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An Elementary Proof of Komlós-Révész Theorem in Hilbert Spaces

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1. Introduction

The Komlós theorem [19] first proved in 1967 has many applications. (See, for example, [2], [4], [5], [6], [9], [10], [11], [15], [18], [27]). A general interpretation is given by Aldous [1]. Several proofs have been given: Chatterji ([13], [14]); Trautner [26]. Its extension to L^1_X where X is an infinite dimensional Banach space has been studied in two directions: one with respect to the weak $\sigma(X, X')$ convergence [3] and [23], the other for the strong convergence by [25] in a unsuccessful way as pointed in [17] in which one finds a strong version of Komlós theorem in super-reflexive Banach spaces ([17], Theorem 6). Extension of Garling's theorem to Mosco convergence for convex weakly random sets in separable super-reflexive Banach spaces is stated recently by [12]. Let us mention also a recent extension of Slenk's theorem [24] via a regular method of summability [7] in L^1_H where H is a Hilbert space.

In this paper, we aim to give an elementary proof of Komlós theorem in Hilbert space for the strong convergence as well as we give a new proof of Révész's theorem [22]. All we need are some truncation techniques in the seminal proof of Komlós theorem [19] that will be summarized in two lemmas below. Apart from these facts our proof involves only elementary mathematical tools and does not appeal to martingale techniques as it was done in ([17], [19], [25]).

2. Notations and Preliminaries

We will use the following notations.

- $(\Omega, \mathcal{F}, \mu)$ is a complete probability space.
- X is a Banach space.

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- $L^1_X = L^1_X(\Omega, \mathcal{F}, \mu)$ is the Banach space of all Bochner integrable mappings $f : \Omega \to X$.
- H is a Hilbert space.
- $L^2_H = L^2_H(\Omega, \mathcal{F}, \mu)$ is the Hilbert space of all strongly measurable mappings $f : \Omega \to H$ such that $\|f(.)\|^2$ is μ-integrable.
- $\langle ., . \rangle$ is the inner product in the Hilbert spaces H and L^2_H.
- If $f : \Omega \to X$ is a mapping and a, b are positive real numbers, we denote:
  $\{\|f\| > a\} = \{\omega \in \Omega : \|f(\omega)\| > a\}$.
  $\{a \leq \|f\| < b\} = \{\omega \in \Omega : a \leq \|f(\omega)\| < b\}$.
  and we set: $F_a(f)(\omega) = f(\omega)$ if $\|f(\omega)\| < a$ and $F_a(f)(\omega) = 0$ if $\|f(\omega)\| \geq a$.

The following simple result is useful.

Lemma 2.1. *Let (x_n) be a weakly convergent sequence in X and x its limit. Then there exists an integer N such that*

$$\|x\| \leq 2 \inf_{n \geq N} \|x_n\|.$$

Proof. As (x_n) converges weakly to x, we have $\|x\| \leq \liminf_n \|x_n\|$. If $\liminf_n \|x_n\| = 0$, then the result is obvious. Now assume that $\liminf_n \|x_n\| > 0$, then

$$\|x\| < 2 \liminf_n \|x_n\| = \sup_{N \geq 1} 2 \inf_{n \geq N} \|x_n\|.$$

Hence there exists $N \geq 1$ such that $\|x\| \leq 2 \inf_{n \geq N} \|x_n\|$. □

The next lemma was used by Komlós. We extend it to Banach spaces and give a simple proof.

Lemma 2.2. *Let (u_n) be a bounded sequence in L^1_X and (v_n) a sequence in L^1_X. Suppose that, for each $k \in \mathbb{N}^*$, the sequence $(F_k(u_n))_n$ converges weakly to v_k in L^1_X. Then the sequence (v_n) converges strongly in L^1_X and μ-a.e.*

Proof. Put $v_0 = 0$. It is enough to prove that the series $\sum_{k \geq 1} \|v_k - v_{k-1}\|_1$ is convergent. For each $k \in \mathbb{N}^*$, the sequence $(F_k(u_n) - F_{k-1}(u_n))_n$ converges weakly in L^1_X to $v_k - v_{k-1}$. By Lemma 2.1, there exists $m_k \in \mathbb{N}^*$ such that:

$$\|v_k - v_{k-1}\|_1 \leq 2 \inf_{n \geq m_k} \|F_k(u_n) - F_{k-1}(u_n)\|_1.$$

Let $N \in \mathbb{N}^*$ and $n \geq \max(m_1, m_2, \ldots, m_N)$. Then

$$\begin{aligned} \sum_{k=1}^{N} \|v_k - v_{k-1}\|_1 &\leq 2 \sum_{k=1}^{N} \|F_k(u_n) - F_{k-1}(u_n)\|_1 \\ &\leq 2\|u_n\|_1 \\ &\leq 2 \sup_{p \geq 1} \|u_p\|_1 \end{aligned}$$

and therefore $\sum_{k=1}^{+\infty} \|v_k - v_{k-1}\|_1 \leq 2 \sup_{p \geq 1} \|u_p\|_1 < +\infty$. □

The proof of the next lemma is similar to that given by Komlós.

