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BMO Regularity for One-Dimensional Minimizers of some Lagrange Problems*

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We extend our results about a class of non-regular Lagrange problems of Calculus of Variations showing that the derivative of minimizers are in BMO. For this class we give also some results of optimality relative to the Tonelli set, of the type recently given by Ball-Mizel, by using results of Harmonic Analysis.

Keywords: Calculus of Variations, One-dimensional problems, Reverse Jensen Inequalities, Tonelli set, BMO, Orlicz Spaces.

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1. Introduction

Let $L : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Borel function, where I denotes an interval of $\mathbb{R}$ (for semplicity we will assume $I = [0, 1]$). We will consider in the following solutions of the so-called Lagrange problem, that is, absolute minimizers of the one dimensional problem of the Calculus of Variations

$$\mathcal{F}: \quad u \in W^{1,1}(I, \mathbb{R}^n), \quad u(0) = A, \quad u(1) = B \quad \to \quad \mathcal{F}(u) = \int_I L(t, u, u')dt.$$

In the paper [13] we proved BMO regularity results for some Lagrange problems of the type considered in [1], for which minimizers need not necessarily be Lipschitz. The Lagrangians $L(t, s, z)$ considered in these papers verify, besides some standard conditions in order to get the existence of minimizers, a growth condition of the type power in z:

$$g(t, s)|z|^\alpha - c_2 \ \le \ L(t, s, z) \ \le \ g(t, s)|z|^\alpha + c_4 \qquad \forall (t, s, z) \in I \times \mathbb{R}^n \times \mathbb{R}^n$$

with $\alpha > 1$, $c_2, c_4 > 0$, g continuous and positive.

Aim of this paper is to extend the results of [13] for Lagrangians satisfying growth conditions of the type

$$g(t, s)\Phi(|z|) - c_2 \ \le \ L(t, s, z) \ \le \ g(t, s)\Phi(|z|) + c_4 \qquad \forall (t, s, z) \in I \times \mathbb{R}^n \times \mathbb{R}^n \quad (1.1)$$

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with an N-function Φ (i.e. a nonnegative convex function on $\mathbb{R}^+$ such that $\Phi(0) = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$, $\lim_{t\to+\infty} \frac{\Phi(t)}{t} = +\infty$) such that $\Phi^{\frac{1}{\alpha}}$ is convex for some $\alpha > 1$, $c_2, c_4 \geq 0$. We will assume throughout the paper that L verifies (1.1) with g continuous and bounded function, greater than a positive constant, L l.s.c. in (s, z), L convex in z so that absolute minimizers of $\mathcal{F}$ exist, as shown in [16]. The first classical results about the Lagrange problem are due to Tonelli [25, 26]: if the Lagrangian $L(t, s, z)$ is a nonnegative $\mathcal{C}^3$ function on $\mathbb{R}^3$ satisfying $L_{zz} > 0$ and the superlinear growth condition $L(t, s, z) \geq \varphi(z)$ where $\varphi(z)/|z| \to \infty$ as $|z| \to \infty$, it is possible to show that absolute minimizers exist, and any minimizer u has a finite or infinite derivative u' at every point of I. Moreover $u' : I \to \mathbb{R} \cup \{-\infty, +\infty\}$ is continuous and the Tonelli set defined by $E = \{x \in I : |u'(x)| = \infty\}$ is a closed set of zero measure [3].

About the Lagrangians of the type (1.1) we recall that if $L(t, s, z) = g(t, s)\Phi(z)$ with Φ strictly convex and g positive, continuous and locally Lipschitz in s uniformly in t, in [8] it is shown that absolute minimizers exist and are of class C^1.

We will essentially improve the results of [13] also in the case of Φ power function: we will deduce some BMO results for derivative of minimizers that we found to be in EXP. Moreover, some of our results may be extended to Lagrangians which have growth in z bigger than powers, for instance exponential growth. We will give some examples and we will state some results of optimality about the Tonelli set of minimizers, of the type of Ball-Mizel [3], Davie [11].

We will follow the same technique used in [13], based on properties of functions verifying reverse integral inequalities; nevertheless, this paper contains some results of independent interest (see Lemma 2.1 about rearrangements and Lemma 2.6 about sufficient conditions to get BMO regularity of functions) and some applications of classical results of Harmonic Analysis to the Calculus of Variation (see Section 4).

2. Preliminaries

Throughout the paper we will assume that Φ is an N-function such that $\Phi^{\frac{1}{\alpha}}$ is convex for some $\alpha > 1$. Let $E \subset \mathbb{R}$ be a bounded interval, and let $L_\Phi = L_\Phi(E)$ be the set of all (equivalence classes modulo equality a.e. in E of) measurable functions f satisfying $\Phi\left(\frac{|f|}{t}\right) \in L^1(E)$ for some $t > 0$. Following [19, 20, 18] we define

$$\| f \|_\Phi = \inf\left\{t > 0 : \int_E \Phi\left(\frac{|f(x)|}{t}\right) dx \ \leq \ 1\right\} \quad , \quad \forall f \in L_\Phi(E).$$

The functional $\| \cdot \|_\Phi$ is called Luxembourg norm and it may be checked that it is a Banach function norm.

