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Existence of Regular Solutions for a One-Dimensional Simplified Perfect-Plastic Problem with a Unilateral Gradient Constraint

Thierry Astruc
Laboratoire d'Analyse Non-Linéaire Appliquée, UFR Sciences et Techniques
Université de Toulon et du Var, BP 132, 83130 La Garde cedex, France.
e-mail: astruc@univ-tlv.fr

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This work is devoted to the study of the existence of "regular" solutions for a one-dimensional problem with unilateral constrained gradient in Perfect-Plasticity.
The particularity of this problem consists in the fact, that the tools usually employed to prove the Inf-Sup equality between the displacement problem and the stress problem do not work.
In a first part, we establish this equality by the mean of a penalty method which employs the theory of the convex functions of measures.
In a second part, we find the regular limit loads, between which the displacement problem possesses at least a solution which is in $W^{1,1}$, verifiyng the boundary conditions and the constraint on the gradient. We give an example where these loads are infinite.

Ce travail est consacré à l'étude de l'existence de solutions régulières pour un problème unidimensionnel, avec un gradient contraint unilatéral en Plasticité Parfaite.
La particularité de ce problème réside en ceci que, les outils habituellement employés pour montrer l'égalité entre les problèmes en déplacement et en contrainte ne fonctionnent plus.
Dans une première partie, on établit cette égalité au moyen d'une méthode de pénalité, qui utilise la théorie des fonctions convexes de mesure.
Dans un deuxième temps, on trouve les charges limites régulières, en deça desquelles le problème en déplacement admet des solutions qui sont dans $W^{1,1}$, et qui vérifient les conditions aux bords et la contrainte sur le gradient. On donne aussi un exemple où ces charges limites sont infinies.

1. Introduction

This paper is devoted to the study of the regularity of the solutions of perfect-plastic problems, for a one-dimensional model with a unilateral constrained gradient. For previous works on plasticity, the reader can consult Suquet [16], Temam [17], Strang Temam [18] and Kohn Temam [13].
In these works, the authors have proved, for a model without constraint on the gradient that, when the load is not "too large", there exists a solution to the displacement problem in some weak sense. In particular, u belongs to $BD = \{u \in L^1(\Omega, \mathbb{R}^N), \epsilon_{ij}(u) \in M^1\}$ – M^1 is the space of bounded measures on Ω, and $\epsilon(u)$ is the symmetric part of the gradient – In the one-dimensional case, it reduces to $BV(]0,1[)$. Until now, only very few things are known about regularity or uniqueness.

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The first original result in that direction concerns the two dimensional case with a load $\lambda = 0$ and was proved by P. Sternberg, G. Williams and W. Ziemer [15]. Let us note that Kohn and Strang [12] have previously been able to construct a "regular" solution for the antiplane shear, in the two dimensional case, and for particular shape of Ω and particular boundary data. Moreover, the load λ was zero.
In a previous work [1,2], the author of this paper has established the existence of regular solutions for

$$Inf(P_\lambda) = \operatorname*{Inf}_{\substack{u\in W^{1,1}(]0,1[) \\ u(o)=\alpha \\ u(1)=\beta}} \left\{ \int_0^1 \Psi(u'(t))\,dt - \lambda \int_0^1 f(t)u(t)\,dt \right\}$$

for λ small enough; more precisely, he has proved the existence of regular limit loads $\underline{\lambda}_r, \bar{\lambda}_r$ such that, for every λ in the segment $\Lambda_r =]-\underline{\lambda}_r, \bar{\lambda}_r[$, $Inf(P_\lambda)$ possesses a solution in $W^{1,1}(]0,1[)$, which verify the boundary conditions.

Here we consider the following one-dimensional problem:

$$Inf(P_\lambda) = \operatorname*{Inf}_{\substack{u\in W^{1,1}(]0,1[) \\ u(o)=\alpha \\ u(1)=\beta \\ u'\leq 1}} \left\{ \int_0^1 \Psi(u'(t))\,dt - \lambda \int_0^1 f(t)u(t)\,dt \right\} \tag{1}$$

where Ψ is a convex continuous function which is at most linear at infinity and coercive on L^1, f is the load and λ is a parameter.
The relaxed form of this problem has been investigated by Bouchitte Suquet [3], and Buttazzo Faina [4,5].
The main result of this paper concerns the existence of regular limit loads for this problem.

Theorem 1.1. $\exists(\underline{\lambda}_r, \bar{\lambda}_r) \in (\overline{\mathbb{R}}^+)^2/$

1. $\forall\lambda \in]-\underline{\lambda}_r, \bar{\lambda}_r[$, *$Inf(P_\lambda)$ has at least a solution in $W^{1,1}(]0,1[)$, which verifies the boundary condition and the constraint on the gradient.*
2. $\Lambda_d = \{\lambda \notin]-\underline{\lambda}_r, \bar{\lambda}_r[$ */ $Inf(P_\lambda)$ have at least one regular solution.$\}$ is at most countable.*

In what follows, we will denote by "regular solution" of $Inf(P_\lambda)$, a solution in $W^{1,1}(]0,1[)$, which verifies the boundary conditions and the constraint on the gradient.
The plan of this paper is as follows:
In Section 2, we precise the assumptions on Ψ, and recall the variational formulations of the displacement and the stress problem.
In Section 3, we establish the Inf-Sup equality between the dual problems. Let us point out, that this result is not a direct corollary of the theory of Convex Analysis, and can be obtained by the mean of a penalty method.
In Section 4, we look for necessary and sufficient conditions to have a solution u in $W^{1,1}(]0,1[)$ which verifies the boundary conditions. For that aim, we use a change of variable in the stress problem. We give a sufficient condition on Ψ^* and f to ensure the existence of a "regular" solution u of the problem.
In Section 5, we introduce the regular limit loads. We will remark that they can be bounded or not, illustrating this by presenting classical mechanical examples.

2. Notations

Let us consider the problem:

$$Inf(P_\lambda) = \operatorname*{Inf}_{\substack{u\in W^{1,1}(]0,1[)\\ u(o)=\alpha\\ u(1)=\beta\\ u'\leq 1}} \left\{ \int_0^1 \Psi(u'(t))\,dt \, - \lambda \int_0^1 f(t)u(t)\,dt \right\} \tag{2}$$

where $f \in L^\infty(]0,1[)$ and

$$C_0(|\xi|-1) \leq \Psi(\xi) \leq C_1(|\xi|+1), \tag{H$_1$}$$

$$\Psi \text{ is convex} \tag{H$_2$}$$

$$\Psi(\xi) \underset{-\infty}{\sim} -\nu\xi \quad \nu \in \mathbb{R}^{*+}. \tag{H$_3$}$$

Let us introduce

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \overline{\mathbb{R}} \\ X &\longmapsto \begin{cases} \Psi(X) & \text{if } X \leq 1 \\ +\infty & \text{elsewhere,} \end{cases} \end{aligned}$$

and

$$Sup(P^*_\lambda) = \operatorname*{Sup}_{\substack{\sigma=-\lambda f\\ \sigma\geq -\nu}} \left\{ - \int_0^1 g^*(\sigma(t))\,dt \, + \beta\sigma(1) - \alpha\sigma(0) \right\}. \tag{3}$$

Remark 2.1. This problem is the weak variational formulation of the following formal equations and inequations of perfect plastic unilateral problem:

$$\begin{cases} u(0) = \alpha \\ u(1) = \beta \\ u' \in \partial g^*(\sigma) \\ \sigma' = -\lambda f \end{cases}$$

