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Decompositions of Compact Convex Sets

Diethard Pallaschke
Institut für Statistik und Mathematische Wirtschaftstheorie der Universität Karlsruhe, Kaiserstr. 12, 76128 Karlsruhe, Germany.
e-mail: lh09@rz.uni-karlsruhe.de

Ryszard Urbański
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland.
e-mail: rich@math.amu.edu.pl

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In a recent paper R. Urbański [13] investigated the mimimality of pairs compact convex sets which satisfy additional conditions, namely the minimal convex pairs. In this paper we consider some different possibilities of decomposing a given compact convex set into smaller compact convex sets which are related by translations or by reflections. Combining our results with the characterization of minimality of convex pairs of compact convex sets given in [13] we prove in the second part of this paper that for the two-dimensional case the following statements:

M: *equivalent minimal pairs of compact convex sets are uniquely determined up to translations* (see [3], [11])

CM: *equivalent convex minimal pairs of compact convex sets are uniquely determined up to translations* (see [13])

are equivalent.

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1. Introduction

In this paper we consider the Rådström-Hörmander lattice [4] of equivalence classes of pairs of nonempty compact convex sets. As in [6] we denote for a real topological vector space X the set of all nonempty compact convex subsets by $\mathcal{K}(X)$ and the set of all pairs of nonempty compact convex subsets by $\mathcal{K}^2(X)$, i.e. $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by: "$(A,B) \sim (C,D)$ *if and only if* $A + D = B + C$" using the Minkowski sum, and a partial order is given by the relation: "$(A,B) \leq (C,D)$ *if and only if* $A \subseteq C$ *and* $B \subseteq D$." The space $\mathcal{K}^2(X)$ has been investigated in series of papers, (see for instance [3], [6], [7], [8], [9], [11], [13]). Pairs of compact convex sets arise in quasidifferential calculus as the sub- and superdifferentials of the directional derivative of a quasidifferentiable function and in formulas for

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the numerical evaluation of the Aumann-Integral (see [2] and [1]).
Let us fix some notations: Let X be a real topological vector space, and X^* be the space of all continuous real valued linear functionals. For two compact convex sets $A, B \in \mathcal{K}(X)$ we will use the notation

$$A \vee B := \operatorname{conv}(A \cup B),$$

where the operation "conv" denotes the convex hull. With $\overline{A}$ we denote the closure of a set A. During the proofs, an easy identity for compact convex sets, which was first observed by A. Pinsker [10] will be used frequently, namely: For $A, B, C \in \mathcal{K}(X)$ we have:

$$(A + C) \vee (B + C) = C + (A \vee B).$$

We will use the abbreviation $A+B\vee C$ for $A+(B\vee C)$ and $C+d$ for $C+\{d\}$ for compact convex sets A, B, C and a point d. Moreover we will write $[a, b]$ instead of $\{a\} \vee \{b\}$.

Finally let us state explicitely the order cancellation law (see [4], [12]).

Let X be real topological vector space and $A, B, C \subseteq X$ compact convex subsets. Then the inclusion

$$A + B \subseteq A + C$$

implies

$$B \subseteq C.$$

If X is a real topological vector space and $A, B \subseteq X$ are compact convex subsets, then A is called a *"summand"* of B if there exists a $C \in \mathcal{K}(X)$ with $A + C = B$.

Thus from the algebraic point of view the set $\mathcal{K}(X)$ of all nonempty compact convex subsets of a real topological vector space X is an *ordered commutative semigroup* $(\mathcal{K}(X), \star, \preceq)$ *with cancellation property* such that for every finite subset the maximum exists. If the partial order $\preceq$ is given by the inclusion and if the semigroup operation by the (formal multiplication) $\star$ with:

$$A \star B := A + B,$$

then we obtain from Pinsker's formula, that the (formal) multiplication and the partial order are related by $A \star (B \vee C) = (A \star B) \vee (A \star C)$.