We will denote by $EXP = EXP(E)$ the Orlicz space corresponding to $\Phi(t) = e^t - t - 1$.

In the following we will consider also the John-Nirenberg space of functions with bounded mean oscillation, i.e. the space of the functions $f \in L^1(E)$ such that

$$\| f \|_* = \sup_{J\subset E} \fint_J \left| f(t) - \fint_J f(s) ds \right| dt < +\infty.$$

This space is called $BMO = BMO(E)$, and is a Banach space under the norm $\|\cdot\|_*$ [24]. It can be proved that $L^\infty(E) \subsetneq BMO(E) \subsetneq EXP(E) \subsetneq \bigcap_{1<p<\infty} L^p(E)$.

Finally we introduce a very useful tool to prove some lemmas in this section. Let f be a measurable function defined in E. The decreasing rearrangement of f is defined to be the function

$$f^*(t) = \inf\{s > 0 : |\{x \in E : |f(x)| > s\}| \leq t\} \qquad \forall\, t \in]0, |E|].$$

There is a wide literature about rearrangements (see for example [6]).

We begin with the following lemma, consequence of a one-dimensional result by Korenovskii [17], who proved the case $\widetilde{M} = 0$.

Lemma 2.1. *Let $E \subset \mathbb{R}$ be a bounded interval, $f \in L_\Phi(E)$ be a nonnegative function, $M > 1$, $\widetilde{M} \geq 0$ be such that*

$$\fint_J \Phi(f)dt \leq M\Phi\left(\fint_J f dt\right) + \widetilde{M} \tag{2.1}$$

for any interval $J \subset E$. Then we have

$$\fint_J \Phi(f^*)dt \;\leq\; M\Phi\left(\fint_J f^* dt\right) + \widetilde{M} \tag{2.2}$$

for any interval $J \subset [0, |E|]$, where f^* denotes the decreasing rearrangement of f.

Proof. Let $\epsilon > 0$. By assumption (2.1) we have

$$\left(1 + \frac{\epsilon M}{M-1}\right)\fint_J \Phi(f)dt \;\leq\; M\left(1 + \frac{\epsilon M}{M-1}\right)\Phi\left(\fint_J f dt\right) + \widetilde{M}\left(1 + \frac{\epsilon M}{M-1}\right)$$

and therefore

$$\begin{aligned}
&\left(1 + \frac{\epsilon M}{M-1}\right)\fint_J \Phi(f)dt + \frac{\widetilde{M}}{M-1}\\
\leq\;& M\left(1 + \frac{\epsilon M}{M-1}\right)\Phi\left(\fint_J f dt\right) + \widetilde{M}\left(1 + \frac{\epsilon M}{M-1}\right) + \frac{\widetilde{M}}{M-1}\\
\leq\;& (1+\epsilon)M\left(1 + \frac{\epsilon M}{M-1}\right)\Phi\left(\fint_J f dt\right) + \frac{\epsilon M\widetilde{M}}{M-1} + \frac{M\widetilde{M}}{M-1}\\
=\;& (1+\epsilon)M\left[\left(1 + \frac{\epsilon M}{M-1}\right)\Phi\left(\fint_J f dt\right) + \frac{\widetilde{M}}{M-1}\right] \qquad \forall J \subset E.
\end{aligned}$$

Let us set

$$\phi_\epsilon(t) = \left(1 + \frac{\epsilon M}{M-1}\right)\Phi(t) + \frac{\widetilde{M}}{M-1},$$

we have shown that

$$\fint_J \phi_\epsilon(f)dt \ \leq \ (1+\epsilon)M\phi_\epsilon\left(\fint_J f dt\right) \qquad \forall J\subset E.$$

By convexity of ϕ_ϵ we have, by the theorem of Korenovskii [17], that

$$\fint_J \phi_\epsilon(f^*)dt \ \leq \ (1+\epsilon)M\phi_\epsilon\left(\fint_J f^* dt\right) \qquad \forall J\subset [0,|E|]$$

and therefore

$$\begin{aligned}&\fint_J \left(1+\frac{\epsilon M}{M-1}\right)\Phi(f^*)dt + \frac{\widetilde{M}}{M-1}\\ &\leq (1+\epsilon)M\left(1+\frac{\epsilon M}{M-1}\right)\Phi\left(\fint_J f^* dt\right) + (1+\epsilon)M\frac{\widetilde{M}}{M-1} \qquad \forall J\subset[0,|E|].\end{aligned}$$

Letting $\varepsilon \to 0$ we get

$$\fint_J \Phi(f^*)dt + \frac{\widetilde{M}}{M-1} \ \leq \ M\Phi\left(\fint_J f^* dt\right) + \frac{M\widetilde{M}}{M-1} \qquad \forall J\subset[0,|E|]. \tag{2.3}$$

i.e. inequality (2.2). □

Let us note that in order to prove Lemma 2.1 we don't need that $\Phi^{\frac{1}{\alpha}}$ is convex for some $\alpha > 1$, but only convexity of Φ.

We will need in the sequel the following higher integrability result from reverse Jensen inequalities, corollary of the following Lemma, proved in [13] (see also [23, 14, 15]).

