1$ and $xy_2 \to p_2qr_2$, then by Lemma 8(b) we have $xq \to r_1{\star\star}$ and $xq \to r_2{\star\star}$, but $r_1 \neq r_2$. Further, since in the ith row all symbols but i appear exactly three times, all first symbols in each row are also different. By Lemma 6(a), the same statements hold for columns.

We start with the blank table, fill in the cells (a, b) and (b, a) by triples cde and edc accordingly, and try to continue. Lemma 8(b) gives two possibilities to fill the cell (a, d): either ecb or ebc. Let's try the second one (see Table 2).

Table 2.

	a	b	c	d	e
a	⊠⊠⊠	cde		ebc	
b	edc	⊠⊠⊠		cae	
c			⊠⊠⊠		
d	cbe	eae		⊠⊠⊠	
e					⊠⊠⊠

The cell (b, d) is filled by either cae or cea (Lemma 8(b) applied to $ba \to edc$). So, the cell (d, b) is filled by either eac or aec. But aec is impossible; otherwise Lemma 8(b) would give us a relation $de \to c{\star\star}$, contrary to $da \to cbe$. Hence, the cells (b, d) and (d, b) are filled as in Table 2. But then the second symbols in (c, a) and (c, b) should both be equal to e: a contradiction.

Hence, the second possibility in Lemma 8(b) never holds, and each relation $xy \to pqr$ implies $xq \to rpy$. This follows from the fact that the relation $ab \to cde$

in Lemma 8 was chosen arbitrarily. Now it's an easy exercise to fill the table and to make sure that it can be done uniquely. □

Now the final contradiction is near. Without loss of generality let us assume that $12 \to 345$. Then Lemma 9 gives us labels of all good vertices, and Lemma 8(a) gives some edges between them. Applying Lemma 8(a) consecutively to relations

$$15 \to 423,\ 25 \to 134,\ 34 \to 125, \text{ and } 14 \to 532,$$

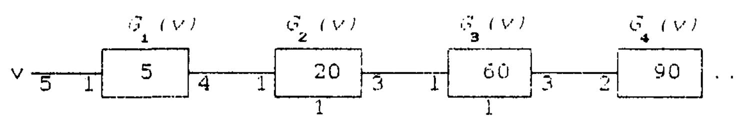
we find the 4-path

$$132 : 542 \ - \ 134 : 243 \ - \ 354 : 245 \ - \ 352 : 125 \ - \ 453 : 123.$$

which, together with vertices 132 and 123, gives us a 7-cycle, contrary to Lemma 1(d). The theorem is proved.

Concluding remarks

1. We have proved, in fact, a stronger result than the theorem. Since only properties of the graph G listed in Lemma 1 were used in the proof, we proved nonexistence of graphs which locally look like



   around every vertex.
2. The proof presented is computer-free, but I think it would not be possible to find it without the help of a computer. In fact, after Lemma 6 was proved, a simple computer search gave 20 nonisomorphic variants for the set of relations $ij \to abc$. The rest of the argument appeared through many attempts to kill these variants.

References

1. A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
2. D.G. Fon-Der-Flaass. "There exists no distance-regular graph with intersection array (5, 4, 3; 1, 1, 2)." to appear in *Europ. J. Combinatorics*.