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Are Some Optimal Shape Problems Convex?

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The optimal shape problem in this paper is to construct plates or beams of minimal weight. The thickness $u(x)$ is variable, but the vertical deformation $y(x)$ should not exceed a certain threshhold. The functions u and y are related to each other via the differential equation $\Delta(bu^p\Delta y) = f$, see (1.2) below. We investigate under which boundary conditions on y the class of admissible thickness functions u is convex. In two out of three cases we give a positive answer, contrary to the common belief that these optimal shape problems are nonconvex. Moreover, under one type of boundary condition, the answer is different for beam and plate. Nonconvexity is shown by means of counterexamples which were found using MAPLE.

Keywords: optimal shape, plate equation, convexity, maximal deformation, beam

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1. Introduction

Subject of this paper is to investigate the convexity of some optimal shape problems, in which the control variable $u(x)$ typically represents the thickness of a beam or plate occupying $\Omega \subset \mathbb{R}^n$, and $y(x)$ stands for its deformation. One wants to minimize the total mass

$$m(u) = \int_\Omega u(x)\ dx \tag{1.1}$$

subject to certain constraints on y. In general these problems are believed to be nonconvex.

However, there are situations in which they are convex after all, and we shall exhibit some of those situations. To be precise we shall assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and that the deformation of a beam ($n = 1$) or plate ($n = 2$) is governed by the equation

$$\Delta(bu^p\Delta y) = f \le 0 \quad \text{in } \Omega \quad , \tag{1.2}$$

which has to be understood in the sense of distributions, i.e.

$$\int_\Omega bu^p\Delta y\Delta\varphi = \int_\Omega f\cdot\varphi \quad \text{for every } \varphi \in C_0^\infty(\Omega) \quad , \tag{1.3}$$

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and where $p \geq 1$, $b > 0$ are constants. $f \in L^1(\Omega)$ represents a load, and the thickness function $u(x) \in L^\infty(\Omega)$ satisfies the constraint

$$0 < a \leq u(x) \leq M \quad . \tag{1.4}$$

The downward deformation of the beam or plate is not allowed to exceed a certain magnitude, i.e. there is a positive constant r such that

$$y(x) \geq -r \quad . \tag{1.5}$$

Notice that (1.5) is an implicit constraint on the set of admissible controls $u(x)$.
We shall suppose that $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and that $y(x)$ satisfies one of the following three kinds of boundary conditions (1.6)–(1.8)

(1.6) $\quad y = \Delta y = 0$ on $\partial\Omega$ $\quad$ (hinged beam or plate)

(1.7) $\quad y = \frac{\partial y}{\partial \nu} = 0$ on Γ_1 , $\quad y_{\nu\nu} = \frac{\partial}{\partial \nu}(bu^p \Delta y) = 0$ on Γ_2
(cantilever beam or partly clamped plate)

(1.8) $\quad y = \frac{\partial y}{\partial \nu} = 0$ on $\partial\Omega$ $\quad$ (clamped beam or plate)

Our main result can be stated as follows.

Theorem 1.1.
a) *Problem* (1.1)–(1.5) *is convex for boundary conditions* (1.6) *of Navier type.*
b) *Problem* (1.1)–(1.5) *is convex for boundary conditions* (1.7) *of mixed type and* $n = 1$, *but in general nonconvex for* $n \geq 2$.
c) *Problem* (1.1)–(1.5) *is in general nonconvex for boundary conditions* (1.8) *of Dirichlet type.*

For the proof we shall try to show that constraint (1.5) is convex and we shall use the Poisson representation formula.