Within this context, the elements of $\mathcal{K}^2(X)$ with respect to the relation $\sim$ can be considered as *fractions.*

There are interesting sub-semigoups of $\mathcal{K}(X)$, namely the semigroup $\mathcal{P}(X)$ consisting of all polytopes, the semigroup $\mathcal{SK}(X)$ consisting of all strict compact convex sets and for $X = \mathbb{R}^n$ the semigroup $\mathcal{B}(X)$ consisting of all closed Euclidean balls.

It has been shown in [13] that a pair (A, B) is convex if and only if $A \vee B$ is a summand of $A + B$. This characterization of convex pairs in terms of the Minkowski sum plays an essential role in the sequel.

More precisely: A pair $(A, B) \in \mathcal{K}^2(X)$ is called *"convex"* if $A \cup B$ is a convex set. If $B \subset A$ or $A \subset B$, then the pair (A, B) is called *"monotone"*. Every monotone pair is also a convex pair. Furthermore a pair $(A, B) \in \mathcal{K}^2(X)$ is called *"minimal"* if and only if for every equivalent pair $(C, D) \in \mathcal{K}^2(X)$ the relation $(C, D) \leq (A, B)$ implies $C = A$ and $B = D$ and analogously we say that a convex pair $(A, B) \in \mathcal{K}^2(X)$ is *"minimal convex"* if and only if for every equivalent convex pair $(C, D) \in \mathcal{K}^2(X)$ the relation $(C, D) \leq (A, B)$ implies $C = A$ and $B = D$.

In [8] the following notation was introduced: For $A, B, S \in \mathcal{K}(X)$, we say that S *"separates"* the sets A and B if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \emptyset$. In this notation the following result was proved in [13].

Theorem 1.1. *Let X be a real topological vector space and $A, B \in \mathcal{K}(X)$. Then the following statements are equivalent:*

(i.) *The set $A \cup B$ is convex*
(ii.) *The set $A \cap B$ separates the sets A and B*
(iii.) *The set $A \vee B$ is a summand of the set $A + B$*
(iv.) $A + B = A \vee B + A \cap B$ *and* $A \cap B \neq \emptyset$.

Let us remark, that if $A \cap B$ separates the sets A and B, the basic relationship between the Minkowski sum, the convex hull and the intersection is given by property iv) of Theorem 1.1.

The algebraic analogue of this formula is, that the product of two integers $a, b \in \mathbb{N}$ is equal to the product of its *greatest common divisor* $d(a, b)$ with its *smallest common multiplier* $m(a, b)$. i.e. $a \cdot b = d(a, b) \cdot m(a, b)$, where the order $\preceq$ in $\mathbb{N}$ is given by: $n \preceq m \iff n$ divides m. For this order maximum and minimum exist and are given for $k, l \in \mathbb{N}$ by $k \vee l = m(k, l)$ and $k \wedge l = d(k, l)$.

2. Algebraic Decomposition of Compact Convex Sets

In this section we prove that every summand of a polytope is again a polytope.

Theorem 2.1. *Let (X, τ) be a topological vector space and $A \subset X$ a polytope. Then every summand of A is also a polytop.*

Proof. Since $A \subset X$ is a polytope, i.e.,

$$A = \operatorname{conv}\{a_1, ..., a_n\}, \quad a_1, ..., a_n \in X$$

it is contained in a finite-dimensional subspace $Y \subset X$ on which the induced topology is locally convex. Hence for every index $i \in \{1, ..., n\}$ there exists a continuous linear functional $f_i \in Y^*$ such that $a_i = H_{f_i}(A) = \{x \in A \mid f_i(x) = \sup_{x' \in A} f_i(x')\}$.