Lemma 2.2. *Let $E \subset \mathbb{R}$ be a bounded interval, and let $f \in L^\alpha(E)$, $f \geq 0$, $\alpha > 1$, $M > 1$, $\widetilde{M} \geq 0$ be such that*

$$\fint_J f(t)^\alpha dt \ \leq \ M\left(\fint_J f(t)dt\right)^\alpha + \widetilde{M} \qquad \forall\ J\subset E.$$

Then the following holds:

$$\left(\fint_J f(t)^q dt\right)^{\frac{\alpha}{q}} \ \leq \ \frac{\alpha}{q\gamma_{\alpha,M}(q)}\fint_J f(t)^\alpha dt + \frac{\widetilde{M}}{\gamma_{\alpha,M}(q)}\ \frac{q-\alpha}{q} \quad \forall q\in[\alpha,\beta(\alpha,M)[, \quad \forall J\subset E$$

where $\beta(\alpha, M)$ is the solution greater than α of the equation:

$$\gamma_{\alpha,M}(x) = 1 - M^\alpha\frac{x-\alpha}{x}\left(\frac{x}{x-1}\right)^\alpha = 0.$$

We remark that we get an estimate of the exponent of integrability in terms of the coefficients appearing in the reverse inequalities. Such estimate is analogous to the one obtained in [12], where the case Φ power, f non-increasing function, $\widetilde{M} = 0$ is proved.

Lemma 2.3. *In the same assumptions of* Lemma 2.1, *we have* $\Phi(f)^{\frac{1}{\alpha}} \in L^q(E)$ $\forall q \in [\alpha, \beta(\alpha, M)[$.

Proof. Put $v = \Phi(f)^{\frac{1}{\alpha}}$. By assumption (2.1) and convexity of $\Phi(f)^{\frac{1}{\alpha}}$ we have

$$\begin{aligned} \fint_J v^\alpha dt &= \fint_J \Phi(f) dt \;\leq\; M \left[\Phi^{\frac{1}{\alpha}}\left(\fint_J f dt\right)\right]^\alpha + \widetilde{M} \\ &\leq M\left(\fint_J \Phi(f)^{\frac{1}{\alpha}} dt\right)^\alpha + \widetilde{M} = M\left(\fint_J v dt\right)^\alpha + \widetilde{M} \qquad \forall J \subset E \end{aligned}$$

Then, by Lemma 2.2, the following holds

$$\begin{aligned} & \left[\fint_J \Phi(f)^{\frac{q}{\alpha}} dt\right]^{\frac{\alpha}{q}} = \left(\fint_J v^q dt\right)^{\frac{\alpha}{q}} \\ \leq\; & \frac{\alpha}{q\gamma_{\alpha,M}(q)} \fint_J v^\alpha dt + \frac{\widetilde{M}}{\gamma_{\alpha,M}(q)} \frac{q-\alpha}{q} \\ =\; & \frac{\alpha}{q\gamma_{\alpha,M}(q)} \fint_J \Phi(f) dt + \frac{\widetilde{M}}{\gamma_{\alpha,M}(q)} \frac{q-\alpha}{q} \qquad \forall q \in [\alpha, \beta(\alpha, M)[, \forall J \subset E. \end{aligned}$$

and therefore the assertion follows. □

Remark 2.4. Since $\lim_{M \to 1} \beta(\alpha, M) = +\infty$, if inequality (2.1) is satisfied with $M = 1$, applying Lemma 2.3 we have that $\Phi(f)^{\frac{1}{\alpha}} \in \bigcap_{1<p<\infty} L^p(E)$. Corollary 2.10 below shows that as a matter of fact, if $M = 1$, we have $\Phi(f)^{\frac{1}{2}} \in BMO(E)$.

Next Lemma, true also in dimension $n > 1$, gives a characterization of BMO that can be shown by using some well-known results, and so we omit its proof (see Corollary 1.5, chap. 8 of [24]).

Lemma 2.5. *Let $Q_0 \subset \mathbb{R}^n$ $(n \geq 1)$ be a cube. If $\alpha > 0$ and $g : Q_0 \to \mathbb{R}$ is a measurable function belonging to $L^p(Q_0)$ $\forall\, 1 \leq p < \infty$, then*

$$\sup_{Q \subset Q_0} \limsup_{p \to \infty} \frac{1}{p^\alpha} \left(\fint_Q |g(x)|^p dx\right)^{\frac{1}{p}} < \infty \Leftrightarrow g^{\frac{1}{\alpha}} \in BMO(Q_0)$$

where the supremum is taken over all cubes $Q \subset Q_0$.

By using Lemma 2.5 one could prove a BMO regularity result for a minimizer u in the case $\Phi(t) = t^\alpha$, $\alpha > 1$. Namely, in the same assumptions of Theorem 4.3 of [13] and with an analogous argument, it is possible to get that $|u'|^{\frac{\alpha}{2}\frac{\mu}{\mu-1}} \in BMO$ $\forall 0 < \mu < \lambda$, where λ is the exponent of Hölder of the function g related to the functional $\mathcal{F}$. In Section 3 below we will prove a more general result, that in the case $\Phi(t) = t^\alpha$, $\alpha > 1$ gives that also $|u'|^{\frac{\alpha}{2}} \in BMO$.