Let us recall some related results from the literature. In [6] Problem (1.1)–(1.5) is studied for $n = 1$ and $p = 3$ and declared as nonconvex under any of the three boundary conditions (1.6), (1.7) or (1.8). Our result shows that the convexity of the problem depends on the type of boundary condition and on the dimension n of the problem. In [1] Problem (1.1)–(1.5) is studied for $p = 2$ and under boundary condition (1.8), but under the additional constraint

$$|\nabla u(x)| \leq C \quad . \tag{1.9}$$

This constraint was introduced for mathematical convenience, because it lead to a compactness property of the set of admissible designs u.
We tried to find out why many colleagues working in optimal shape design believe that problems of this nature are nonconvex. Tracing back the literature we found a paper of Velte and Villaggio [7], who studied a second order problem

$$-(buy')' = f \quad \text{in } (0,1) \tag{1.10}$$

under Dirichlet boundary conditions

$$y(0) = y(1) = 0 \tag{1.11}$$

or mixed Dirichlet-Neumann boundary conditions

$$y(0) = y'(1) = 0 \quad . \tag{1.12}$$

They showed existence of a solution to problem (1.1) (1.4) (1.10) (1.11), but now constraint (1.5) was replaced by

$$|y'(x)| \leq C \quad . \tag{1.13}$$

Constraint (1.13) is nonconvex as stated by [7], see Remark 4.6 at the end of our paper. If however (1.11) is replaced by (1.12) the constraint (1.13) becomes convex, see Remark 4.7.
Other functionals than (1.1) have been considered for instance in [2] or [5]. If y_0 is a given state and ν a nonnegative constant, let

$$J(y,u) := \int_\Omega \{|y-y_0|^2 + \nu u^2\} dx \quad . \tag{1.14}$$

Now suppose that u satisfies (1.4) and (1.9), that y satisfies a version of (1.10) (1.11), namely

$$-\mathrm{div}(u \nabla y) = f \;\text{ in } \Omega \;, \qquad y = g \text{ on } \partial\Omega$$

and that (1.5) is replaced by

$$|y(x)| \leq \delta \quad . \tag{1.15}$$

Then existence of an optimal control was shown in [2]. If however, $\nu = 0$ and assumptions (1.9) and (1.15) are dropped, there does not exist an optimal solution as shown in [5]. Variations of this theme are also contained in [3,4].
Our paper is organized as follows: Section 2 contains proofs of the convexity and Section 3 proofs of the nonconvexity results of Theorem 1.1. Open problems, related observations and remarks are listed in Section 4.

2. Some optimal shape problems are convex

In this section we prove the first half of our main result.
a) Consider first the case of boundary condition (1.6) and introduce

$$g(x) := bu^p(x)\Delta y(x)$$

as a new function. Then g solves

$$-\Delta g(x) = -f(x) \geq 0 \;\text{ in } \Omega \tag{2.1}$$

and

$$g(x) = 0 \;\text{ on } \partial\Omega \quad . \tag{2.2}$$

Consequently, by the Poisson formula,

$$g(x) = -\int_\Omega G(x,z) f(z) dz \geq 0 \quad , \tag{2.3}$$

where G in Green's function for the Laplace operator on Ω under Dirichlet boundary conditions. But by definition of g, (2.3) and (1.6)

$$-\Delta y(x) = -\frac{1}{bu^p(x)}g(x) \leq 0 \ \text{ in } \Omega \tag{2.4}$$

and

$$y(x) = 0 \ \text{ on } \partial\Omega \ . \tag{2.5}$$

Another application of Poisson's formula gives

$$y(x) = -\frac{1}{b}\int_\Omega G(x,z)\frac{g(z)}{u^p(z)}dz \ . \tag{2.6}$$

Now suppose that u_0 gives rise to deformation y_0, so u_0 is mapped to y_0, that u_1 is mapped to y_1 and that $u_\lambda = \lambda u_1 + (1-\lambda)u_0$ is mapped to y_λ. We observe that under each of those controls u_0, u_1 and u_λ the auxiliary function g is the same nonnegative function. Moreover, the mapping $t \to -t^{-p}$ is concave for $t \in [a, M]$, so that

$$\begin{aligned} y_\lambda(x) &= \frac{1}{b}\int_\Omega G(x,z)\left(\frac{-1}{u_\lambda^p(z)}\right)g(z)dz \\ &\geq \frac{1}{b}\int_\Omega G(x,z)\left(\frac{-\lambda}{u_1^p(z)} - \frac{(1-\lambda)}{u_0^p(z)}\right)g(z)dz \\ &\geq \lambda y_1(x) + (1-\lambda)y_0(x) \geq -r \ . \end{aligned} \tag{2.7}$$