3. Convex Duality

Theorem 3.1. *Under the assumptions* (H$_1$)–(H$_3$), (P^*_λ) *is the dual of* (P_λ). *Furthermore,*

$$Sup(P^*_\lambda) = Inf(P_\lambda). \tag{4}$$

Proof. For the convenience of the reader, the proof of this theorem is divided into six steps:

First Step:
This first step consists in establishing that (P^*_λ) is the dual problem of (P_λ).
For that purpose, we will use the notations of I. Ekeland and R. Temam (cf. [10]):

Let V and V^*, Y and Y^* be two couples of dual spaces, $\Lambda : V \longrightarrow Y$ be a linear continuous operator, F and G be two convex functions:

$$F : V \longrightarrow \mathbb{R}$$
$$G : Y \longrightarrow \mathbb{R}$$

and F^* and G^* be their conjugate functions in the sense of Fenchel:

$$\forall p^* \in V^*,\ F^*(p^*) = \operatorname*{Sup}_{p \in V} \{< p^*, p > -F(p)\}$$

let (P) and (P^*) be the two following conjugate problems:

$$(P) = \operatorname*{Inf}_{v \in V} \{F(v) + G(\Lambda(v))\}$$

and

$$(P^*) = \operatorname*{Sup}_{p^* \in Y^*} \{-F^*(\Lambda^* p^*) - G^*(-p^*)\}.$$

Theorem 3.2.

$$-\infty \leq \mathit{Inf}(P) \leq \mathit{Sup}(P^*) < +\infty. \tag{5}$$

Furthermore, if there exists $u_0 \in V$ such that $F(u_0) < +\infty$, and if G is continuous at the point $\Lambda(u_0)$, then

$$\mathit{Inf}(P) = \mathit{Sup}(P^*).$$

In order to apply Theorem 3.2, we take $V = W^{1,1}\,(]0,1[)$, $Y = L^1\,(]0,1[)$, and the operator

$$\begin{aligned} \Lambda : V &\longrightarrow Y \\ u &\longmapsto u'. \end{aligned}$$

The set of admissible displacement is

$$\mathcal{U}_{ad} = \{u \in V / u(0) = \alpha \text{ and } u(1) = \beta\},$$

and the functionals are

$$\begin{aligned} F : V &\longrightarrow \overline{\mathbb{R}} \\ u &\longmapsto \begin{cases} -\lambda \displaystyle\int_0^1 f(t)u(t)\,dt & \text{if } u \in \mathcal{U}_{ad} \\ +\infty & \text{elsewhere,} \end{cases} \\ G : Y &\longrightarrow \overline{\mathbb{R}} \\ u &\longmapsto \int_0^1 g(u). \end{aligned}$$

With these notations, problem (1) may be written as

$$\mathit{Inf}(P_\lambda) = \operatorname*{Inf}_{u \in V} \Big\{F(u) + G(\Lambda u)\Big\}.$$

Remark 4.2 in Chapter III of [10] gives us that the dual problem is

$$Sup(P_\lambda^*) = \operatorname*{Sup}_{p^* \in Y^*} \Big\{ - F^*(\Lambda^* p^*) - G^*(-p^*) \Big\}.$$

The computation of $F^*(\Lambda^*\sigma)$ is classical (cf [2]);

$$F^*(\Lambda^*\sigma) = \begin{cases} \beta\sigma(1) - \alpha\sigma(0) & \text{if } -\sigma \in S_{ad}(\lambda) \\ +\infty & \text{elsewhere,} \end{cases}$$

where $S_{ad}(\lambda) = \{\sigma \in L^\infty / \sigma' + \lambda f = 0\}$.

We use then Proposition 2.5 of [17] to obtain

$$G^*(-\sigma) = \int_0^1 g^*(-\sigma).$$

Finally, we get

$$Sup(P_\lambda^*) = \operatorname*{Sup}_{\substack{\sigma = -\lambda f \\ \sigma \geq -\nu}} \Big\{ - \int_0^1 g^*(\sigma(t))\, dt \, + \beta\sigma(1) - \alpha\sigma(0) \Big\}.$$

Indeed, it is easy to verify that, if $Y < -\nu$, then $g^*(Y) = +\infty$.

Second Step:
This step consists in introducing the penalty method.
Applying Theorem 3.2, we obtain that

$$-\infty \leq Sup(P_\lambda^*) \leq Inf(P_\lambda) \leq +\infty.$$

Since the set of admissible stresses is non-empty, we can write

$$-\infty < Sup(P_\lambda^*) \leq Inf(P_\lambda) \leq +\infty. \tag{6}$$

A first attempt to prove the equality (4) consists in using the following criterion of the above-mentionned theorem of [10]:

$$\begin{cases} \text{There exists } u_0 \in V \text{ such that } F(u_0) < +\infty, \\ G(\Lambda u_0) < +\infty \text{ and } G \text{ being continuous in point } \Lambda u_0. \end{cases} \tag{7}$$

Unfortunately, this criterion is not verified here. We overcome the difficulty by using a penalty method:
Let η be a positive real and let us introduce the function:

$$\begin{aligned} g_\eta : \mathbb{R} &\longrightarrow \mathbb{R} \\ X &\longmapsto \Psi(X) + \frac{1}{2\eta}(X-1)^+. \end{aligned}$$

g_η verify the following properties:

$$\lim_{\eta\to 0} g_\eta(X) = g(X), \quad \forall X \in \mathbb{R} \tag{8}$$

$$g_\eta \le g, \tag{9}$$

which implies that

$$\forall X \in \mathbb{R} \qquad\qquad g_\eta^*(X) \ge g^*(X).$$

Now we define the penalized problem

$$\mathit{Inf}(P_{\lambda,\eta}) = \operatorname*{Inf}_{\substack{u\in W^{1,1}(]0,1[)\\ u(o)=\alpha \\ u(1)=\beta \\ u'\le 1}} \Big\{ \int_0^1 g_\eta(u'(t))\,dt - \lambda \int_0^1 u'(t)\,f(t)\,dt \Big\}, \tag{10}$$

whose dual problem is

$$\mathit{Sup}(P_{\lambda,\eta}{}^*) = \operatorname*{Sup}_{\substack{\sigma\in S_{ad}(\lambda)\\ \sigma\ge -\nu}} \Big\{ -\int_0^1 g_\eta^*(\sigma(t))\,dt + \sigma(1)\beta - \sigma(0)\alpha \Big\}. \tag{11}$$

[10] ensures that $\mathit{Sup}(P_{\lambda,\eta}{}^*) \le \mathit{Inf}(P_{\lambda,\eta})$. It is easy to remark that for this problem, criterion (7) is verified; we have then

$$\mathit{Sup}(P_{\lambda,\eta}{}^*) = \mathit{Inf}(P_{\lambda,\eta}), \tag{12}$$

from which we deduce

$$\begin{aligned} -\infty < {} & \mathit{Inf}(P_{\lambda,\eta}) \\ & = \mathit{Sup}(P_{\lambda,\eta}{}^*) \le \mathit{Sup}(P_\lambda^*) \le \mathit{Inf}(P_\lambda) \le +\infty. \end{aligned} \tag{13}$$

□

Third Step:
This step consists in introducing a penalized form of (P_λ) on BV:

$$\begin{aligned} &\mathit{Inf}(P_R^B) = \\ &\operatorname*{Inf}_{\substack{u\in BV(]0,1[)\\ u'\le 1\\ u(o^+)\le\alpha \\ u(1^-)\ge\beta}} \Big\{ \int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)) + \Psi_\infty(u(0^+) - \alpha) - \lambda \int_0^1 f(t)\,u(t)\,dt \Big\}. \end{aligned} \tag{14}$$

and establishing the following inequality:

$$\mathit{Inf}(P_R^B) \le \varliminf_{\eta\to 0^+} \mathit{Inf}(P_{\lambda,\eta}). \tag{15}$$