Lemma 2.3. *Let* (f_n) *be a bounded sequence in* $L^1_{\mathbb{R}_+}$. *Then there exists a subsequence* (g_n) *of* (f_n) *such that for each subsequence* (h_n) *of* (g_n):

(a) $\sum_{n\geq 1}(1/n^2)\|F_n(h_n)\|_2^2 < +\infty$;

(b) $\sum_{n\geq 1}\mu(\{h_n \geq n\}) < +\infty$.

Proof. Put M $= \sup_n ||f_n||_1$. For each integer $k \geq 1$ the real valued sequence

$$(\mu(\{k-1 \leq f_n < k\}))_n$$

is bounded. Then there exist subsequences (f_n^1), $(f_n^2), \ldots$, $(f_n^k), \ldots$ of (f_n), where (f_n^{k+1}) is a subsequence of (f_n^k), and a sequence $(p_k)_k$ in [0,1] such that

$$\forall k \geq 1, \lim_n \mu(\{k-1 \leq f_n^k < k\}) = p_k$$

and

$$\forall k \geq 1, \forall n \geq 1, \mu(\{k-1 \leq f_n^k < k\}) < p_k + \frac{1}{k^3}.$$

Put $g_n = f_n^{n^2}$ and let (h_n) be a subsequence of (g_n). Then

(i) $\forall k \geq 1, \lim_n \mu(\{k-1 \leq h_n < k\}) = p_k$,

and

(ii) $\forall n \in \mathbb{N}^*$, $\forall k$ with $1 \leq k \leq n^2$, $\mu(\{k-1 \leq h_n < k\}) < p_k + \frac{1}{k^3}$.

One has

$$\sum_{k=1}^{N} p_k = \lim_n \sum_{k=1}^{N} \mu(\{k-1 \leq h_n < k\}) \leq \mu(\Omega),$$

and

$$\sum_{k=1}^{N}(k-1)p_k = \lim_n \sum_{k=1}^{N}(k-1)\mu(\{k-1 \leq h_n < k\}) \leq M.$$

Then $\sum_{k=1}^{+\infty} p_k \leq 1$ and $\sum_{k=1}^{+\infty}(k-1)p_k \leq M$.

(a) One has

$$\|F_n(h_n)\|_2^2 \leq \sum_{k=1}^{n} k^2\mu(\{k-1 \leq h_n < k\}) < \sum_{k=1}^{n} k^2(p_k + \frac{1}{k^3}).$$

Hence

$$\begin{aligned}
\sum_{n\geq 1}\frac{1}{n^2}\|F_n(h_n)\|_2^2 &< \sum_{n\geq 1}[\frac{1}{n^2}\sum_{k=1}^{n}k^2(p_k+\frac{1}{k^3})] \\
&= \sum_{k\geq 1}[k^2(p_k+\frac{1}{k^3})\sum_{n=k}^{+\infty}\frac{1}{n^2}] \\
&\leq 2\sum_{k\geq 1}k(p_k+\frac{1}{k^3}) \\
&= 2(\sum_{k\geq 1}(k-1)p_k+\sum_{k\geq 1}p_k+\sum_{k\geq 1}\frac{1}{k^2}) \\
&\leq 2(M+1+2) < +\infty.
\end{aligned}$$

(b) One has

$$\begin{aligned}
\mu(\{h_n\geq n\}) &= \sum_{k=n+1}^{n^2}\mu(\{k-1\leq h_n<k\})+\mu(\{h_n\geq n^2\}) \\
&< \sum_{k=n+1}^{n^2}(p_k+\frac{1}{k^3})+\frac{1}{n^2}\|h_n\|_1 \\
&\leq \sum_{k\geq n+1}p_k+\sum_{k\geq n+1}\frac{1}{k^3}+\frac{M}{n^2} \\
&\leq \sum_{k\geq n+1}p_k+\frac{1}{2n^2}+\frac{M}{n^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n\geq 1}\mu(\{h_n\geq n\}) &< \sum_{n\geq 1}[\sum_{k\geq n+1}p_k+\frac{2M+1}{2n^2}] \\
&= \sum_{k\geq 2}(k-1)p_k+\sum_{n\geq 1}\frac{2M+1}{2n^2} < +\infty.
\end{aligned}$$

□

3. Proofs of Komlós and Révész theorems in Hilbert spaces

Theorem 3.1. *Let (f_n) be a bounded sequence in L^1_H. Then there exists a subsequence (g_n) of (f_n) and a μ-integrable function g such that*

$$\lim_k \frac{1}{k}\sum_{n=1}^{k}h_n(\omega)=g(\omega)\quad \mu\text{-a.e.}$$

for each subsequence (h_n) of (g_n).