In the following we need of other conditions ensuring $BMO(Q_0)$. We notice that if $p \geq 2$, then it is easy to prove that for all $f \in L^p(Q_0)$ the inequality

$$\fint_Q f^p dx \;\leq\; \left(\fint_Q f\,dx\right)^p + k \tag{2.4}$$

for any cube $Q \subset Q_0$, with k independent of Q, implies $f^{\frac{p}{2}} \in BMO(Q_0)$: in fact, we have (see the proof of Lemma 3.5 of [13]):

$$\fint_Q \left| f(x)^{\frac{p}{2}} - \left(\fint_Q f^{\frac{p}{2}}\right)\right| dx \;\leq\; \left(\fint_Q \left| f(x)^{\frac{p}{2}} - \left(\fint_Q f^{\frac{p}{2}}\right)\right|^2 dx\right)^{\frac{1}{2}}$$

$$= \;\left[\fint_Q \left(f(x)^p - \left(\fint_Q f^{\frac{p}{2}}\right)^2\right) dx\right]^{\frac{1}{2}} \;\leq\; \left[\fint_Q \left(f(x)^p - \left(\fint_Q f\right)^p\right) dx\right]^{\frac{1}{2}} \;\leq\; k^{\frac{1}{2}}.$$

If $1 < p < 2$, inequality (2.4) still implies $f^{\frac{p}{2}} \in BMO(Q_0)$, as shown by the following lemma.

Lemma 2.6. *Let $Q_0 \subset \mathbb{R}^n$ $(n \geq 1)$ be a cube, $p \in]1, 2]$, $f \in L^p(Q_0)$. Then*

$$\fint_Q f^p dx \;\leq\; \left(\fint_Q f\,dx\right)^p + k \;\; \forall Q \subset Q_0 \Rightarrow \fint_Q f^p dx \;\leq\; \left(\fint_Q f^{\frac{p}{2}} dx\right)^2 + \frac{k}{p-1} \;\; \forall Q \subset Q_0$$

and therefore $f^{\frac{p}{2}} \in BMO(Q_0)$.

Proof. We have, by the Hölder and Young inequalities, that for any $Q \subset Q_0$

$$\begin{aligned}\fint_Q f^p dx \;&\leq\; \left(\fint_Q f\,dx\right)^p + k = \left(\fint_Q f^{p-1} f^{2-p} dx\right)^p + k \\ &\leq \left(\fint_Q (f^{p-1})^{\frac{p}{2(p-1)}}\,dx\right)^{2(p-1)} \left(\fint_Q (f^{2-p})^{\frac{p}{2-p}}\,dx\right)^{2-p} + k \\ &= \left[\left(\fint_Q f^{\frac{p}{2}} dx\right)^2\right]^{p-1} \left(\fint_Q f^p dx\right)^{2-p} + k \\ &\leq (p-1)\left(\fint_Q f^{\frac{p}{2}} dx\right)^2 + (2-p)\fint_Q f^p dx + k\end{aligned}$$

and therefore

$$(p-1)\fint_Q f^p dx \;\leq\; (p-1)\left(\fint_Q f^{\frac{p}{2}} dx\right)^2 + k$$

from which the assertion follows. □

Remark 2.7. By using Lemma 2.5 one can easily show that the converse of Lemma 2.6 is true, namely, if $f^{\frac{p}{2}} \in BMO(Q_0)$, $p > 1$, then there exists $k > 0$ such that

$$\fint_Q f^p dx \;\leq\; \left(\fint_Q f\,dx\right)^p + k \quad \forall Q \subset Q_0.$$

By using Lemma 2.6, we have the following generalization of Lemma 3.5 of [13], that can be proved essentially in the same way:

Lemma 2.8. *Let $p \geq \alpha > 1$, and let $h \in \mathcal{C}^{0,\frac{\alpha}{p}}(I)$, $h > 0$. If $f \in L^p(I)$ is a nonnegative function such that*

$$\fint_J h(t)f(t)^\alpha dt \;\leq\; \left(\fint_J h(t)dt\right)\left(\fint_J f(t)dt\right)^\alpha + \widetilde{M}$$

for any interval $J \subset I$, with $\widetilde{M} \geq 0$ independent of J, then $f^{\frac{\alpha}{2}} \in BMO(I)$.

Corollary 2.9. *Let $p \geq \alpha > 1$, and let $h \in \mathcal{C}^{0,\frac{\alpha}{p}}(I)$, $h > 0$. If f is a nonnegative measurable function such that $\Phi^{\frac{1}{\alpha}}(f) \in L^p(I)$ and*

$$\fint_J h(t)\Phi(f(t))dt \;\leq\; \left(\fint_J h(t)dt\right)\Phi\left(\fint_J f(t)dt\right) + \widetilde{M} \tag{2.5}$$

for any interval $J \subset I$, with $\widetilde{M} \geq 0$ independent of J, then $\Phi(f)^{\frac{1}{2}} \in BMO(I)$.