Let us assume, that $A = B + C$. Since for every $i \in \{1, ..., n\}$ we have:

$$a_i = H_{f_i}(A) = H_{f_i}(B) + H_{f_i}(C)$$

it follows that for every index $H_{f_i}(B)$ and $H_{f_i}(C)$ are points. Hence for every index $i \in \{1, ..., n\}$ we have:

$$a_i = H_{f_i}(A) \in \operatorname{conv}\{H_{f_1}(B), H_{f_2}(B), ..., H_{f_n}(B)\} + H_{f_i}(C).$$

Let us put

$$B' := \operatorname{conv}\{H_{f_1}(B), H_{f_2}(B), ..., H_{f_n}(B)\}$$

and

$$C' := \operatorname{conv}\{H_{f_1}(B), H_{f_2}(B), ..., H_{f_n}(B)\},$$

then for every index $i \in \{1, ..., n\}$ we have:

$$a_i = H_{f_i}(A) \in B' + H_{f_i}(C),$$

and by Pinsker's formula we obtain:

$$\begin{aligned} A &\subset (B' + H_{f_1}(C)) \vee (B' + H_{f_2}(C)) \vee .... \vee (B' + H_{f_n}(C)) \\ &= (B' + H_{f_1}(C) \vee H_{f_2}(C)) \vee .... \vee (B' + H_{f_n}(C)) \\ &= (B' + H_{f_1}(C) \vee H_{f_2}(C) \vee ... \vee H_{f_n}(C)) = B' + C' \subset A. \end{aligned}$$

Therefore

$$A = B' + C' \;=\; B + C \;\subset\; B + C',$$

and from the order law of cancellation it follows that

$$C \subset C' \quad \text{and} \quad B \subset B'.$$

Hence we obtain that $C = C'$ and $B = B'$ and since B' and C' are polytopes the theorem is proved. □

Remark 2.2. Observe that in the semigroup of polytopes every element can only be decomposed by polytopes, i.e, for every $A \in \mathcal{P}(X)$ the equation $A = B + C$ considered as an equation in $\mathcal{K}(X)$ in the variables B and C has only solutions in $\mathcal{P}(X)$. In this sense the semigroup of polytopes is algebraic closed with respect to the Minkowski sum. The same is true for the semigroup $\mathcal{SK}(X)$ consisting of all strict compact convex sets, but not for the semigroup $\mathcal{B}(X)$ consisting of all closed Euclidean balls for $X = \mathbb{R}^n$, see for instance the example B) in [7]. Thus the sub-semigroups $\mathcal{P}(X)$ and $\mathcal{SK}(X)$ of $\mathcal{K}(X)$ behave as the multiplicative semigroup $(2 \cdot \mathbb{N} + 1, \,\cdot\,)$ of odd integers which is an algebraic closed sub-semimigroup $(\mathbb{N}, \,\cdot\,)$ of integers.

3. Special Set Theoretic Decompositions of Compact Convex Sets

In this section we characterize compact convex sets which can be represented in a special way as the union of other compact convex sets. We begin with the following proposition:

Proposition 3.1. *Let (X, τ) be a topological vector space $x \in X$ and $A, B \in \mathcal{K}(X)$. Then $A \cup (B + x)$ is convex if and only if $A \vee (B + x)$ is a summand of $A + B$.*

Proof. From Theorem 1.1 it follows that $A \cup (B + x)$ is convex if and only if $A \vee (B + x)$ is a summand of $A + B + x$. But this is equivalent to the fact that $A \vee (B + x)$ is a summand of $A + B$. □

Corollary 3.2. *Let (X, τ) be a topological vector space, $x \in X$ and $A \in \mathcal{K}(X)$. Then $A \cup (A + x)$ is convex if and only if the interval $I = [0, x]$ is a summand of A.*

Proof. By assumption we have $A \cup (A+x) = A \vee (A+x)$. From Pinsker's formula follows that $A \vee (A+x) = (A + \{0\}) \vee (A + \{x\}) = A + [0, x]$. Now for $B = A$ we deduce from Proposition 3.1 that there exists a $C \in \mathcal{K}(X)$ such that

$$A + A \;=\; A \vee (A + x) + C \;=\; A + [0, x] + C,$$

and hence by the order cancellation law we get that $A = [0, x] + C$.
The inverse implication is obvious. □

Corollary 3.3. *Let (X, τ) be a topological vector space. If $A \in \mathcal{K}(X)$ and interval $I = [a, b]$ is summand of A then $A \cup (A + b - a)$ is convex.*