Proof. Step 1. We may assume without loss of generality that (f_n) is a sequence of simple functions. To see this, take for each n a simple function f'_n such that the sequence $(f_n - f'_n)$ converges strongly in L^1_H and μ-a.e. to the null function. Then (f'_n) is bounded in L^1_H and for every subsequence (h_n) of (f_n), we have

$$\frac{1}{k}\sum_{n=1}^{k} h_n(\omega) = \frac{1}{k}\sum_{n=1}^{k}(h_n - h'_n)(\omega) + \frac{1}{k}\sum_{n=1}^{k} h'_n(\omega)$$

where $h'_n = f'_{p_n}$ if $h_n = f_{p_n}$. So, $\frac{1}{k}\sum_{n=1}^{k} h_n(\omega)$ converges μ-a.e. if and only if $\frac{1}{k}\sum_{n=1}^{k} h'_n(\omega)$ converges μ-a.e. because $\frac{1}{k}\sum_{n=1}^{k}(h_n - h'_n)(\omega)$ converges to 0 μ-a.e., and, when the convergence occurs, they have the same limit function.

Using Lemma 2.3 and extracting a subsequence if necessary, we may suppose that

$$\sum_{n\geq 1}\frac{1}{n^2}\|F_n(h_n)\|_2^2 < +\infty \quad \text{and} \quad \sum_{n\geq 1}\mu(\{\|h_n\| \geq n\}) < +\infty$$

for every subsequence (h_n) of (f_n). For each integer $k \in \mathbb{N}^*$, the sequence $(F_k(f_n))_{n\geq 1}$ is bounded in L^2_H, and therefore, is weakly relatively compact. Then there exist subsequences (f^k_n) of (f_n) and functions u_k of L^2_H such that (f^{k+1}_n) is a subsequence of (f^k_n) and, for each $k \geq 1$ we have

$$\lim_n F_k(f^k_n) = u_k$$

with respect to the weak topology $\sigma(L^2_H, L^2_H)$, and, using Lemma 2.1, we can have

$$\|u_k\|_2 \leq 2\|F_k(f^k_n)\|_2, \quad \forall n \geq 1.$$

Put $f'_n = f^n_n \quad (n \geq 1)$. Then

$$\lim_n F_k(f'_n) = u_k$$

for $\sigma(L^2_H, L^2_H)$, and

$$\|u_k\|_2 \leq 2\|F_k(f'_n)\|_2, \qquad \forall n \geq k.$$

Put $\varepsilon_k = 1/2^k$ $(k \geq 1)$. Since $\|u_k\|_\infty \leq k$, for each integer $k \geq 1$, there exists a simple function v_k such that

$$\|v_k\|_\infty \leq k,$$
$$\lim_k (u_k(\omega) - v_k(\omega)) = 0 \ \mu\text{-a.e.}$$
$$\|u_k - v_k\|_2 \leq \min(\inf_{n\geq k}\|F_k(f'_n)\|_2, \frac{\varepsilon_{k-1}}{4k}).$$

Observe that if $\inf_{n\geq k}\|F_k(f'_n)\|_2 = 0$, then $u_k = 0$ and therefore we can take $v_k = 0$. Then we have

$$\begin{aligned}\|v_k\|_2 &\leq \|v_k - u_k\|_2 + \|u_k\|_2\\ &\leq \|F_k(f'_n)\|_2 + 2\|F_k(f'_n)\|_2, \quad \forall n \geq k\\ &= 3\|F_k(f'_n)\|_2, \quad \forall n \geq k.\end{aligned}$$

Moreover

$$\begin{aligned}\|F_k(f'_n) - v_k\|_2 &\leq \|F_k(f'_n)\|_2 + \|v_k\|_2\\ &\leq 4\|F_k(f'_n))\|_2, \quad \forall n \geq k.\end{aligned}$$

If (h_n) is a subsequence of (f'_n), then $\sum_{k\geq 1}(1/k^2)\|F_k(h_k) - v_k\|_2^2 < +\infty$ because $\sum_{k\geq 1}(1/k^2)\|F_k(h_k)\|_2^2 < +\infty$.