Proof. We have

$$\begin{aligned}
& \fint_J h(t)\left[\Phi^{\frac{1}{\alpha}}(f(t))\right]^\alpha dt \\
\leq\; & \left(\fint_J h(t)dt\right)\left[\Phi^{\frac{1}{\alpha}}\left(\fint_J f(t)dt\right)\right]^\alpha + \widetilde{M} \\
\leq\; & \left(\fint_J h(t)dt\right)\left[\fint_J \Phi^{\frac{1}{\alpha}}(f(t))dt\right]^\alpha + \widetilde{M} \qquad \forall J \subset I.
\end{aligned}$$

From this inequality, by using Lemma 2.8, we get $\left[\Phi^{\frac{1}{\alpha}}(f)\right]^{\frac{\alpha}{2}} \in BMO(I)$, and the assertion follows. □

Corollary 2.10. *If h is a positive and Hölder continuous function on I and f is a nonnegative function such that $\Phi^{\frac{1}{\alpha}}(f) \in \bigcap_{1<p<\infty} L^p(I)$ and verifying inequality (2.5), then $\Phi^{\frac{1}{2}}(f) \in BMO(I)$.*

3. BMO regularity for minimizers

We begin with the following theorem, that is an extension of the main result of [1] by Ambrosio-Ascenzi. The functional $\mathcal{F}$ considered in the following verifies the assumptions mentioned in the introduction, and therefore the Lagrangian L is a Borel function such that

$$c_1\Phi(|z|) - c_2 \;\leq\; L(t,s,z) \;\leq\; c_3\Phi(|z|) + c_4 \qquad (c_1, c_3 > 0; c_2, c_4 \geq 0) \tag{3.1}$$

We remark that we make such assumptions only in order to get existence of absolute minimizers, but they are not necessary to prove Theorem 3.1. Nevertheless, assuming the existence of a minimizer u in the proof of Theorem 3.1, the continuity of the positive function g ensure that inequality (3.1) is true with the constants $c_1 = \min_{I\times K} g$, $c_2 = \max_{I\times K} g$, where $K \subset\subset \mathbb{R}^n$ is the closed convex hull of the range of u.

Let us note also that we don't use the assumption $\Phi \in \Delta_2$, and therefore in this theorem Lagrangians with exponential growth are allowed.

Theorem 3.1. *Let Φ be an N-function such that $\Phi^{\frac{1}{\alpha}}$ is convex for some $\alpha > 1$, and let us assume that there is a continuous positive function g on $I \times \mathbb{R}^n$ such that*

$$\lim_{|z|\to+\infty} \frac{L(t,s,z)}{\Phi(|z|)} = g(t,s) \tag{3.2}$$

uniformly in the compact subsets of $I \times \mathbb{R}^n$. Then any minimizer $u : I \times \mathbb{R}^n \to \mathbb{R}$ of the functional $\mathcal{F}$ satisfies the condition

$$\Phi(|u'|) \in \bigcap_{1<p<\infty} L^p(I)$$

and therefore u is λ -Hölder continuous for any $\lambda < 1$.

Proof. We begin by following the same argument of the proof of Theorem 1.1 of [13], that we give here for completeness. We denote by $K \subset\subset \mathbb{R}^n$ the closed convex hull of the range of u, and we set $v = |u'| \in L_\Phi(I)$. First of all we will show that for any $M' > 1$ there exist $h \in N, C > 0$, some positive constants m_i $(i = 1, ..., h)$, such that if

$$E_i = \left\{ t \in I : \frac{i-1}{h} \le t \le \frac{i}{h} \right\} \qquad (i = 1, ..., h)$$

then for any interval $J = [a,b] \subset E_i$ the following inequalities hold:

$$\fint_J \Phi(v(t))dt \le M'\Phi\left(\fint_J v(t)dt\right) + \frac{2C}{m_i} \tag{3.3}$$

Let us consider the modulus of continuity of $g_{|_{I\times K}}$

$$\omega_g(\sigma) = \sup_{\substack{dist((t_1,s_1),(t_2,s_2))\le\sigma \\ (t_1,s_1),(t_2,s_2)\in I\times K}} |g(t_1,s_1) - g(t_2,s_2)|$$

and the modulus of continuity of u

$$\omega_u(\delta) = \sup_{|t_1-t_2|\le\delta} |u(t_1) - u(t_2)| \quad .$$

Since $g_{|_{I\times K}}$ and u are uniformly continuous, we have

$$\omega_g(2\omega_u(\delta)) \downarrow 0 \quad \text{as} \quad \delta \downarrow 0. \tag{3.4}$$

Now, for any $J = [a,b] \subset I$, let w be the function

$$w(t) = \begin{cases} u(t) & \text{if } t \in I \setminus J \\ u(a) + \left(\fint_J u'(t)dt \right)(t-a) & \text{if } t \in J \end{cases} .$$

We have

$$\begin{aligned} |u(t) - w(t)| &\leq \left| u(t) - \left[u(a) + \frac{u(b)-u(a)}{b-a}(t-a) \right] \right| \\ &\leq \quad |u(t)-u(a)| + |u(b)-u(a)| \left| \frac{t-a}{b-a} \right| \ \leq \ 2\omega_u(b-a) \quad \forall t \in J \end{aligned}$$

and therefore:

$$|g(t,w(t)) - g(t,u(t))| \ \leq \ \omega_g(|w(t)-u(t)|) \ \leq \ \omega_g(2\omega_u(b-a)) \qquad \forall t \in I \ . \tag{3.5}$$