For $\eta > 0$ we define:

$$J_{\lambda,\eta}(u) = \int_0^1 g_\eta(u') - \lambda \int_0^1 fu. \tag{16}$$

Using the definition of $Inf(P_{\lambda,\eta})$, we can choose u_η in $\mathcal{U}_{ad}$ such that

$$J_{\lambda,\eta}(u_\eta) \leq Inf(P_{\lambda,\eta}) + \eta. \tag{17}$$

Lemma 3.3. *The three following assertions hold :*

$$(u_\eta)_{\eta>0} \textit{ is bounded in } W^{1,1}(]0,1[). \tag{18}$$

$$\int_0^1 \Psi(u'_\eta) \textit{ is bounded in } \mathbb{R}. \tag{19}$$

$$\begin{aligned} &\int_0^1 \frac{1}{2\eta}(u'_\eta - 1)^+ \textit{ is bounded:} \\ &\exists C_2 > 0,\ \int_0^1 (u' - 1)^+ \leq C_2 \eta. \end{aligned} \tag{20}$$

Proof of Lemma 3.3. For $\lambda_1 > \lambda$, we write:

$$\begin{aligned} J_{\lambda,\eta}(u) &= (1 - \frac{\lambda}{\lambda_1}) \int_0^1 \Psi(u') + (1 - \frac{\lambda}{\lambda_1}) \int_0^1 \frac{1}{2\eta}(u' - 1)^+ - \frac{\lambda}{\lambda_1}\lambda_1 \int_0^1 fu \\ &\quad + \frac{\lambda}{\lambda_1} \int_0^1 \Psi(u') + \frac{\lambda}{\lambda_1} \int_0^1 \frac{1}{2\eta}(u' - 1)^+ \\ &= \frac{\lambda}{\lambda_1} J_{\lambda_1,\eta}(u) + (1 - \frac{\lambda}{\lambda_1}) \int_0^1 \Psi(u') + (1 - \frac{\lambda}{\lambda_1}) \int_0^1 \frac{1}{2\eta}(u' - 1)^+. \end{aligned} \tag{21}$$

From this last equality, we can deduce that

$$J_{\lambda,\eta}(u) - \frac{\lambda}{\lambda_1} J_{\lambda_1,\eta}(u) \geq (1 - \frac{\lambda}{\lambda_1}) \int_0^1 \Psi(u').$$

Furthermore, $J_{\lambda,\eta}(u_\eta) \leq Inf(P_{\lambda,\eta}) + \eta \leq Inf(P_\lambda) + \eta$ and $J_{\lambda,\eta}(u_\eta) \geq Inf(P_{\lambda_1,\eta}) \geq Inf(P_{\lambda_1,1})$.
Then, since $1 - \frac{\lambda}{\lambda_1} > 0$, we deduce that

$$\int_0^1 \Psi(u'_\eta) \text{ is bounded in } \mathbb{R}.$$

Using the assumption (H_1), we conclude that u'_η is bounded in L^1. This fact, joined with the boundary conditions, ensures that u_η is bounded in $W^{1,1}(]0,1[)$.
This result, joined to (21), gives (20). □

Let us now introduce

$$\tilde{u}_\eta : \mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto \begin{cases} \alpha & \text{if } t \leq 0 \\ u_\eta & \text{if } t \in [0,1] \\ \beta & \text{if } t \geq 1. \end{cases}$$

Properties (18) and (20) still hold for the sequence $(\tilde{u}_\eta)_{\eta>0}$. Since it is bounded in $BV(\mathbb{R})$, we can extract from it a (weakly) converging subsequence which has a limit v in $BV(\mathbb{R})$. Moreover, the subsequence $(\Psi(u'_\eta))_{\eta>0}$ is bounded in $M^1(\mathbb{R})$; so we can once more extract a (weakly) converging subsequence. Let us denote by μ its limit and by $(u'_{\sigma(\eta)})_{\eta>0}$ the sequence after this double extraction. One can remark that v verifies $v(t) = \alpha$ for $t < 0$ and $v(t) = \beta$ for $t > 1$.
Let us denote by u the restriction of v to $[0,1]$.

$$v' = u'\chi_{]0,1[} + (\beta - u(1^-))\delta_{\{1\}} + (u(0^+) - \alpha)\delta_{\{0\}}.$$

The theory of functions of measures gives then

$$\int_{\mathbb{R}} \Psi(v') = \int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)) + \Psi_\infty(u(0^+) - \alpha),$$

where $u(1^-) = \lim\limits_{x\to 1^-} u(x)$, $u(0^+) = \lim\limits_{x\to 0^+} u(x)$, and $\Psi_\infty(x) = \lim\limits_{t\to+\infty} \dfrac{\Psi(tx)}{t}$.
Using Lemma 2.1 in [8], we have

$$\Psi(v') \leq \mu \leq \underline{lim}_{\eta\to 0^+} \Psi(\tilde{u}'_{\sigma(\eta)}). \tag{22}$$

Furthermore, (20) ensures that $v' \leq 1$. Let us verify that $u(1^-) \geq \beta$ and $u(0^+) \leq \alpha$, i.e. that u is allowed to have only negative jump.
Choosing $(\epsilon_n)_n$ a sequence which has 0 as limit, so that v has no mass in $1-\epsilon_n$, we have, for all $n > 0$, $-v(1-\epsilon_n) + \beta = \displaystyle\int_{1-\epsilon_n}^{1+\epsilon_n} v'(t)\,dt \leq 2\epsilon_n$. Hence, we find that $u(1^-) = v(1^-) \geq \beta$.
Similarly, $u(0^+) = v(0^+) \leq \alpha$.
Then, from (15) and noticing that $\displaystyle\int_0^1 f u_{\sigma(\eta)} \to \int_0^1 f u$, we get:

$$\begin{aligned} \underline{lim}_{\eta\to 0^+} Inf(P_{\lambda,\eta}) &\geq \underline{lim}_{\eta\to 0^+} \int_0^1 \Psi(u'_{\sigma(\eta)}) - \lambda \int_0^1 f u_{\sigma(\eta)} + \frac{1}{2\eta}\int_0^1 (u'_{\sigma(\eta)} - 1)^+ \\ &\geq \int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)) + \Psi_\infty(u(0^+) - \alpha) - \lambda \int_0^1 f(t)u(t)\,dt \\ &\geq Inf(P_R^B) \end{aligned}$$

We now introduce the relaxed form of (P_λ) on $W^{1,1}$

$$
Inf(P_R^W) = \tag{23}
$$
$$
\underset{\substack{u\in W^(]0,1[)\\ u'\leq 1\\ u(o^+)\leq\alpha\\ u(1^-)\geq\beta}}{\mathrm{Inf}} \Big\{\int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)) + \Psi_\infty(u(0^+)-\alpha) - \lambda\int_0^1 f(t)\,u(t)\,dt\Big\}.
$$

Fourth Step:
This step consists in establishing the equality between $Inf(P_R^B)$ and $Inf(P_R^W)$.

Proposition 3.4. *$\forall u \in BV(]0,1[)$ such that $u' \leq 1$, $u(0^+) \leq \alpha$ and $u(1^-) \geq \beta$, $\exists (u_n)_{n\in\mathbb{N}} \in (W^{1,1}(]0,1[))^{\mathbb{N}}$, $u'_n \leq 1$, $u_n(0^+) \leq \alpha$ and $u_n(1^-)\geq\beta$ such that*

$$
\begin{aligned}
&u_n \to u \quad L^1 \textit{ strongly}\\
&u'_n \rightharpoonup u \quad M^1 \textit{ vaguely,}
\end{aligned}
$$

and

$$
\int_0^1 \Psi(u'_n) + \Psi_\infty(u_n(0^+)-\alpha) + \Psi_\infty(\beta - u_n(1^-))
$$
$$
\longrightarrow \int_0^1 \Psi(u') + \Psi_\infty(u((0^+)-\alpha) + \Psi_\infty(\beta - u(1^-))).
$$

Proof.
We establish first the following lemma.