Step 2. Let $\mathcal{F}_n$ be the smallest sub-σ-algebra of $\mathcal{F}$ such that the functions $f'_1, \ldots f'_n$; $v_1, \ldots v_n$ are $\mathcal{F}_n$-measurable. Then $\mathcal{F}_n$ is finite because all these functions are simple functions. Let us prove that there exist integers $n_1 < n_2 < \ldots < n_p < n_{p+1} < \ldots$ such that $n_1 = 1$ and, for all $p \geq 2$ and all $2 \leq k \leq p$:

$$\sup_{1\leq q\leq p-1}\ \sup_{1\leq l\leq k-1}\ \sup_{B\in\mathcal{F}_{n_{p-1}}} |\langle 1_B(F_l(f'_{n_q}) - v_l), (F_k(f'_{n_p}) - v_k)\rangle| < \varepsilon_{k-1}. \tag{$*$}$$

We proceed by recurrence. Let us suppose $p \geq 2$ and that $n_1 < n_2 < \ldots < n_{p-1}$ have been obtained. Then, for each $k \in \{2, \ldots, p\}$, the sequence $(F_k(f'_n) - u_k)_n$ converges weakly to 0 in L^2_H, and, the set

$$\{1_B(F_l(f'_{n_q}) - v_l) : 1 \leq q \leq p-1,\ 1 \leq l \leq k-1,\ B \in \mathcal{F}_{n_{p-1}}\}$$

is a finite set of L^2_H. Then there exists an integer N_k such that, for all $n \geq N_k$ we have

$$\sup_{1\leq q\leq p-1}\ \sup_{1\leq l\leq k-1}\ \sup_{B\in\mathcal{F}_{n_{p-1}}} |\langle\ 1_B(F_l(f'_{n_q}) - v_l), (F_k(f'_n) - u_k)\rangle| < \frac{\varepsilon_{k-1}}{2}.$$

Let $n_p > \max(n_{p-1}, N_2, \ldots, N_p)$, then for each $2 \leq k \leq p, 1 \leq q \leq p-1,\ 1 \leq l \leq k-1$ and $B \in \mathcal{F}_{n_{p-1}}$, we have

$$\begin{aligned}|\langle 1_B(F_l(f'_{n_q}) - v_l), (F_k(f'_{n_p}) - v_k)\rangle| &\leq |\langle 1_B(F_l(f'_{n_q}) - v_l), (F_k(f'_{n_p}) - u_k)\rangle|\\ &\quad + |\langle 1_B(F_l(f'_{n_q}) - v_l), u_k - v_k\rangle|\\ &< \frac{\varepsilon_{k-1}}{2} + \|F_l(f'_{n_q}) - v_l\|_2\|u_k - v_k\|_2\\ &\leq \frac{\varepsilon_{k-1}}{2} + 2l\frac{\varepsilon_{k-1}}{4k}\\ &\leq \varepsilon_{k-1}.\end{aligned}$$

Then $(*)$ is proved.

Put $g_p = f'_{n_p}$ $(p \geq 1)$. Then the sequence $(g_n)_{n\geq 1}$ is bounded in L^1_H and, for each $k \geq 1$, the sequence $(F_k(g_n))_{n\geq 1}$ converges weakly in L^1_H to u_k. By Lemma 2.2, there exists $g \in L^1_H$ such that (u_k) converges to g μ-a.e.

Let (h_n) be a subsequence of (g_n) and let us prove that $\frac{1}{k}\sum_{n=1}^{k} h_n(\omega)$ converges to $g(\omega)$ μ-a.e.

Put $S_k = \sum_{n=1}^{k}(1/n)(F_n(h_n) - v_n)$ and let us first show that $(S_k)_k$ converges μ-a.e. Let $\varepsilon > 0$ and denote, for $m \in \mathbb{N}^*$,

$$A_m = \{\sup_{j\geq 1}\|S_{m+j} - S_m\| \leq \varepsilon\}.$$

Applying the Cauchy criterion, it is enough to show that $\lim_m \mu(A_m) = 1$. Denote

$$A_{m,0} = \Omega \quad , \quad A_{m,k} = \{ \sup_{1 \le j \le k} \|S_{m+j} - S_m\| \le \varepsilon \}$$

$$B_{m,k} = A_{m,k-1} - A_{m,k} = \{ \sup_{1 \le j \le k-1} \|S_{m+j} - S_m\| \le \varepsilon, \ \|S_{m+k} - S_m\| > \varepsilon \}.$$

Then $(A^c_{m,n})_{n \ge 1}$ is an increasing sequence of measurable sets, $\bigcup_{n \ge 1} A^c_{m,n} = A^c_m$ and $(B_{m,k})_{1 \le k \le n}$ is a $\mathcal{F}$-partition of $A^c_{m,n}$.