This shows that the constraint (1.5) is convex and proves Theorem 1.1 a).
b) Let us now suppose that $n = 1$, $\Omega = (0,1)$ and that boundary conditions (1.7) hold:

$$y(0) = y'(0) = 0 \ , \tag{2.8}$$

$$y''(1) = (bu^py'')'(1) = 0 \ . \tag{2.9}$$

In this case obvious integrations and (2.9) lead to

$$bu^py''(\zeta) = \int_\zeta^1\int_\eta^1 f(\xi)d\xi d\eta \leq 0 \tag{2.10}$$

so that $y(x)$ is concave. Two more integrations and use of (2.8) leads to

$$y(x) = \frac{1}{b}\int_0^x\int_0^w u^{-p}(\zeta)\int_\zeta^1\int_\eta^1 f(\xi)d\xi d\eta \ d\zeta dw \ . \tag{2.11}$$

Since the integrand is concave in u we can argue as in a) and conclude that the first part of Theorem 1.1 b) holds.

3. Some optimal shape problems are nonconvex

In this section we shall provide examples for which the constraint (1.5) is nonconvex.

First we consider Problem (1.1)–(1.5) under boundary condition (1.8) in the case $b = n = 1$. Example 3.1 proves Theorem 1.1 c).

Example 3.1. Let $\Omega = (0,1)$, let $d \in (0, \frac{1}{2})$ be a parameter and u_0, u_1 and $u_{1/2}$ be the following thickness functions

$$u_0(x,d) := c \cdot \chi_{[0,d)}(x) + 1 \cdot \chi_{[d,1]}(x) \quad , \tag{3.1}$$

$$u_1(x,d) := 1 \cdot \chi_{[0,1-d]}(x) + c \cdot \chi_{(1-d,1]}(x) \quad , \tag{3.2}$$

$$u_{1/2}(x,d) := [u_0(x,d) + u_1(x,d)]/2 \quad . \tag{3.3}$$

Notice that u_0 is nonsymmetric, that u_1 is its reflection and that $u_{1/2}$ is a convex combination of u_0 and u_1. Choose $f(x) = -\delta_{1/2}(x)$, a symmetric point load in the center of Ω. Then for given p and d one can explicitly integrate (1.2) (1.8) and one can compare $A := \min_{x\in[0,1]} y_{1/2}(x,d)$ with $B := \min_{x\in[0,1]} y_1(x,d)$ to check (1.5). Here y_λ is the deformation caused under control u_λ, $\lambda = 0, 1, 1/2$.
The following table shows values of p, c and d for which $A < B$ and provides four counterexamples to the convexity of constraint (1.5).

p	3	2	1.5	1.2
c	$\sqrt[3]{100}$	$\sqrt[2]{100}$	$\sqrt[1.5]{200}$	$\sqrt[1.2]{100,000}$
d	0.45	0.45	0.475	0.4995
B	$-4.25155 \cdot 10^{-4}$	$-4.25155 \cdot 10^{-4}$	$-2.19496 \cdot 10^{-4}$	$-4.65267 \cdot 10^{-7}$
A	$-6.21587 \cdot 10^{-4}$	$-4.89005 \cdot 10^{-4}$	$-2.21025 \cdot 10^{-4}$	$-4.69828 \cdot 10^{-7}$
$B - A$	$1.96431 \cdot 10^{-4}$	$6.38491 \cdot 10^{-5}$	$1.52878 \cdot 10^{-5}$	$4.56055 \cdot 10^{-9}$

Table 1: Nonconvexity of (1.5)

The numbers in Table 1 were obtained from looking at plots of $u_0(x,d)$ and $\min_{x\in[0,1]} u_{1/2}(x,d)$.