Lemma 3.5. *Let u be in $BV(]0,1[)$, such that $u'\leq 1$, $u(0^+) < \alpha$ and $u(1^-) > \beta$, then there exists $(u_n)_{n\in\mathbb{N}} \in (W^{1,1}(]0,1[))^{\mathbb{N}}$ such that*

$$
\begin{aligned}
&u_n(0^+) \leq \alpha,\\
&u_n(1^-) \geq \beta\\
&u_n \to u \textit{ in } L^1 \textit{ strongly}\\
&|u'_n| \rightharpoonup |u'| \textit{ in } M^1 \textit{ vaguely,}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \Psi(u'_n) + \Psi_\infty(\beta - u_n(1^-)) + \Psi_\infty(u_n(0^+)-\alpha)\\
&\qquad \to \int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)) + \Psi_\infty(u(0^+)-\alpha)
\end{aligned} \tag{24}
$$

Proof. Let ρ in $C^\infty(\mathbb{R})$ be non-negative, even function with support in $]0,1[$, such that

$$
\int_{\mathbb{R}} \rho = 1, \tag{25}
$$

and ρ_n be defined as $\rho_n(x) = n\rho(nx)$. We extend u as follows

$$\tilde{u} = \begin{cases} \alpha & \text{on }]-\infty, 0] \\ u & \text{on }]0, 1[\\ \beta & \text{on } [1, +\infty[, \end{cases} \tag{26}$$

Then $\tilde{u} \in BV(\mathbb{R})$, $\tilde{u}' \leq 1$ and $\tilde{u}' = 0$ outside $[0,1]$. We define $u_n = \rho_n * \tilde{u}$. Since $\tilde{u}' \leq 1$,

$$u_n' = \rho_n * \tilde{u}' \leq 1.$$

In addition,

$$\rho_n * \tilde{u} \to u \text{ in } L^1 \text{ strongly} \quad \text{and}$$
$$\rho_n * \tilde{u}' \rightharpoonup \tilde{u}' \text{ in } M^1 \text{ vaguely.}$$

Ψ being convex, proper, and having a linear growth at infinity (H_1–H_3), the theory of convex functions of measures ensures that

$$\int_{\mathbb{R}} \Psi(u_n') \to \int_{\mathbb{R}} \Psi(\tilde{u}'), \tag{27}$$

which is equivalent to (24). (cf. [8] or [17]).
Furthermore, $\forall x \in \mathbb{R}$,

$$\begin{aligned} u_n(x) &= \int_{\mathbb{R}} \rho_n(x-t)\tilde{u}(t)\,dt \\ &= \int_{x-\frac{1}{n}}^{x} n\rho(n(x-t))\tilde{u}(t)\,dt. \end{aligned}$$

$\forall x \in [0, \frac{1}{n}[$,

$$u_n(x) = \int_{x-\frac{1}{n}}^{0} n\rho(n(x-t))\tilde{u}(t)\,dt + \int_{0}^{x} n\rho(n(x-t))\tilde{u}(t)\,dt.$$

Using the assumptions on the boundary conditions verified by u, let $\epsilon > 0$ be such that $\tilde{u}(0^+) \leq \alpha - \epsilon$ and $\tilde{u}(1^-) \geq \beta - \epsilon$. Let δ be a positive real such that

$$\forall t \in]0, \delta], \tilde{u}(t) < \alpha - \frac{\epsilon}{2}$$
$$\forall t \in [1-\delta, 1[, \tilde{u}(t) > \beta + \frac{\epsilon}{2}.$$

For $n > \frac{1}{\delta}$,

$$\begin{aligned} u_n(x) &< \int_{x-\frac{1}{n}}^{0} n\rho(n(x-t))\alpha\,dt + \int_{0}^{x} n\rho(n(x-t))\alpha\,dt. \\ &< \alpha \end{aligned}$$

Hence, $u_n(0^+) = \lim_{x\to 0^+} u_n(x) \leq \alpha$.

Let us take now $x = 1 - y$, with $y < \frac{\delta}{2}$,

$$\begin{aligned} u_n(1-y) &= \int_{1-y-\frac{1}{n}}^{1-y} n\rho(n(1-y-t))\tilde{u}(t)\,dt \\ &= \int_{y}^{y+\frac{1}{n}} n\rho(n(t-y))\tilde{u}(1-t)\,dt. \end{aligned}$$

For $n_0 > \frac{2}{\delta}$, we have $\forall t \in [y, y+\frac{1}{n}]$, $\tilde{u}(1-t) > \beta$.
Then $u_n(1-y) > \beta$ and $u_n(1^-) = \lim\limits_{y\to 0^+} u_n(1-y) \geq \beta$. □

Let us assume now that $u \in BV(]0,1[), u' \leq 1$, and $u(0^+) = \alpha$ or $u(1^-) = \beta$. We want to generalize the result above to this assumption.
Let u be in $BV(]0,1[), u' \leq 1$, $u(0^+) = \alpha$ and $u(1^-) = \beta$. Let us introduce $\epsilon > 0$ and $\delta > 0$ such that

$$|u(\delta^+) - \alpha| + |u((1-\delta)^-) - \beta| < \epsilon, \tag{28}$$

$$\left| \int_0^\delta |\Psi(u')| + \int_{1-\delta}^1 |\Psi(u')| \right| < \epsilon. \tag{29}$$

We can then define

$$u_\epsilon = \begin{cases} \alpha & \text{on }]-\infty, 0[\\ \frac{\alpha-\epsilon}{2} & \text{on } [0,\delta] \\ u & \text{on }]\delta, 1-\delta[\\ \beta + \frac{\epsilon}{2} & \text{on } [1-\delta, 1] \\ \beta & \text{on }]1, +\infty[. \end{cases}$$

Then $u_\epsilon \to \tilde{u}$ in BV strongly.
Furthermore,

$$\begin{aligned} \left| \int \Psi(\tilde{u}') - \int \Psi(u'_\epsilon) \right| &\leq \left| \int_{[0,1]} (\Psi(\tilde{u}') - \Psi(u'_\epsilon)) \right| \\ &\leq \left| \Psi_\infty(u_\epsilon(0^+) - \alpha) + \Psi_\infty(-u_\epsilon(1^-) + \beta) \right| \\ &\quad + \left| \int_0^1 \Psi(u') - \int_\delta^{1-\delta} \Psi(u') \right| + \left| \Psi_\infty(u_\epsilon(\delta^+) - (\alpha - \frac{\epsilon}{2})) \right| \\ &\quad + \left| \Psi_\infty(-u_\epsilon((1-\delta)^-) + (\beta + \frac{\epsilon}{2})) \right| \\ &\leq \epsilon + 4\,|\Psi_\infty(\epsilon)| \\ &\leq (4C+1)\epsilon. \end{aligned}$$

We obtain then the convergence of

$$\int_0^1 \Psi(u'_\epsilon) + \Psi_\infty(u_\epsilon(0^+) - \alpha) + \Psi_\infty(-u_\epsilon(1^-) + \beta) \quad \text{to}$$

$$\int_0^1 \Psi(u') + \Psi_\infty(u(0^+) - \alpha) + \Psi_\infty(-u(1^-) + \beta).$$