On the other hand, by assumption (3.2), for any $0 < \epsilon < \min\limits_{I \times K} g$ there exists $\Lambda > 0$ such that

$$[g(t,s) - \epsilon]\Phi(|z|) \ \leq \ L(t,s,z) \ \leq \ [g(t,s)+\epsilon]\Phi(|z|) \quad \forall (t,s) \in I \times K, \quad \forall z : |z| > \Lambda$$

and by assumption (3.1) we have

$$-c_2 \ \leq \ L(t,s,z) \ \leq \ c_3\Phi(\Lambda) + c_4 \quad \forall (t,s) \in I \times K, \quad \forall z : |z| < \Lambda$$

and therefore

$$\begin{aligned} & [g(t,s)-\epsilon]\Phi(|z|) - c_2 - \left(\max_{I \times K} g - \epsilon \right) \Phi(\Lambda) \ \leq \ L(t,s,z) \\ \leq \quad & [g(t,s)+\epsilon]\Phi(|z|) + c_3\Phi(\Lambda) + c_4 \quad \forall (t,s,z) \in I \times K \times \mathbb{R}^n \ . \end{aligned}$$

Setting

$$C = \max \left\{ c_2 + \left(\max_{I \times K} g - \epsilon \right) \Phi(\Lambda), c_3\Phi(\Lambda) + c_4 \right\},$$

we get

$$[g(t,u(t)) - \epsilon]\Phi(|u'(t)|) - C \ \leq \ L(t,u(t),u'(t)) \qquad \forall t \in I \tag{3.6}$$

$$L(t,w(t),w'(t)) \ \leq \ [g(t,w(t)) + \epsilon]\Phi(|w'(t)|) + C \qquad \forall J \subset I, \quad \forall t \in I \tag{3.7}$$

and by (3.5), inequality (3.7) becomes

$$L(t,w(t),w'(t)) \ \leq \ [g(t,u(t)) + \omega_g(2\omega_u(b-a)) + \epsilon]\Phi(|w'(t)|) + C \quad \forall t \in I \ . \tag{3.8}$$

By using the continuity of $g(t,u(t))$ and (3.4), for any $M' > 1$ there exist ϵ, $0 < \epsilon < \min\limits_{I \times K} g$, $h \in N$ such that, if we pose

$$\begin{aligned} E_i &= \left\{ t \in I : \frac{i-1}{h} \ \leq \ t \ \leq \ \frac{i}{h} \right\} \qquad \forall i = 1, ..., h \\ M_i &= \max_{E_i} g(t,u(t)) + \omega_g\left(2\omega_u\left(\frac{1}{h} \right) \right) + \epsilon \\ m_i &= \min_{E_i} g(t,u(t)) - \epsilon \end{aligned}$$

we have

$$M_i \ \leq \ M' m_i \ . \tag{3.9}$$

Then, for any $i = 1, ..., h$, from (3.6) and (3.8) we have

$$m_i \Phi(v(t)) - C \ \leq \ L(t, u(t), u'(t)) \qquad \forall t \in E_i \tag{3.10}$$

$$L(t, w(t), w'(t)) \ \leq \ M_i \Phi(|w'(t)|) + C \qquad \forall J \subset E_i, \quad \forall t \in E_i \tag{3.11}$$

and therefore, integrating over $J \subset E_i$, from (3.10) and (3.11) we get

$$m_i \int_J \Phi(v(t))dt - C|J| \ \leq \ M_i \int_J \Phi \left(\left| \fint_J u'(\tau)d\tau \right| \right) dt + C|J| \quad \forall J \subset E_i$$

i.e.

$$m_i \fint_J \Phi(v(t))dt \ \leq \ M_i \Phi \left(\fint_J v(t)dt \right) + 2C \quad \forall J \subset E_i, \quad \forall i = 1, ..., h$$

from which, by (3.9), we get (3.3).

Now, applying Lemma 2.3, we get $\Phi^{\frac{1}{\alpha}} \in L^q(I) \quad \forall q \in [\alpha, \beta(\alpha, M')[$ and therefore, since $\lim_{M' \to 1} \beta(\alpha, M') = +\infty$, we have $\Phi^{\frac{1}{\alpha}}(v) \in \bigcap_{1<p<\infty} L^p(I)$ from which the assertion follows. □

Next Lemma will be used to prove the *BMO* regularity result for Lagrangians of the type (1.1): we prove that every minimizer satisfies an inequality of the type (2.5). We remark that in some cases also exponential growth of Φ are allowed.

Lemma 3.2. *Let L be a Borel function such that*

$$g(t,s)\Phi(|z|) - c_2 \ \leq \ L(t,s,z) \ \leq \ g(t,s)\Phi(|z|) + c_4 \qquad \forall (t,s,z) \in I \times \mathbb{R}^n \times \mathbb{R}^n$$

where Φ is a N-function verifying the Δ_2 condition with constant c_Φ and such that $\Phi^{\frac{1}{\alpha}}$ is convex for some $\alpha > 1$, g is $\lambda-$Hölder continuous on every compact subset of $I \times \mathbb{R}^n$, $g > 0$, $c_2, c_4 \geq 0$. Then every minimizer u of $\mathcal{F}$ is such that the function $v = |u'|$ satisfies the following inequality

$$\fint_J g(t,u(t))\Phi(v(t))dt \leq \fint_J g(t,u(t))dt \Phi \left(\fint_J v(t)dt \right) + T$$

for any interval $J \subset I$, where $T = T(c_2, c_4, \lambda, g, u, c_\Phi)$ is independent of J. The assumption that Φ verifies Δ_2 condition can be dropped if g depends only on t.