If the first of these functions exceeds the second for some (x,d), we have a counterexample, see Figure 1, which was constructed using MAPLE.

The computations leading to Table 1 show that for $p \searrow 1$ the set of parameters (c,d) for which the problem is nonconvex becomes smaller and smaller and seems to converge to $(\infty, 1/2)$. Recall that c had to be less than M. In this context we refer also to Open Problem 4.1 of this paper.

It remains to prove the second half of Theorem 1.1 b). Example 3.2 will provide a counterexample to the convexity of constraint (1.5) if $n \geq 2$ and if Problem (1.1)–(1.5) is studied under boundary conditions (1.7).

Example 3.2. Let $\Omega = (0,1) \times (0,1)$, $\Gamma_1 := \{(x_1,x_2) \mid x_1 \in \{0,1\},\ x_2 \in (0,1)\}$ and $\Gamma_2 := \partial\Omega \setminus \Gamma_1$. Then we have Dirichlet conditions on the vertical parts of the boundary. If the load $f(x_1,x_2)$ is independent of x_2, so is the solution of (1.2) (1.7). Therefore we can modify Example 3.1. We set

$$f(x_1,x_2) := -\delta_{\{1/2,x_2\}}(x) \quad , \tag{3.4}$$

and

$$u_\lambda(x_1,x_2) := u_\lambda(x_1) \quad , \qquad \lambda \in \{0,1,1/2\} \ . \tag{3.5}$$

Now we can use Table 1 again to conclude that constraint (1.5) is nonconvex, since the deformations $y_\lambda(x_1,x_2)$ are independent of x_2. This concludes the proof of Theorem 1.1 .

Figure 1: $y_0(x,d)$ and $\min_{x\in[0,1]} y_{1/2}(x,d)$

4. Open problems, observations and remarks

Open Problem 4.1. As stated at the end of Example 3.1, we were not able to produce a simple counterexample to the convexity of constraint (1.5) when $p = n = 1$ and under the boundary condition (1.8). Therefore the convexity of constraint (1.5) remains an open problem.

Open Problem 4.2. Consider again problem (1.1)–(1.5) under boundary condition (1.8) and for $\Omega = (0,1)$, $p \geq 1$, but under the additional constraint that the load $f(x)$ is symmetric, i.e. $f(x) = f(1-x)$ and that the control $u(x)$ is symmetric. In this situation we could not produce a counterexample to nor prove convexity of (1.5).

Observation 4.3. The introduction of symmetry can sometimes generate convexity. Let $\Omega \in \mathbb{R}^2$ be a thin annulus of large diameter, $\Omega = \{x \in \mathbb{R}^2 \mid 0 < R < |x| < R+\varepsilon\}$ with $R \gg 1$, and suppose that Γ_1 is the outer and Γ_0 the inner boundary of Ω or vice versa. Moreover assume that f and u are radially symmetric. Then solving (1.1)–(1.5) under boundary condition (1.7) is essentially a one-dimensional problem. In view of Theorem 1.1 b) we can expect (1.5) to be a convex constraint. This observation supports the convexity conjecture in Open Problem 4.2.

Observation 4.4. In Section 2 we have shown more than we needed. If u_0 and u_1 are two controls and u_λ their convex combination, then the corresponding supports states y_0, y_1 and

y_λ satisfy the pointwise inequality

$$y_\lambda(x) \geq \lambda y_1(x) + (1-\lambda) y_0(x) \quad . \tag{4.1}$$

In fact, the convexity statements of Theorem 1.1 remain true if the nonlinearity u^p is replaced by a more general function $h(u)$ with the properties that h is positive and $-h^{-1}(u)$ is concave on $[a, M]$. This follows from an inspection of (2.7) and (2.11).
To prove convexity of (1.5) we need only a comparison of the L^∞-norms.