One has

$$\begin{aligned}
\|1_{B_{m,k}}(S_{m+n} - S_m)\|_2^2 &= \|1_{B_{m,k}}(S_{m+k} - S_m) + 1_{B_{m,k}}(S_{m+n} - S_{m+k})\|_2^2 \\
&\ge \|1_{B_{m,k}}(S_{m+k} - S_m)\|_2^2 \\
&\qquad + 2\langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle \\
&\ge \varepsilon^2 \mu(B_{m,k}) + 2\langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\|S_{m+n} - S_m\|_2^2 &\ge \sum_{k=1}^{n} \|1_{B_{m,k}}(S_{m+n} - S_m)\|_2^2 \\
&\ge \varepsilon^2 \mu(A^c_{m,n}) + 2\sum_{k=1}^{n} \langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle.
\end{aligned}$$

But

$$\|S_{m+n} - S_m\|_2^2 = \|\sum_{k=1}^{n} \frac{F_{m+k}(h_{m+k}) - v_{m+k}}{m+k}\|_2^2 = \sum_{k=1}^{n} \frac{\|F_{m+k}(h_{m+k}) - v_{m+k}\|_2^2}{(m+k)^2}$$

$$+ \ 2 \sum_{1 \le i < j \le n} \frac{1}{m+i}\frac{1}{m+j}\langle F_{m+i}(h_{m+i}) - v_{m+i}, F_{m+j}(h_{m+j}) - v_{m+j}\rangle.$$

Whence we get the following estimation

$$\begin{aligned}
\mu(A^c_{m,n}) \le \frac{1}{\varepsilon^2}[&\sum_{k=1}^{n} \frac{1}{(m+k)^2}\|F_{m+k}(h_{m+k}) - v_{m+k}\|_2^2 \\
&+ 2 \sum_{1 \le i < j \le n} \frac{1}{m+i}\frac{1}{m+j}|\langle F_{m+i}(h_{m+i}) - v_{m+i}, F_{m+j}(h_{m+j}) - v_{m+j}\rangle| \\
&+ 2\sum_{k=1}^{n} |\langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle|].
\end{aligned}$$

Now we check that $\lim_{m\to\infty}(\lim_{n\to\infty} \mu(A^c_{m,n})) = 0$. We shall use the foregoing majorations.

(i) As the series $\sum_{k\geq 1}(1/k^2)\ \|F_k(h_k)-v_k\|_2^2$ is convergent, then $\sum_{k=1}^{n}(1/(m+k)^2)$ $\|F_{m+k}(h_{m+k})-v_{m+k}\|_2^2$ converges to 0 when $m\to+\infty$ uniformly in n.

(ii) To get the following estimation put for $1\leq i<j\leq n : h_{m+j}=f'_{n_p}, k=m+j, h_{m+i}=f'_{n_q}, l=m+i$ and $B=\Omega$. Then $p\geq 2$ and, as (h_n) is a subsequence of (f'_{n_p}), then $2\leq k\leq p$. We have easily $1\leq q\leq p-1$ and $1\leq l\leq k-1$. We can then apply $(*)$. This gives

$$\sum_{1\leq i<j\leq n}\frac{1}{m+i}\,\frac{1}{m+j}|\langle F_{m+i}(h_{m+i})-v_{m+i},F_{m+j}(h_{m+j})-v_{m+j}\rangle|$$

$$\begin{aligned}\leq \sum_{1\leq i<j\leq n}\frac{1}{m+i}\,\frac{1}{m+j}\,\varepsilon_{m+j-1} &\leq \frac{1}{m}\sum_{2\leq j\leq n}\ \sum_{1\leq i\leq j-1}\frac{1}{m+j}\,\varepsilon_{m+j-1}\\ &=\frac{1}{m}\sum_{2\leq j\leq n}\frac{j-1}{m+j}\,\varepsilon_{m+j-1}\\ &\leq\frac{1}{m}\sum_{2\leq j\leq n}\varepsilon_{m+j-1}\leq\frac{1}{m}\,\varepsilon^{-m}\end{aligned}$$

which converges to 0 when $m\to+\infty$ uniformly in n.