Proof. Let $A = [a, b] + B$ for some $B \in \mathcal{K}(X)$. Then $A - a \;=\; [0, b - a] + B$ and this implies that that $A \;=\; [0, b - a] + B + a$. Hence by Corollary 3.2 for $x = b - a$ we get that $A \cup (A + b - a)$ is convex. □

Let us put:

$$\mathcal{K}_t(X) := \{A \in \mathcal{K}(X) |\text{ there exists } \; B \in \mathcal{K}(X) \text{ and } x \in X \setminus \{0\} \text{ with } A = B \cup (B + x)\}.$$

Proposition 3.4. *Let (X, τ) be a topological vector space. Then $A \in \mathcal{K}_t(X)$ if and only if there exists a non trivial interval I for which I is summand of A.*

Proof. *Necessity* " $\Rightarrow$:" Let $A = B \cup (B + x)$ for some $B \in \mathcal{K}(X)$ and $x \in X$ be convex. Then clearly $A = B \vee (B + x) = B + [0, x]$ and $I := [0, x]$ is summand of A.
Sufficiency. " $\Leftarrow$" Let $I = [a, b]$ an interval with $a \neq b$ which is a summand of A. Then $A = B + [a, b]$ for some $B \in \mathcal{K}(X)$.

Let us denote by

$$A_1 := \; B + [a, \frac{a+b}{2}] \;=\; B + b + [a - b, \frac{a-b}{2}]$$

and put

$$I_1 := \; [a - b, \frac{a-b}{2}] \quad \text{ and } \; x_0 := \frac{b-a}{2}.$$

Then I_1 is a summand a A_1 and by using Corollary 3.3 we obtain that $A_1 \cup (A_1 + \frac{b-a}{2})$ is convex. Hence

$$\begin{aligned} A_1 \cup (A_1 + x_0) &= A_1 \vee (A_1 + x_0) \;=\; A_1 + \{0\} \vee \{x_0\} \;=\; A_1 + [0, x_0] \\ &= B + [a, \frac{a+b}{2}] + [0, \frac{b-a}{2}] \;=\; B + a + [0, \frac{b-a}{2}] + [0, \frac{b-a}{2}] \\ &= B + a + [0, b - a] = B + [a, b] \;=\; A. \end{aligned}$$

□

Let us put:

$$\mathcal{K}_s(X) := \{A \in \mathcal{K}(X) \;|\; \text{ there exists } \; B \in \mathcal{K}(X) \; \text{ with } \; A = B \cup (-B) \text{ and } B \neq A\}.$$

Proposition 3.5. *Let (X, τ) be a locally convex topological vector space and $A \in \mathcal{K}(X)$ with* card$A > 1$. *Then $A \in \mathcal{K}_s(X)$ if and only if $A \ = \ -A$.*

Proof. *Necessity.* " $\Rightarrow$" If $A \in \mathcal{K}_s(X)$ then $A = B \cup (-B)\}$ for some $B \in \mathcal{K}(X)$ and hence $A \ = \ -A$.

Sufficiency. " $\Leftarrow$" Let $A \in \mathcal{K}(X)$ with card$A > 1$. Hence there exists an element $x_0 \in A$ with $x_0 \neq 0$. Now choose an $f \in X^*$ with $f(x_0) > 0$ and put $B := \{a \in A \mid f(a) \geq 0\}$. Obviously $B \neq \emptyset$, $B \neq A$ and $A = B \cup (-B)$. □

4. Equivalence of Minimality and Convex Minimality for the case of dimension ≤ 2

We begin with the following observation:

Proposition 4.1. *Let (X, τ) be a locally convex topological vector space, $f \in X^*$ and $A, B, K \in \mathcal{K}(X)$ such that $H_f(A) = A$, $H_f(B) = \{b\}$ and $H_f(K) = \{k\}$. Then the pair $(A, B + K)$ is minimal.*