We are then lead to conditions of the Lemma 6.
If only one of the boundary conditions holds, we proceed in a similar way, adapting the construction of u_ϵ. □

Let us prove now that

$$Inf(P_R^W) = Inf(P_R^B). \tag{30}$$

We have

$$Inf(P_R^W) \leq Inf(P_R^B).$$

In order to prove the reverse inequality, let us introduce $(u_\eta)_{\eta>0}$ as a sequence of admissible functions for $Inf(P_R^B)$, such that

$$\begin{aligned} J(u_\eta) = \int_0^1 \Psi(u'_\eta) + \Psi_\infty(u_\eta(0^+) - \alpha) + \Psi_\infty(-u_\eta(1^-) + \beta) \\ \leq Inf(P_R^B) + \eta \end{aligned} \tag{31}$$

According to Proposition 4.5, there exists a sequence (v_η) of $W^{1,1}(]0,1[)$ such that $\forall \eta > 0$, $v'_\eta \leq 1$, $v_\eta(0^+) \leq \alpha$, $v_\eta(1^-) \geq \beta$, and

$$|J(v_\eta) - J(u_\eta)| < \eta.$$

Then

$$J(v_\eta) \leq Inf(P_R^B) + 2\eta,$$

i.e.

$$Inf(P_R^W) \leq Inf(P_R^B) + 2\eta \qquad \forall \eta > 0,$$

which is equivalent to (30).

Fifth Step:
This steps consists in establishing the equalities:

$$\mathrm{Inf}(P_R^W) = \mathrm{Inf}(P_R^B) = Inf(P_\lambda)$$

Thanks to step 4, it is sufficient to prove that

$$Inf(P_R^W) = Inf(P_\lambda) \tag{32}$$

holds. The inequality

$$Inf(P_R^W) \leq Inf(P_\lambda) \tag{33}$$

is immediate. For the reverse one, let u in $W^{1,1}(]0,1[)$ be such that $u' \leq 1$, $u(0^+) < \alpha$ and $u(1^-) > \beta$, and u_n be defined as

$$u_n = \begin{cases} \alpha + nx(u(\frac{1}{n}) - \alpha) & \text{if } x \in [0, \frac{1}{n}] \\ u(x) & \text{if } x \in [\frac{1}{n}, 1 - \frac{1}{n}] \\ \beta + n(1-x)(u(1-\frac{1}{n}) - \beta) & \text{if } x \in [1-\frac{1}{n}, 1]. \end{cases}$$

We get $u_n(0) = \alpha$ and $u_n(1) = \beta$. Furthermore, n being large enough, $u_n' \leq 1$. Indeed, n being large enough, $u(\frac{1}{n}) - \alpha \leq 0$ –(since $u(0^+) < \alpha$)–, and $u(1-\frac{1}{n}) - \beta \geq 0$. Thus, u_n is an admissible displacement for (P_λ).
The computation of $\int \Psi(u_n')$ gives

$$\int_0^1 \Psi(u_n') = \frac{1}{n}\Psi(n(u(\frac{1}{n}) - \alpha)) + \int_{\frac{1}{n}}^{1-\frac{1}{n}} \Psi(u') + \frac{1}{n}\Psi(-n(u(1-\frac{1}{n}) - \beta)),$$

which is converging to $\Psi_\infty(u(0^+) - \alpha) + \displaystyle\int_0^1 \Psi(u') + \Psi_\infty(\beta - u(1^-)$.
On the other hand, the assumptions on f ensure that $\lambda \int f u_n \to \lambda \int f u$.

Now, if only one of the boundary conditions holds, the construction of u_n must be slightly modified.
For example, if $u(0^+) = \alpha$ and $u(1^-) > \beta$,

$$u_n = \begin{cases} u & \text{on } [0, 1-\frac{1}{n}] \\ \beta + n(1-x)(u(1-\frac{1}{n}) - \beta) & \text{if } x \in [1-\frac{1}{n}, 1]. \end{cases}$$

Let then (u_η) be a minimizing sequence of $Inf(P_R^W)$, such that

$$J(u_\eta) \leq Inf(P_R^W) + \eta.$$

Then there exists v_η in $W^{1,1}(]0,1[)$ such that $v_\eta' \leq 1$, $v_\eta(0) = \alpha$, $v_\eta(1) = \beta$, and

$$|J(u_\eta) - J(v_\eta)| < \eta \ ;$$

thus $J(v_\eta) \leq Inf(P_R^W) + 2\eta$, for all $\eta > 0$. Finally, from (33), we get (32).

Sixth Step:
Using (14), (15) and (30), we get the following inequalities:

$$Inf(P_\lambda) \leq \liminf(P_{\lambda,\eta}) \leq Sup(P_\lambda^*) \leq Inf(P_\lambda). \tag{34}$$

which induces

$$Inf(P_\lambda) = Sup(P_\lambda^*). \tag{35}$$

4. Optimization of the Stress Problem

This section is devoted to the study of the solutions of the stress problem, from which we will derive solutions of the displacement problem, under particular conditions.
First, we will establish some useful relations.

4.1. Expression of Ψ^*

Proposition 2.5 of [17] is equivalent to Krasnosels'kii theorem and permits to give an expression of Ψ^*. We can notice that g^* is finite and continuous on $[-\nu, +\infty[$ and that g^* has an upper bound on its domain.
Let us compute g^* for the standart examples studied in [2].

1. The rigid-plastic model.
   Here $\Psi(X) = |X|$, and

$$g(X) = \begin{cases} +\infty & \text{if } X > 1 \\ |X| & \text{if } X \leq 1. \end{cases}$$

   Then

$$g^*(Y) = \begin{cases} Y - 1 & \text{if } Y \geq 1 \\ 0 & \text{if } |Y| \leq 1 \\ +\infty & \text{elsewhere.} \end{cases}$$

2. The strictly convex function.
   $\Psi(X) = \sqrt{1+X^2}$. We have then

$$g(X) = \begin{cases} +\infty & \text{if } X > 1 \\ \sqrt{1+X^2} & \text{if } X \leq 1. \end{cases}$$

   Then

$$g^*(Y) = \begin{cases} Y - \sqrt{2} & \text{if } Y \geq \frac{1}{\sqrt{2}} \\ -\sqrt{1-Y^2} & \text{if } Y \in [-1, \frac{1}{\sqrt{2}}] \\ +\infty & \text{elsewhere.} \end{cases}$$

3. The elasto-plastic model.

$$\Psi(X) = \begin{cases} \frac{1}{2}|X|^2 & \text{if } |X| \leq 1 \\ |X| - \frac{1}{2} & \text{elsewhere.} \end{cases}$$

   Thus

$$g(X) = \begin{cases} +\infty & \text{if } X > 1 \\ \frac{1}{2}|X|^2 & \text{if } |X| \leq 1 \\ -X - \frac{1}{2} & \text{elsewhere.} \end{cases}$$

Then

$$g^*(Y) = \begin{cases} +\infty & \text{if } Y < -1 \\ \frac{1}{2}Y^2 & \text{if } |Y| \leq 1 \\ Y - \frac{1}{2} & \text{elsewhere.} \end{cases}$$

4.2. The Extremality Relation

Let (u, σ) be solution of $(Inf(P_\lambda), Sup(P^*_\lambda))$.