(iii) Now observe that the sets $B_{m,k}$ belong to the smallest sub-σ-algebra Σ of $\mathcal{F}$ such that the functions $h_1,\dots,h_{m+k};v_1,\dots,v_{m+k}$ are Σ-measurable. Then by using similar arguments to those given before and by taking $B=B_{m,k}$, we can apply $(*)$ to obtain the following estimation

$$\begin{aligned}&|\langle 1_{B_{m,k}}(S_{m+k}-S_m),(S_{m+n}-S_{m+k})\rangle|\\ &\qquad=|\langle 1_{B_{m,k}}\sum_{i=m+1}^{m+k}\frac{F_i(h_i)-v_i}{i},\sum_{j=m+k+1}^{m+n}\frac{F_j(h_j)-v_j}{j}\rangle|\\ &\qquad\leq\sum_{i=m+1}^{m+k}\ \sum_{j=m+k+1}^{m+n}\frac{1}{i}\frac{1}{j}|\langle 1_{B_{m,k}}(F_i(h_i)-v_i),F_j(h_j)-v_j\rangle|\\ &\qquad\leq\sum_{i=m+1}^{m+k}\ \sum_{j=m+k+1}^{m+n}\frac{1}{i}\frac{1}{j}\varepsilon_{j-1}.\end{aligned}$$

By adding from $k = 1$ to n we get

$$\begin{aligned}
\sum_{k=1}^{n} |\langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle| &\leq \sum_{k=1}^{n} \sum_{i=m+1}^{m+k} \sum_{j=m+k+1}^{m+n} \frac{1}{i}\,\frac{1}{j}\,\varepsilon_{j-1} \\
&\leq \sum_{k=1}^{n} \sum_{i=m+1}^{m+k} \frac{1}{i}\,\frac{1}{m+k+1} \sum_{j=m+k+1}^{m+n} \varepsilon_{j-1} \\
&\leq \sum_{k=1}^{n} \sum_{i=m+1}^{m+k} \frac{1}{i}\,\frac{1}{m+k+1}\,\frac{1}{2^{m+k-1}} \\
&\leq \sum_{k=1}^{n} \frac{1}{m+k+1} \sum_{i=m+1}^{m+k} \frac{1}{m+1}\,\frac{1}{2^{m+k-1}} \\
&= \sum_{k=1}^{n} \frac{k}{m+k+1}\,\frac{1}{m+1}\,\frac{1}{2^{m+k-1}} \\
&\leq \frac{1}{m+1}\,\frac{1}{2^{m-1}}.
\end{aligned}$$

Since the last term converges to 0 when $m \to +\infty$ uniformly in n, $(S_k)_{k\geq 1}$ converges μ-a.e. Therefore from Kronecker's lemma, we deduce that $\frac{1}{k}\sum_{n=1}^{k}(F_n(h_n) - v_n)$ converges to 0 μ- a.e.

We have

$$\frac{1}{k}\sum_{n=1}^{k} h_n = \frac{1}{k}\sum_{n=1}^{k}(F_n(h_n) - v_n) + \frac{1}{k}\sum_{n=1}^{k}(v_n - u_n) + \frac{1}{k}\sum_{n=1}^{k} u_n + \frac{1}{k}\sum_{n=1}^{k}(h_n - F_n(h_n))$$

and

$\frac{1}{k}\sum_{n=1}^{k}(F_n(h_n) - v_n)$ converges to 0 μ-a.e.,

$\frac{1}{k}\sum_{n=1}^{k}(v_n - u_n)$ converges to 0 μ -a.e.,

$\frac{1}{k}\sum_{n=1}^{k} u_n$ converges to g μ-a.e.,

$\frac{1}{k}\sum_{n=1}^{k}(h_n - F_n(h_n))$ converges to 0 μ a.e. because $\sum_{n\geq 1}\mu(\{h_n \geq n\}) < +\infty$.

Then $\frac{1}{k}\sum_{n=1}^{k} h_n$ converges to g μ-a.e. □

The following is an extension of Révész' theorem [22] to functions with values in a Hilbert space.

Theorem 3.2. *Let (f_n) be a bounded sequence in L^2_H. Then there exists a subsequence (g_n) of (f_n) and $f \in L^2_H$ such that if $\sum_{n\geq 1} a_n^2 < +\infty$, then $\sum_{n\geq 1} a_n(h_n - f)$ converges μ-a.e. for every subsequence (h_n) of (g_n).*