Proof. Let $(A', B') \leq (A, B + K)$ and (A', B') be equivalent to $(A, B + K)$. Then

$$A + B' \ = \ B + K + A'.$$

By assumption there exists a functional $f \in X^*$ such that

$$H_f(A) = A, \quad H_f(B) = \{b\} \ \text{ and } \ H_f(K) = \{k\}.$$

Hence from

$$H_f(A) + H_f(B') \ = \ H_f(B + K) + H_f(A')$$

and $H_f(B + K) = \{k + b\}$ it follows that for every $b' \in H_f(B')$

$$A + b' - x \subset A', \quad \text{ with } \quad x = k + b.$$

Therefore we obtain $b' - x = 0$ and $A = A'$ and from the law of cancellation $B' = B + K$. □

Let us now formulate the following hypothesis:

M: *equivalent minimal pairs of compact convex sets are uniquely determined up to translations*

CM: *equivalent convex minimal pairs of compact convex sets are uniquely determined up to translations*

MM: *equivalent monotone minimal pairs of compact convex sets are uniquely determined up to translations*

Theorem 4.2. *For $X = \mathbb{R}^n$ with $n \leq 2$ the hypothesis* M, CM *and* MM *are equivalent.*

Proof. We will first show the equivalence of M and MM

"M $\Longrightarrow$ MM" Let $X = \mathbb{R}^n$ with $n \leq 2$, $A, B \in \mathcal{K}(X)$ and suppose that the pair (A, B) is monotone, i.e., $B \subset A$. From the order cancellation law follows that every pair (C, D)

which is equivalent to (A,B) is also monotone. Hence the hypothesis M implies the hypothesis MM.

"MM $\Longrightarrow$ M" Let $X = \mathbb{R}^n$ with $n \leq 2$, $A,B,C,D \in \mathcal{K}(X)$ and suppose that (A,B) and (C,D) are equivalent minimal pairs.
We have to show that hypothesis M holds.
Therefore we consider the following cases:

i) Let us consider the case where one of the sets A,B,C,D has a nonempty interior. For instance, suppose that *int* $A \neq \emptyset$.

   Then there exists a $\lambda > 0$ such that $A \subset B + \lambda \cdot B$. From $A + D = B + C$ follows that $A + D \subset B + D + \lambda \cdot B$ and hence we obtain that $B + C \subset B + D + \lambda \cdot B$. From the order cancellation law we deduce that $C \subset D + \lambda \cdot B$.
   Now let (C', D') be an equivalent minimal pair to $(C, D + \lambda \cdot B)$ with $C' \subset C$ and $D' \subset D + \lambda \cdot B$.
   First observe that the pair $(A, B + \lambda \cdot B)$ (see [6]) is a minimal pair equivalent to (C', D'). Since $A \subset B + \lambda \cdot B$ we have $C' \subset D'$. From the assumption that equivalent convex minimal pairs are related by translations we get $C' = A + x$ and $D' = B + \lambda \cdot B + x$. But $C' \subset C$ and hence we obtain that $A + x \subset C$ and $B + x \subset D$. Since the pair (A,B) is minimal, it follows that $A + x = C$ and $B + x = D$.

ii) All sets A,B,C,D have empty interior.

   a) If A,B are parallel intervals then there exists a point $y \in X$ such that $A \subset B + y$. Since (A,B) and (C,D) are equivalent and minimal and therefore related by translations, we conclude that $C \subset D + y$. Since the pairs $(A, B + y)$ and $(C, D + y)$ are also minimal and equivalent, there exists a point $x \in X$ such that $A = C + x$ and $B + y = D + y + x$ and thus we conclude that $A = C + x$ and $B = D + x$.
   Although it is not used in this proof, let us mention the following fact: If $A = [a_1, a_2]$ is parallel to a shorter interval than $B = [b_1, b_2]$ then (A,B) is equivalent to $(a_1, [b_1, b_2 - a_2 + a_1])$ so that by minimality of (A,B) the interval $A = [a_1, a_1]$ reduces to a singleton.

   b) If A,B are not parallel then there exists a ball $K = B(0,r)$ such that $A \subset B + K$. Hence $C \subset D + K$ and by Proposition 4.1 $(A, B + K)$ and $(C, D + K)$ are minimal pairs equivalent to (A,B). Hence $A = C + x$ and $B + K = D + K + x$ which implies $B = D + x$.