Proposition 4.1. *The three following assertions hold and are equivalent*

$$\Psi(u'(t)) + g^*(\sigma(t)) - \sigma(t)\, u'(t) = 0 \quad ae\ on\ [0,1]. \tag{36}$$

$$u'(t) \in \partial g^*(\sigma(t)) \quad ae\ on\ [0,1]. \tag{37}$$

$$\sigma(t) \in \partial g(u'(t)) \quad ae\ on\ [0,1]. \tag{38}$$

Proof. From the Inf-Sup equality, we obtain that

$$\int_0^1 g(u'(t))\, dt - \lambda \int_0^1 f(t)\, u(t)\, dt = \beta\sigma(1) - \alpha\sigma(0) - \int_0^1 g^*(\sigma(t))\, dt.$$

Since $\sigma \in S_{ad}(\lambda)$, $\sigma'(t) = -\lambda\, f(t)$ and $\forall t \in [0,1]$,

$$\int_0^1 (g(u'(t)) + g^*(\sigma(t)))\, dt + \int_0^1 \sigma'(t)\, u(t)\, dt = \beta\sigma(1) - \alpha\sigma(0).$$

Hence, intregrating by parts,

$$\int_0^1 \sigma'(t)u(t)\, dt = -\int_0^1 \sigma(t)u'(t)\, dt + \beta\sigma(1) - \alpha\sigma(0).$$

Therefore, we obtain

$$\int_0^1 \left[g(u'(t)) + g^*(\sigma(t)) - \sigma(t)u'(t)\right]\, dt = 0 \tag{39}$$

Using the definition of g^*,

$$\begin{aligned} g^*(\sigma(t)) &= \mathrm{Sup}_{X\in\mathbb{R}}\{X\sigma(t) - g(X)\} \\ &\geq u'(t)\sigma(t) - g(u'(t)). \end{aligned}$$

We derive from this inequality that $g(u'(t)) + g^*(\sigma(t)) - \sigma(t)u'(t)$ has a constant sign for almost all t in $[0,1]$. Therefore, (39) implies that

$$g(u'(t)) + g^*(\sigma(t)) - \sigma(t)u'(t) = 0,$$

for almost every t in $[0,1]$.

The three assertions are equivalent since g is convex, lower semi-continuous and proper (see [10]). □

4.3. Limit Loads

The main difference between this problem and the non-constrained problem is that here, there is no finite limit load. In other words, for every λ in $\mathbb{R}$, there exists indeed an admissible stress for (P^*_λ), i.e. a stress which is in $S_{ad}(\lambda)$ and verifies $\sigma \geq -\nu$.

– $\lambda > 0$. Let σ_1 be defined as

$$\sigma_1(x) = -\nu - \lambda \int_b^x f(t)\,dt,$$

where $b \in B_M = \{x \in [0,1]. \int_0^x f(t)\,dt$ is maximum $\}$. $(B_M \neq \emptyset)$ Then

$$\forall x \in [0,1], \quad \int_b^x f(t)\,dt \leq 0. \tag{40}$$

– $\lambda < 0$. Let σ_2 be defined as

$$\sigma_2(x) = -\nu - \lambda \int_a^x f(t)\,dt$$

where $a \in B_m = \{x \in [0,1]. \int_0^x f(t)\,dt$ is minimum$\}$.

We can easily check that in both cases $\sigma \in S_{ad}(\lambda)$ and $\sigma \geq -\nu$ *a.e*, and since σ is bounded, $\int g^*(\sigma) < \infty$.

4.4. Stress problem Optimization

As we did in [2], we now take $\sigma_\lambda(b)$ as a new parameter that we denote by X. Then

$$\sigma_\lambda(X,t) = X - \lambda \int_b^t f(s)\,ds. \tag{41}$$

We introduce the functional

$$G_\lambda(X) = -\int_0^1 g^*(\sigma_\lambda(X,t))\,dt + \sigma_\lambda(X,1)\beta - \sigma_\lambda(X,0)\alpha \tag{42}$$

which is defined and continuous on $D_{G_\lambda} = [-\nu, +\infty[$. Maximizing the functional in (P^*_λ) is equivalent to look for the "critical" points of G_λ.

Proposition 4.2. *G_λ is right and left differentiable on D_{G_λ} and*

$$G'_{\lambda,d}(X) = \beta - \alpha - \int_0^1 {g^*}'_d(\sigma_\lambda(X,t))\,dt \tag{43}$$

$$G'_{\lambda,g}(X) = \beta - \alpha - \int_0^1 {g^*}'_g(\sigma_\lambda(X,t))\,dt. \tag{44}$$

Proof. g having the same properties than Ψ, this proof is identical to the corresponding one in [2]. □

4.4.1. Behaviour at infinity of $G'_{\lambda,g}(X)$

Let $X > \Psi'(1)$. Then $\forall t \in [0,1]$, $\sigma_\lambda(X,t) > \Psi'(1)$, and $g^*(\sigma_\lambda(X,t)) = 1$. Finally, we obtain

$$G'_{\lambda,g}(X) = \beta - \alpha - 1. \tag{45}$$

We must distinguish three cases:

1. $\underline{\beta - \alpha > 1:}$
   G^* is non decreasing, and $Sup(P^*_\lambda) = +\infty$. In fact, there are no admissible displacements
   $$\left\{u \in W^{1,1}(]0,1[)/ {}_{\substack{u(0)=\alpha \\ u(1)=\beta}} \text{ and } u' \leq 1\right\} = \emptyset$$
   when $u(1) - u(0) > 1$.
2. $\underline{\beta - \alpha = 1:}$
   It is easy to see that only u defined by
   $$u(t) = \alpha + (\beta - \alpha)t$$
   is admissible and
   $$Inf(P_\lambda) = \int_0^1 \Psi(\alpha + (\beta - \alpha)t)\, dt - \lambda \int_0^1 f(t)(\alpha + (\beta - \alpha)t)\, dt.$$
3. $\underline{\beta - \alpha < 1:}$ (46)
   We have then
   $$\lim_{X \to +\infty} G'_{\lambda,g}(X) < 0 \tag{47}$$

4.4.2. Behaviour near $-\nu$ of $G'_{\lambda,d}(X)$

We must distinguish three cases according to the sign of $\lim_{X \to -\nu^+} G'_{\lambda,d}(X)$.

I) $\underline{\lim_{X \to -\nu^+} G'_{\lambda,d}(X) > 0}$ (48)

Proposition 4.3. *Under (48), $Inf(P_\lambda)$ has at least one regular solution.*

Proof. We will need the following lemma:

Lemma 4.4. *Let f be a continuous function on $[a, +\infty[$, everywhere right and left differentiable on $]a, +\infty[$, such that:*

$$\begin{cases} f'_d \text{ et } f'_g \text{ are non-decreasing,} \\ \lim_{x \to a^+} f'_d(x) < 0, \\ \lim_{x \to +\infty} f'_g(x) > 0. \end{cases}$$

Then there exists $x_0 \in]a, +\infty[$ *such that*

$$\begin{cases} f_g'(x_0) \leq 0 \\ f_d'(x_0) \geq 0. \end{cases}$$

Proof. This lemma is a generalized version of Lemma 2.3 from [2]. □

(47)–(48) being verified, we can apply Lemma 4.4, and find X_0 in $]-\nu, +\infty[$, such that $G'_{\lambda,g}(X_0) \leq 0$ and $G'_{\lambda,d}(X_0) \geq 0$. Then G_λ achieves its upperbound on X_0.
Introducing $(a, b) \in (\mathbb{R}^*)^2$ such that

$$\begin{cases} a + b = 1 \\ aG'_{\lambda,g}(X_0) + bG'_{\lambda,d}(X_0) = 0, \end{cases}$$

we define

$$\begin{aligned} v :\ [0,1] &\to \mathbb{R} \\ t &\mapsto a\, g_g^{*\prime}(\sigma_\lambda(X_0,t)) + b\, g_d^{*\prime}(\sigma_\lambda(X_0,t)). \end{aligned}$$