Proof. Without loss of generality, we may suppose that (f_n) converges weakly to f in L^2_H. Choose a sequence (f'_n) of simple functions with $\|f_n - f - f'_n\|_2 \leq \frac{1}{2^n}$. Then (f'_n) is bounded in L^2_H and converges weakly to 0. For every subsequence (h_n) of (f_n) we have

$$\sum_{n\geq 1} \|a_n(h_n - f - h'_n)\|_1 \leq \sum_{n\geq 1} \|a_n(h_n - f - h'_n)\|_2$$
$$\leq (\sup_{n\geq 1} |a_n|) \sum_{n\geq 1} \|(h_n - f - h'_n)\|_2 < +\infty$$

where $h'_n = f'_{p_n}$ if $h_n = f_{p_n}$. It follows that $\sum_{n\geq 1} a_n(h_n - f - h'_n)$ converges μ-a.e. So, $\sum_{n\geq 1} a_n(h_n - f)$ converges μ-a.e. if and only if, $\sum_{n\geq 1} a_n h'_n$ converges μ-a.e. We can then suppose that (f_n) is a sequence of simple functions converging weakly to 0 in L^2_H. Now we will use several notations and results in the proof of Theorem 3.1. Put $\varepsilon_k = \frac{1}{k2^k}$. Let $\mathcal{F}_n$ be the smallest sub-σ-algebra of $\mathcal{F}$ such that the functions $f_1, \ldots, f_n$ are $\mathcal{F}_n$-measurable. Then $\mathcal{F}_n$ is finite. As (f_n) converges weakly to 0 there exists a subsequence (f_{n_p}) of (f_n) where $f_{n_1} = f_1$ and, for all $p \geq 2$

$$\sup_{1\leq q\leq p-1} \sup_{B\in\mathcal{F}_{n_{p-1}}} |\langle 1_B f_{n_q}, f_{n_p}\rangle| < \varepsilon_{p-1} \tag{1}$$

Put $g_p = f_{n_p}$ $(p \geq 1)$ and let (h_n) be a subsequence of (g_n). Denote $S_k = \sum_{n=1}^k a_n h_n$ for every $k \geq 1$. We will prove that (S_k) converges μ-a.e by proceeding as in the proof of Theorem 3.1. Introducing the sets A_m, $A_{m,k}$ and $B_{m,k}$ as in the proof of Theorem 3.1, we have

$$\mu(A^c_{m,n}) \leq \frac{1}{\varepsilon^2}[\sum_{k=1}^n a^2_{m+k}\|h_{m+k}\|_2^2 + 2\sum_{1\leq i<j\leq n} |a_{m+i}a_{m+j}||\langle h_{m+i}, h_{m+j}\rangle|$$
$$+ 2\sum_{k=1}^n |\langle 1_{B_{m,k}}(S_{m+k} - S_m), (S_{m+n} - S_{m+k})\rangle|]$$

We have the following estimations

$$\sum_{k=1}^n a^2_{m+k}\|h_{m+k}\|_2^2 \leq \sup_{p\geq 1}\|h_p\|_2^2 \sum_{k=1}^n a^2_{m+k}. \tag{2}$$

and

$$\sum_{1\leq i<j\leq n} |a_{m+i}a_{m+j}||\langle h_{m+i}, h_{m+j}\rangle| \leq \sum_{1\leq i<j\leq n} |a_{m+i}||a_{m+j}|\varepsilon_{m+j-1}$$
$$\leq (\sup_{p\geq m} a_p)^2 \sum_{2\leq j\leq n}\sum_{1\leq i\leq j-1} \varepsilon_{m+j-i}$$
$$= (\sup_{p\geq m} a_p)^2 \sum_{2\leq j\leq n} (j-1)\varepsilon_{m+j-1}$$
$$\leq (\sup_{p\geq m} a_p)^2 \sum_{2\leq j\leq n} \frac{1}{2^{m+j-1}}. \tag{3}$$

and

$$
\begin{aligned}
&\sum_{k=1}^{n} |\langle 1_{B_{m,k}}(S_{m+k}-S_m),(S_{m+n}-S_{m+k})\rangle| \\
&\qquad \le \sum_{k=1}^{n}\sum_{i=m+1}^{m+k}\sum_{j=m+k+1}^{m+n} |a_i||a_j||\langle 1_{B_{m,k}}h_i, h_j\rangle| \\
&\qquad \le (\sup_{p\ge m} a_p)^2 \sum_{k=1}^{n}\sum_{i=m+1}^{m+k}\sum_{j=m+k+1}^{m+n} \frac{1}{(j-1)2^{j-1}} \\
&\qquad \le (\sup_{p\ge m} a_p)^2 \sum_{k=1}^{n}\sum_{i=m+1}^{m+k} \frac{1}{m+k}\,\frac{1}{2^{m+k-1}} \\
&\qquad = (\sup_{p\ge m} a_p)^2 \sum_{k=1}^{n} \frac{k}{m+k}\,\frac{1}{2^{m+k-1}} \\
&\qquad \le (\sup_{p\ge m} a_p)^2 \frac{1}{2^m}\sum_{k=1}^{n} \frac{1}{2^{k-1}}. \qquad (4)
\end{aligned}
$$

By (2), (3), (4), we see that $\mu(A^c_{m,n})$ converges to 0 when $m \to +\infty$ uniformly in n. That finishes the proof. □

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References

[1] D. J. Aldous: Limit theorems for subsequences of arbitrarily-dependent sequences of random variables, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 40 (1977) 59–82.