Observation 4.5. One might be tempted to believe in the following qualitative statement: The thicker the beam, the smaller its deformation. In other words, if $u_0(x) \leq u_1(x)$, then $|y_0(x)| \leq |y_1(x)|$. This statement is wrong as can be seen from the following example of the beam under Dirichlet boundary conditions, i.e. Problem (1.2)(1.8) with $f(x) = -\delta_{1/2}(x)$:

$$u_0(x) := 40 \cdot \chi_{[0,0.46]}(x) + 1 \cdot \chi_{(0.46,1]}(x) \quad ,$$
$$u_1(x) := 40 \cdot \chi_{[0,0.5]}(x) + 1 \cdot \chi_{(0.5,1]}(x) \quad ,$$

The minima of the functions $y_0(x)$ and $y_1(x)$ are -0.000855 and -0.000865, and therefore the stronger beam u_1 can lead to a larger deformation y_1 (in L^∞). The qualitative shapes of y_0 and y_1 are plotted in Figure 2.

Figure 2: $y_0(x)$ and $y_1(x)$

Remark 4.6. Problem (1.1) (1.4) (1.10) (1.13) of Villaggio and Velte is in general nonconvex under boundary condition (1.11). For the proof of this fact choose constants $C = 1/100$, $b = 100$, $M \geq 100$ and $a \leq 1$, load $f(x) = 2\delta_{1/2}(x)$ and control functions

$$u_0(x) = 1 \quad \text{and} \quad u_1(x) = \chi_{[0,1/2]}(x) + \frac{3}{2}\,\chi_{(1/2,3/4)}(x) + 100\,\chi_{[3/4,1]}(x) \quad .$$

Then u_0 and u_1 are admissible, but $u_{1/2}(x) = [u_0(x) + u_1(x)]/2$ is not admissible because the constraint

$$|y'(x)| \le \frac{1}{100} \tag{1.13}$$

is violated for the state $y_{1/2}$ corresponding to $u_{1/2}(x)$. In fact, a straightforward but lengthy calculation shows that equality holds a.e. in (1.13) for y_0 and that y_1 satisfies (1.13), too, while $y'_{1/2}(x) = 0.01135 > 1/100$ for $x \in (1/2, 3/4)$.

Remark 4.7. Problem (1.1) (1.4) (1.10) (1.13) of Villagio and Velte is convex under boundary condition (1.12). To prove this one integrates (1.10) and obtains

$$bu(x)y'(x) = -\int_0^x f(z)dz + A$$

where A is a constant. A second integration gives

$$y(x) = \frac{1}{b}\int_0^x \frac{1}{u(\xi)}(A - \int_0^\xi f(z)dz)d\xi + B \quad .$$

The boundary conditions $y(0) = y'(1) = 0$ imply

$$A = \int_0^1 f(\xi)d\xi \quad \text{and} \quad B = 0 \quad ,$$

so that

$$u(x)y'(x) = \frac{1}{b}\int_x^1 f(\zeta)d\zeta \tag{4.2}$$

and

$$y(x) = \frac{1}{b}\int_0^x \frac{1}{u(\xi)}\int_\xi^1 f(z)dzd\xi \quad . \tag{4.3}$$

Now we are in a situation to show that constraint (1.13) is convex. If u_0 and u_1 satisfy (1.1) (1.4) (1.10) (1.13) and (1.12), we know from (1.13) and (4.2) that

$$u_i(x) \ge \max\left\{a \ ; \ \frac{1}{b \cdot C} \,|\int_x^1 f(\zeta)d\zeta|\right\} \quad , \tag{4.4}$$

for $i \in \{0, 1\}$. Consequently (4.4) holds for $i \in [0, 1]$ and

$$u_\lambda = \lambda u_1(x) + (1 - \lambda)u_0(x) \quad .$$

But then we know from (4.2) that $|y'_\lambda(x)| \le C$, where y_λ is the state corresponding to u_λ. This proves the convexity of constraint (1.13).

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