One can then verify that

$$v(t) \in \partial g^*(\sigma_\lambda(X_0,t)), \quad \forall t \in [0,1], \tag{49}$$

$$v \in L^1(]0,1[). \tag{50}$$

We define

$$\begin{aligned} u :\ [0,1] &\to \mathbb{R} \\ t &\mapsto \alpha + \int_0^t v(s)\, ds. \end{aligned}$$

$$u \in W^{1,1}(]0,1[), \tag{51}$$

and verify the boundary conditions since

$$\begin{aligned} u(1) &= \alpha + a \int_0^1 g_g^{*\prime}(\sigma_\lambda(X_0,t))\, dt + b \int_0^1 g_d^{*\prime}(\sigma_\lambda(X_0,t))\, dt \\ &= aG'_{\lambda,g}(X_0) + bG'_{\lambda,d}(X_0) + (a+b)\,\beta \\ &= \beta. \end{aligned}$$

Moreover, $\forall x \in [-\nu, +\infty[$, $g_g^{*\prime} \leq 1$ and $g_d^{*\prime} \leq 1$. Thus

$$\forall t \in [0,1], \quad u'(t) = v(t) \leq 1. \tag{52}$$

Hence u is an admissible displacement and u and $\sigma_\lambda(X_0, .)$ verify the extremality relation (49). Finally, u is a solution of $Inf(P_\lambda)$. □

II) $\underline{\lim_{X\to-\nu^+} G'_{\lambda,d}(X) = 0}$ (53)

Proposition 4.5. *Under (53), Inf(P_λ) has at least one regular solution.*

Proof. The proof needs the following lemma:

Lemma 4.6.

$$\lim_{x\to-\nu^+} \int_0^1 g_d^{*\prime}(\sigma_\lambda(X,t))\,dt = \int_0^1 g_d^{*\prime}(\sigma_\lambda(-\nu,t))\,dt. \tag{54}$$

Proof. From (52), we get $\lim_{x\to-\nu^+} \int_0^1 g_d^{*\prime}(\sigma_\lambda(X,t))\,dt = \beta-\alpha$. Moreover, the function $g_d^{*\prime}(\sigma_\lambda(X,t))$ being monotonous with respect to X, one can apply the Lebesgue's monotonous convergence Theorem from which derives the result. □

Under the assumption (53), G_λ is maximized in $X_0 = -\nu$.
Let

$$\begin{aligned} v\ :\ &[0,1] \to \overline{\mathbb{R}} \\ &t \ \ \mapsto \lim_{x\to-\nu^+} g_d^{*\prime}(\sigma_\lambda(X,t)), \end{aligned}$$

and

$$\begin{aligned} u\ :\ &[0,1] \to \mathbb{R} \\ &t \ \ \mapsto \alpha + \int_0^1 v(s)\,ds. \end{aligned}$$

We can easily obtain that u is an admissible displacement and $(u,\sigma_\lambda(-\nu,.))$ verify the extremality relation. □

III) $\underline{\lim_{X\to-\nu^+} G'_{\lambda,d}(X) = -\epsilon\ ;\ \epsilon>0}$ (55)

Proposition 4.7. *Assume (55) and let B_M be defined by*

$$B_M = \{x\in[0,1],/\int_0^x f(t)\,dt \ \ \textit{is maximum}\ \}.$$

a) *If $Mes(B_M)=0$, Inf(P_λ) has at least one regular solution if and only if*

$$\begin{aligned} \int_0^1 Inf\{\partial g^*(\sigma_\lambda(-\nu,t))\,dt\} &\le \beta-\alpha \\ &\le \int_0^1 Sup\{\partial g^*(\sigma_\lambda(-\nu,t))\,dt\}. \end{aligned} \tag{56}$$

b) *If $Mes(B_M)\neq 0$, Inf(P_λ) has at least one regular solution if and only if Ψ_0 is linear.*

Proof. This proof is similar to the proof of Proposition 2.5 in [2]. Let us recall the steps that we established for it.
Let $(X_i)_{i\in I}$ be the points of discontinuity of $g^{*\prime}$, and T_i, T, b_i^+, b_i^- and λ_i be defined as:
$T_i = \{t \in [0,1] / \sigma_\lambda(-\nu, t) = X_i\}$, $T = \underset{i\in I}{\cup} T_i$, $b_i^- = g_g^{*\prime}(X_i)$, $b_i^+ = g_d^{*\prime}(X_i)$
and $\lambda_i = Mes(T_i)$.
(47)–(46) et (55) ensure that G_λ achieves its upperbound in $X_0 = -\nu$.

a) $\underline{Mes(B_M) = 0}$

$$\forall t \in [0,1]\backslash\{B_M \cup T\}, \qquad \lim_{X\to -\nu^+} g_d^{*\prime}(\sigma_\lambda(X,t)) = g_d^{*\prime}(\sigma_\lambda(-\nu,t))$$

We have then

$$\int_{[0,1]\backslash\{B_M\cup T\}} g_g^{*\prime}(\sigma_\lambda(-\nu,t))\,dt = \beta - \alpha + \epsilon - \sum_{i\in I} \lambda_i b_i^+.$$

If u is solution of $Inf(P_\lambda)$, from (47) we get

$$\begin{aligned} &\forall t \in T_i, && b_i^- \le u'(t) \le b_i^+ \\ &\forall t \in [0,1]\backslash\{B_M \cup T\}, && u'(t) = g_g^{*\prime}(\sigma_\lambda(-\nu,t)) = g_d^{*\prime}(\sigma_\lambda(-\nu,t)) \end{aligned}$$

and

$$\beta - \alpha + \epsilon - \sum_{i\in I} \lambda_i (b_i^+ - b_i^-) \le \int_0^1 u'(t)\,dt \le \beta - \alpha + \epsilon \tag{57}$$

$$\sum_{i\in I} \lambda_i (b_i^+ - b_i^-) \le \epsilon. \tag{58}$$

We can easily verify this is a necessary and sufficient condition, by introducing

$$\begin{aligned} v \;:\; & [0,1] \to \mathbb{R} \\ t \quad & \mapsto \begin{cases} g^{*\prime}(\sigma_\lambda(-\nu,t)) & \forall t \in [0,1]\backslash\{B_M \cup T\} \\ \mu b_i^+ + (1-\mu) b_i^- & \forall t \in T_i \\ 0 & \forall t \in B_M \end{cases} \end{aligned}$$

where $\mu = \frac{\epsilon}{\sum\limits_{i\in I}\lambda_i(b_i^+ - b_i^-)}$,

and

$$\begin{aligned} u \;:\; & [0,1] \to \mathbb{R} \\ t \quad & \mapsto \alpha + \int_0^t v(s)\,ds \end{aligned}$$

Since, for every $t \in T_i$, $Inf\{\partial g^*(\sigma_\lambda(-\nu,t)) = b_i^-$, $Sup\{\partial g^*(\sigma_\lambda(-\nu,t)) = b_i^+$,

$$(57) \iff \int_0^1 Inf\{\partial g^*(\sigma_\lambda(-\nu,t))\,dt\} \le \beta - \alpha \le \int_0^1 Sup\{\partial g^*(\sigma_\lambda(-\nu,t))\,dt\}.$$

b) $\underline{Mes(B_M) \neq 0}$

If Ψ_0 is strictly convex, $\partial g^*(-\nu) = \{-\infty\}$. There are no regular solution to $Inf(P_\lambda)$.
If Ψ_0 is linear, $\partial g^*(-\nu) = [-\infty, a_0]$. We have already seen that

$$\lim_{X \to -\nu^+} g_g^{*\prime}(\sigma_\lambda(X,t)) = g_g^{*\prime}(\sigma_\lambda(-\nu,t)). \tag{59}$$

We then distinguish two cases:

α) $\sum_{i \in I} \lambda_i (b_i^+ - b_i^-) \geq \epsilon$

Let $\mu = \frac{\epsilon}{\sum_{i\in I} \lambda_i (b_i^+ - b_i^-)}$, v be defined by

$$\begin{aligned} v \ : \ [0,1] &\to \mathbb{R} \\ t \quad &\mapsto \begin{cases} g^{*\prime}(\sigma_\lambda(-\nu,t)) & \forall t \in [0,1] \backslash \{B_M \cup T\} \\ \mu b_i^+ + (1-\mu) b_i^- & \forall t \in T_i \\ a_0 & \forall t \in B_M, \end{cases} \end{aligned}$$

and

$$\begin{aligned} u \ : \ [0,1] &\to \mathbb{R} \\ t \quad &\mapsto \alpha + \int_0^t v(s)\, ds. \end{aligned}$$

Then u is a regular solution of $Inf(P_\lambda)$.