[2] A. Amrani, C. Castaing: Weak compactness in Pettis integration, 1996. (submitted)

[3] E. J. Balder: Infinite-Dimensional Extension of a theorem of Komlós, Probab. Theory Rel. Fields 81 (1989) 185–188.

[4] E. J. Balder: Unusual applications of a.e. convergence, In: G. A. Edgar, L. Sucheston (eds): Almost Everywhere Convergence, Academic Press, New York, 1989, 31–53.

[5] E. J. Balder: New sequential compactness results for spaces of scalarly integrable functions, J. Math. Anal. Appl. 151 (1990) 1–16.

[6] E. J. Balder: On equivalence of strong and weak convergence in L^1 spaces under extreme point condition, Israel J. Math. 75 (1991) 21–47.

[7] H. Benabdellah, C. Castaing: Weak compactness and convergences in $L^1_E(\mu)$, C. R. Acad. Sc. Paris, t. 321, Série I (1995) 165–170.

[8] H. Benabdellah, C. Castaing: Weak compactness and convergences in $L^1_E(\mu)$, Preprint Université Montpellier II (1995/03).

[9] A. V. Bukhvalov, G. Ya. Lozanovskii: On sets closed with respect to convergence in measure in spaces of measurable functions, Dokl. Akad. Nauk. SSSR 212 (1973) 1273–1275. English translation: Soviet Math. Dokl. 14 (1973) 1563–1565.

[10] A. V. Bukhvalov: Optimization without compactness and its applications, Operator Theory: Advances and Applications 75 (1995) 95–112.

[11] C. Castaing: Weak compactness and convergences in Bochner and Pettis integration, Vietnam J. Math. 3(3) (1996).

[12] C. Castaing, F. Ezzaki: Convergences of convex weakly compact random sets in super-reflexive Banach spaces, Preprint Université Montpellier II, 1996.

[13] S. D. Chatterji: A general strong law, Invent. Math. 9 (1970) 235–245.

[14] S. D. Chatterji: A subsequence principle theory, Jahresberichte der Deutschen Math. Vereinigung 87 (1985) 91–107.

[15] J. Diestel, W. M. Ruess, W. Schachermayer: Weak compactness in $L^1(\mu, X)$, Proc. AMS 118(2) (1993) 447–453.

[16] N. Etemadi: An elementary proof of the strong law of large numbers, Z. Wahrsch. Verw. Geb. 55 (1981) 119–122.

[17] D. J. H. Garling: Subsequence principles for vector-valued random variables, Math. Proc. Camb. Philos. Soc. 86 (1979) 301–311.

[18] A. Ionescu-Tulcea: On pointwise convergence, compactness and equicontinuity II, Advances Math. 12 (1974) 171–177.

[19] J. Komlós: A generalization of a problem of Steinhaus, Acta Math. Acad. Sci. Hungar. 18 (1967) 217–229.

[20] M. Loève: Probability theory, 4th ed., Springer, New York, 1977.

[21] G. Pisier: Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1975) 326–350.

[22] P. Révész: On a problem of Steinhaus, Acta. Math. Acad. Sci. Hungar. 16 (1965) 310–318.

[23] M. Saadoune: Une nouvelle extension en dimension infinie du Théorème de Komlós, Applications: Compacité faible dans L^1_X, convergence en mesure, 1996. Preprint (submitted)

[24] W. Szlenk: Sur les suites faiblement convergentes dans l'espace L, Studia Math. 25 (1965) 337–341.

[25] A. M. Suchanek: On almost sure convergence of Cesaro averages of subsequences of vector-valued functions, J. Multivariate Anal. 8 (1978) 589–597.

[26] R. Trautner: A new proof of the Komlós-Révész-theorem, Prob. Theory Rel. Fields 84 (1990) 281–287.

[27] M. Valadier: La loi forte des grands nombres comme conséquence d'autres théorèmes, Séminaire Analyse Convexe, Exposé 15 (1991).

[28] P. Walters: Ergodic theory, Lecture Notes in Mathematics 458, Springer-Verlag, Berlin, 1975.