Proof. Let $J = [a,b]$, and let w be the function that agrees with u in $I \backslash J$, and that is linear in J, so that $u(a) = w(a)$, $u(b) = w(b)$, $w'_{|J} \equiv \fint_J u'(t)dt$. We have

$$\begin{aligned}
\fint_J g(t,u(t))\Phi(v(t))dt - c_2 \quad &\leq \quad \fint_J L(t,u(t),u'(t))dt \\
\leq \ \fint_J L(t,w(t),w'(t))dt \quad &\leq \quad \fint_J g(t,w(t))dt \Phi \left(\fint_J v(t)dt \right) + c_4 \\
= \ \fint_J g(t,u(t))dt \Phi \left(\fint_J v(t)dt \right) + c_4 &+ \fint_J [g(t,w(t)) - g(t,u(t))]dt \Phi \left(\fint_J v(t)dt \right).
\end{aligned}$$

Now we have just to prove that

$$\sup_J \left| \fint_J [g(t,w(t)) - g(t,u(t))]dt \Phi\left(\fint_J v(t)dt\right)\right| \leq T = T(c_2, c_4, \lambda, g, u, c_\Phi)$$

If g depends only on t, the assertion is obvious. We note that for every $0 < \mu < \lambda$

$$\begin{aligned} \Phi\left(\fint_J v(t)dt\right) &\leq c_5 + c_6 \left(\fint_J v(t)dt\right)^q \\ &\leq c_5 + c_6 \frac{1}{|J|^q}\left[\int_J v(t)^{\frac{\mu}{q}} |J|^{1-\frac{\mu}{q}} dt\right]^q = c_5 + c_6 \frac{1}{|J|^\mu} \parallel v \parallel^q_{L^{\frac{q}{\mu}}} \end{aligned}$$

for some $c_5, c_6 > 0$, $q > 1$ (the existence of q is ensured by the assumption $\Phi \in \Delta_2$). On the other hand we have

$$\fint_J [g(t,w(t)) - g(t,u(t))]dt \leq 2^\lambda [g]_{0,\lambda} [u]^\lambda_{0,\frac{\mu}{\lambda}} |J|^\mu$$

and therefore we get the assertion. □

By using Lemma 3.2 and Corollary 2.10 we have the following

Theorem 3.3. *If Φ verifies the Δ_2 condition and g is Hölder continuous on every compact subset of $I \times \mathbb{R}^n$, then every minimizer u of $\mathcal{F}$ is such that the function $v = |u'|$ satisfies $\Phi(v)^{\frac{1}{2}} \in BMO(I)$. The assumption that Φ verifies the Δ_2 condition can be dropped if g depends only on t.*

Example 3.4. Let $1 < \alpha < 2$, $1 < p < 2\alpha^{-1}$, $w(x) = |\log|x||^p$ and

$$L(t,s,z) = \begin{cases} |z|^\alpha & \text{if } s = v(t) \\ |z|^\alpha + M & \text{otherwise} \end{cases} \qquad \forall\, (t,s,z) \in [-1,1] \times \mathbb{R} \times \mathbb{R},$$

where $v(t) = \int_{-1}^{t} w(x)dx$. In [13] we showed that if $M > 0$ is a constant sufficiently large, then v is a minimizer of $\mathcal{F}$ and $w = v' \notin BMO(-1,1)$. As a matter of fact we have $w^{\frac{\alpha}{2}} \in BMO(-1,1)$, in conformity with the assertion of Theorem 3.3.

4. The Tonelli set

The first examples of Lagrangians in which the Tonelli set E is nonempty where given by Ball and Mizel in [3, 4] and have been studied in [10]. Similar examples are in [21, 22]. In [11] and [3] it is shown that Tonelli's result (see Section 1) is optimal in the sense that any closed set of measure zero is the singular set E corresponding to a suitable L. Moreover, other properties of the Tonelli set E are studied in [5], and in [9], in which it is shown that if L is of polynomial type in its three arguments, then E is a suitable closed set of measure zero, namely, countable with at most a finite number of points of accumulation.

All the examples and theorems of previous authors study Lagrangians with continuous second derivatives in all their arguments. In [8] Lagrangians not necessarily differentiable are considered, and it is shown that also in a more general setting the Tonelli set is a closed set of measure zero; in [2] Lipschitz regularity for minimizers has been proved for Lagrangians with superlinear growth, independent of t, and measurable in s.

Now let us consider Lagrangians of the type (1.1). As observed in [8], the absolutely continuity of minimizers ensures that the Tonelli set E is a set of measure zero.

We remark that in spite of BMO regularity of minimizers that we proved in Theorem 3.3, we substantially cannot prove any further property of the Tonelli set, in fact we will show the following

Theorem 4.1. *Any closed set of measure zero of I is the singular set of an absolute minimizer of some functional corresponding to a suitable Lagrangian L verifying* (1.1).