β) $\sum_{i \in I} \lambda_i (b_i^+ - b_i^-) < \epsilon$

We take $\mu = \epsilon - \sum_{i \in I} \lambda_i (b_i^+ - b_i^-)$. Let

$$\begin{aligned} v \ : \ [0,1] &\to \mathbb{R} \\ t \quad &\mapsto \begin{cases} g^{*\prime}(\sigma_\lambda(-\nu,t)) & \forall t \in [0,1] \backslash \{B_M \cup T\} \\ b_i^+ & \forall t \in T_i \\ a_0 - \frac{\mu}{Meas(B_M)} & \forall t \in B_M, \end{cases} \end{aligned}$$

and

$$\begin{aligned} u \ : \ [0,1] &\to \mathbb{R} \\ t \quad &\mapsto \alpha + \int_0^t v(s)\, ds \end{aligned}$$

is a solution of $Inf(P_\lambda)$. $\square$

Of course, we have analogous results if λ is negative.

4.5. The case $\lambda = 0$

Like in [2], the problem with no load play a role which is slightly different. It will ensure that the set of regular loads will be non-empty.

$$Inf(P_0) = \operatorname*{Inf}_{\substack{u \in W^{1,1}(]0,1[) \\ u(o)=\alpha \\ u(1)=\beta \\ u' \leq 1}} \{\int_0^1 \Psi(u'(t))\,dt\}. \tag{60}$$

$$Sup(P_0^*) = \operatorname*{Sup}_{X \geq -\nu} \{(\beta - \alpha)\,X - g^*(X)\}. \tag{61}$$

Proposition 4.8. *$Inf(P_0)$ has at least one regular solution.*

Proof. We argue as in proof of Proposition 3.1 in [2]. Let us denote by $G_0(X) = (\beta - \alpha)\,X - g^*(X)$.

$$D_{G_0} = [-\nu, +\infty]$$

G_0 is continuous, everywhere right and left differentiable and

$$G'_{0,\delta} = \beta - \alpha - {g^*_\delta}'(X)$$

where δ denotes g or d. Moreover $\lim\limits_{X \to +\infty} G'_{0,g}(X) = \beta - \alpha - 1$. We obtain the natural condition that $\beta - \alpha$ must be less than 1.

I) $\lim\limits_{x \to -\nu^+} G'_{0,d}(X) \geq 0$.

Arguing as we did in case I and II, we find that there exists a regular solution for $Inf(P_\lambda)$.

II) $\lim\limits_{x \to -\nu^+} G'_{0,d}(X) < 0. \qquad (62)$

Then G_0 is maximized in $-\nu$. Ψ_0 can not be strictly convex, else the sub-differential of g^* in $-\nu$ would be reduced to $\{-\infty\}$. It is thus linear, and $\lim\limits_{X \to -\nu^+} {g^*_d}'(X) = a_0$.

We get:

$$(62) \Longleftrightarrow \beta - \alpha < a_0$$

We can see that $u(t) = \alpha + (\beta - \alpha)\,t$ is a regular solution of $Inf(P_0)$. $\square$

5. Regular Limit Loads

In this section, we establish the link between the previous conditions of existence of a regular solution and λ.

Let us suppose that $\beta - \alpha < 1$ and denote

$$\Lambda_c^+ = \{\lambda \in \mathbb{R}^+ / (48) \text{ or } (53) \text{ hold } \} \cup \{0\}$$

Theorem 5.1. $\exists(\underline{\lambda}_r, \bar{\lambda}_r) \in (\overline{\mathbb{R}}^+)^2/$

1. $\forall\lambda \in] -\underline{\lambda}_r, \bar{\lambda}_r[$, *Inf*$(P_\lambda)$ *have at least one regular solution.*
2. $\Lambda_d = \{\lambda \notin]-\underline{\lambda}_r, \bar{\lambda}_r[/$ *Inf*(P_λ) *have at least one regular solution.*$\}$ *is at most countable.*

Remark 5.2.

1. If $\beta - \alpha > 1$, *Inf*(P_λ) has no regular solution.
2. The countability of Λ_d is a sharp condition; it is possible to construct some example, when ψ is not strictly convex, where Λ_d is not finite.

Lemma 5.3. Λ_c^+ *is a segment of* $\mathbb{R}^+$.

Proof. $0 \in \Lambda_c^+$. Let $\lambda > 0$, $\lambda \in \Lambda_c^+$ and λ_1 be in $]0, \lambda[$.

$$\lim_{X \to -\nu^+} G'_{\lambda,d}(X) \geq 0,$$

where $G'_{\lambda,d}(X) = \beta - \alpha - \int_0^1 g_d^{*\prime}(\sigma_\lambda(X,t))\,dt$ (cf. (43)).

Moreover $\sigma_\lambda(X,t) = X - \lambda \int_b^t f(s)\,ds$. Using the monotony of $g_d^{*\prime} \circ \sigma_\lambda$, we claim, proceeding like in [2], that

$$\lim_{X \to -\nu^+} G'_{\lambda_1,d}(X) \geq 0.$$

Thus λ_1 belongs to Λ_c^+. □

Proof. The proof of the Theorem 5.1 is exactly the same as those of [2]. □

We can notice that Λ_c^+ is not always bounded in $\mathbb{R}$. Let us consider the case of the rigid-plastic bar (cf. section 2.1), under a force which has the value 1 on $[0, \frac{1}{2}]$ and 0 elsewhere, with boundary conditions such that $\beta < \alpha$.
We have then $B_M = [\frac{1}{2}, 1]$ and $B_m = \{0\}$.
The stress has the following expression

$$\begin{aligned}\sigma_\lambda(X,t) &= X - \lambda \int_{\frac{1}{2}} tf(s)\,ds \\ &= \begin{cases} X + \lambda(\frac{1}{2} - t) & \text{if } t \in [0, \frac{1}{2}] \\ X & \text{elsewhere.} \end{cases}\end{aligned}$$

We have to dinstinguish two cases

1. $\lambda < 4$
   Then $\forall t$, $\sigma_\lambda(-1,t) < 1$ and

   $$\lim_{X \to -1^+} G'_{\lambda,d}(X) = \beta - \alpha < 0.$$

   Since meas$(B_M) \neq 0$, Proposition 4.7 ensures the existence of a regular solution to (P_λ).

2. $\lambda \geq 4$

   We have this time

$$G'_{\lambda,d}(-1) = \beta - \alpha - \int_0^{\frac{1}{2}-\frac{2}{\lambda}} dt$$
$$= \beta - \alpha - (\frac{1}{2} - \frac{2}{\lambda}) < 0$$

   which ends to the same conclusion.

We find, in the end, on this example that $\Lambda_r^+ = [0, +\infty[$.

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