Finally we prove the equivalence of M and CM: In [13] it is shown that hypothesis M implies CM. Since every equivalent pair of a monotone pair is also monotone and since monoton pairs are convex, we deduce that the hypothesis CM implies hypothesis MM. Now the implication MM $\Longrightarrow$ CM can be shown in the same way as in the proof Theorem 4.9 in [13]. This completes the proof of the equivalences. □

5. Examples

Example 5.1. In [13] Theorem 4.5 it was shown, that for an arbitrary topological vector space every class $[A,B]$ contains a minimal convex pair (A_0, B_0).

From the fact that a convex pair is minimal, it does not follow that the whole class is convex. More pecisely: We give now an example of a pair (A_0, B_0) which is minimal and minimally convex and for which there exists an equivalent pair which is not convex.

Let $X := \mathbb{R}^2$ and $A, B, E, E', F, F' \in \mathcal{K}(X)$ as indicated in Figure 1. Let us put $C := A \cup E,\ D := B \cup F$ and $C' := A \cup E',\ D' := B \cup F'$. Then we have:

$$(A, B) \sim (C, D) \sim (C', D')$$

where (A, B) is minimal and convex and the pairs (C, D) and (C', D') are nonconvex.

Obviously the pair (A, B) is convex and the minimality follows from Theorem 2.1 in [7]. The equivalences

$$(A, B) \sim (C, D) \sim (C', D')$$

can be shown as follows:
We have:

$$C \;=\; A \cup E, \quad D \;=\; B \cup F \quad \text{and} \quad E \;=\; F - x.$$

Observe that

$$1.)\ D - x \;=\; (B - x) \cup E \qquad \text{and} \qquad 2.)\ C \;=\; A \cup E$$

are convex, hence it follows:

$$3.)\ B - x + E \;=\; E \cap (B - x) \;+\; (D - x)$$

$$4.)\ A + E \;=\; A \cap E \;+\; A \vee E \;=\; A \cap E \;+\; C.$$

But

$$A \cap E \;=\; E \cap (B - x) \;=\; [-x, 0] \;=: I$$

and therefore

$$\begin{aligned} B + E \; &= I + D \\ A + E \; &= I + C \end{aligned}$$

which implies

$$B + C \;=\; A + D$$

i.e., $(A, B) \sim (C, D)$.

Example 5.2. If a pair (A, B) consists of strictly convex sets, then an equivalent minimal pair does not necessarily consist of strictly convex sets as the following example shows:

Let $X := \mathbb{R}^2$ and $A_0, B_0, C \in \mathcal{K}(X)$ as indicated in Figure 2. Let us put $A := A_0 \cup C$, $B := B_0 \cup C$. Then the pairs (A, B) and (A_0, B_0) are equivalent, the pair (A_0, B_0) is minimal and does not consist of strictly convex sets, while the pair (A, B) consists of strictly convex sets.

The equivalence of $(A, B) \sim (A_0, B_0)$ can be seen as follows:

Since the sets $A := A_0 \cup C$, $B := B_0 \cup C$ are convex, we have:

$$\begin{aligned} A_0 + C \; &= A + A_0 \cap C \\ B_0 + C \; &= B + B_0 \cap C, \end{aligned}$$

hence $A_0 + B \;=\; B_0 + A$ and therefore $(A_0, B_0) \sim (A, B)$.

The minimality of (A_0, B_0) follows from Lemma 5.1 in [3] which was proved by J. Grzybowski and can be used for a characterization of minimality for the two-dimensional case.

An alternative to Example 5.2 is to take a ball $K = B(0, r)$ and the sets A, B from Example 5.1. Then $A + K$ and $B + K$ are strictly convex and (A, B) is a minimal pair equivalent to $(A + K, B + K)$.

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