We will prove Theorem 4.1 by constructing functionals with prescribed Tonelli closed set of measure zero. The corresponding examples given by Davie [11] are not of our type: his examples are either with L having not superlinear growth, or with L having a non positive asimptotic behaviour.

The examples we give are of the type of [1], and are issued by using some results of Harmonic Analysis about the (A_p) class of weights of Muckenhoupt (see [24]).

Let us recall some basic definitions: the Hardy-Littlewood local maximal function of a function $f \in L^1_{loc}(I)$ is defined by

$$Mf(t) = \sup_{\substack{t\in J \\ J\subset I}} \fint_J f(t)dt$$

and the (A_p) condition of Muchenhoupt is defined by

$$f \in (A_p) \quad \Leftrightarrow \quad \sup_{J\subset I} \left(\fint_J f(t)dt\right)\left(\fint_J f(t)^{-\frac{1}{p-1}}dt\right)^{p-1} \; < \; +\infty \qquad (p>1),$$

$$f \in (A_1) \quad \Leftrightarrow \quad \exists k>0 \quad \text{such that } \sup_{J\subset I} \quad \fint_J f(t)dt \quad \le \quad k \quad \operatorname*{ess\,inf}_J f$$

where each supremum is taken over all intervals $J \subset I$. It can be easily shown that if $1 \le r < s$, the class (A_r) is contained in the class (A_s). We will use the following well known results, the first of which is by Coifman-Rochberg [7, 24]:

Theorem 4.2. *If $f \ge 0$, $f \in L^1_{loc}(I)$, then for each $0 \le \epsilon < 1$ we have $(Mf)^\varepsilon \in A_1$.*

Theorem 4.3. *If $f \ge 0$, $f \in L^1_{loc}(I)$, then $\log f \in BMO(I)$ if and only if there exists $\eta > 0$ such that $f^\eta \in (A_2)$.*

Proof of Theorem 4.1: Let E be a closed subset of I having measure zero, let $U_0 = I$, and let $U_1, U_2, ..., U_n, ...$ a sequence of open subsets of $\mathbb{R}$ such that $|U_n| \le 2^{-n}$, $\overline{U_{n+1}} \subset U_n$ and $\bigcap\limits_{n=1}^{\infty} U_n = E$, and let $f = \sum\limits_{n=0}^{\infty} \chi_{U_n \cap I}$.

We have

$$\int_I f(t)dt = \sum_{n=0}^{\infty} |U_n \cap I| \ \leq \ \sum_{n=0}^{\infty} 2^{-n} = 2 < +\infty \tag{4.1}$$

and therefore $f \in L^1(I)$, $f = +\infty$ on E, $1 \leq f < +\infty$ on $I \backslash E$. The Hardy-Littlewood local maximal function of f is therefore such that $Mf = +\infty$ on E (because $Mf(t) \geq f(t)$ a. e. in I). Now we claim that $Mf < +\infty$ on $I \backslash E$. In fact, let $t_0 \in I \backslash E$. Since

$$I \backslash E = I \backslash \bigcap_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} I \backslash U_n,$$

there exists $M \in \mathbb{N}$ such that $t_0 \notin U_M$, and therefore $t_0 \notin \overline{U_{M+1}}$. Let $\mathcal{I}$ be an open interval centered in t_0 such that $\mathcal{I} \cap \overline{U_{M+1}} = \emptyset$. We have $\mathcal{I} \cap U_n = \emptyset \quad \forall n > M$ and therefore $\chi_{U_n \cap I}(t) = 0 \quad \forall t \in \mathcal{I}, \quad \forall n > M$. From this it follows that $f(t) \leq M \quad \forall t \in \mathcal{I}$ and therefore

$$⨍_J f(t)dt \leq M$$

for any interval $J \subset \mathcal{I}$.

Now let J be a fixed interval in I containing t_0, it is $|J| < \frac{1}{2}|\mathcal{I}|$ or $|J| \geq \frac{1}{2}|\mathcal{I}|$. In the first case we have $J \subset \mathcal{I}$ and therefore $⨍_J f(t)dt \leq M$, in the second case we have

$$⨍_J f(t)dt \leq \frac{2}{|\mathcal{I}|} \int_I f(t)dt < \infty.$$

In this way we get $Mf < +\infty$ on $I \backslash E$.

Now let us note that also the function $(Mf)^{\frac{1}{2}}$ is such that $(Mf)^{\frac{1}{2}} = +\infty$ on E, $(Mf)^{\frac{1}{2}} < +\infty$ on $I \backslash E$, and by Theorem 4.2 the function $(Mf)^{\frac{1}{2}}$ belongs to the (A_2) class of Muckenhoupt. Put

$$v = \log Mf,$$

we have $v \geq 0$ because $f \geq 1$, $v = +\infty$ on E, $v < +\infty$ on $I \backslash E$ and by Theorem 4.3 $v \in BMO(I)$. By using Remark 2.7, there exists $k > 0$ such that

$$⨍_J v(t)^2 dt \leq \left(⨍_J v(t)dt \right)^2 + k$$

for any interval $J \subset I$. The function v satisfies the condition (see [1]) to be the derivative of a minimizer of some functional corresponding to a suitable Lagrangian L verifying (1.1), and therefore we get the assertion. $\square$

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