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Homogenization of Periodic Finsler Metrics

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We prove an homogenization result in $W^{1,1}$ and in BV for a sequence (F_ε) of functionals of the form

$$F_\varepsilon(u) = \int_0^1 f(\frac{u}{\varepsilon}, u')\ dt$$

where ε is a positive parameter which tends to zero, $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is $[0,1)^n$-periodic in the first variable, convex in the second variable and satisfies a suitable growth condition of order one.

Under the additional assumption that $f(x, \cdot)$ is positively 1-homogeneous, we show how our result is equivalent to the analogous homogenization result (dealt with by Acerbi and Buttazzo) in which growth conditions of order $p > 1$ are considered.

Keywords: Homogenization, Γ-convergence, BV-functions

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1. Introduction

Let us consider a functional $F : W^{1,1}((0,1); \mathbb{R}^n) \to [0, +\infty)$ of the type:

$$F(u) = \int_0^1 f(u, u')\ dt\,, \tag{1.1}$$

where the Borel function $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is assumed to be convex and positively 1-homogeneous in the second variable and satisfies the following growth condition:

$$\lambda|\xi| \le f(x,\xi) \le \Lambda|\xi| \qquad \text{for every } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

with $0 < \lambda \le \Lambda < +\infty$. It is known that the integral functional defined in (1.1) represents the length of a curve u associated to the Finsler metric f (see [9]). Note the absence of regularity (apart from Borel measurability) in the first variable; the main reason being the fact that in some problems of geometry, physics and engineering (such as, for instance, fields theory, general relativity, geometrical optics etc...) we have to consider metrics with singularities. We recall that the case of a Riemannian manifold falls within this setting taking f of the special form $f(x,\xi) = \sqrt{g_{ij}(x)\xi^i\xi^j}$.

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The main result of this paper is an homogenization theorem concerning a sequence of functionals of the type (1.1). Actually, we will study a more general situation.

Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the following properties:

$$f(x, \cdot) \quad \text{is convex for every } x \in \mathbb{R}^n; \tag{1.2}$$

$$f(\cdot, \xi) \quad \text{is } [0,1)^n\text{-periodic for every } \xi \in \mathbb{R}^n; \tag{1.3}$$

$$\lambda|\xi| \leq f(x, \xi) \leq \Lambda(1 + |\xi|) \quad \text{for every } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{1.4}$$

where $0 < \lambda \leq \Lambda < +\infty$ are fixed constants. For every $\varepsilon > 0$ and $u \in W^{1,1}((0,1); \mathbb{R}^n)$, set

$$F_\varepsilon(u) = \int_0^1 f\left(\frac{u}{\varepsilon}, u'\right)\, dt.$$

We shall prove that, for ε going to zero, F_ε tends (technically, Γ-converges on $W^{1,1}((0,1); \mathbb{R}^n)$ with respect to the L^1-topology) to the functional defined by

$$F(u) = \int_0^1 \phi(u')dt \qquad \text{for every } u \in W^{1,1}((0,1); \mathbb{R}^n), \tag{1.5}$$

where $\phi : \mathbb{R}^n \to [0, +\infty)$ is a convex function satisfying (1.4) and given by

$$\phi(\xi) = \lim_{\varepsilon \to 0^+} \left[\inf \left\{ \int_0^1 f(\frac{u}{\varepsilon}, u')\, dt : u \in W^{1,1}((0,1); \mathbb{R}^n),\ u(0) = 0,\ u(1) = \xi \right\} \right],$$

for every $\xi \in \mathbb{R}^n$ (see Theorem 3.1). An analogous result in Sobolev spaces $W^{1,p}((0,1); \mathbb{R}^n)$, with $p > 1$, was proved by Acerbi and Buttazzo in [1].

In the study of minimum problems involving the functionals F_ε the linear growth (1.4) does not allow the use of the direct methods of the Calculus of Variations, due to the lack of compactness of minimizing sequences. This naturally leads to extend F_ε to the whole space $BV((0,1); \mathbb{R}^n)$ of functions of bounded variation, with $F_\varepsilon = +\infty$ outside $W^{1,1}((0,1); \mathbb{R}^n)$. Also for such extensions we give an homogenization theorem, proving that the limit is just the L^1-lower semicontinuous envelope on $BV((0,1); \mathbb{R}^n)$ of the functional which coincides with F, given in (1.5), on $W^{1,1}((0,1); \mathbb{R}^n)$, and takes the value $+\infty$ otherwise (see Theorem 5.2).

Finally, a particular attention is given to the case of periodic Finsler metrics (i.e. when f is positively 1-homogeneous in the second variable). Indeed, in this case, the homogenization result can be proved in a simpler way, by using a reparametrization technique; moreover, it is possible to show the equivalence with the analogous result on the homogenization of metrics with growth $p > 1$ proved by Acerbi and Buttazzo in [1] (see Remark 4.3).

The paper is organized as follows: in Section 2 we recall some definitions and state some results on Γ-convergence. In Section 3, we prove the homogenization result in $W^{1,1}((0,1); \mathbb{R}^n)$ and in Section 4 we specialize it in the positively 1-homogeneous case. Finally, in Section 5 and 6 we study the homogenization in $BV((0,1); \mathbb{R}^n)$ without and with boundary conditions, respectively.

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2. Notation, preliminaries and first results

Throughout the paper $n \geq 1$ is a fixed integer. We set $I = (0,1)$. We denote by $\mathcal{A}(\mathbb{R})$ the family of all bounded open subsets of $\mathbb{R}$. If B is a Borel subset of $\mathbb{R}$, $|B|$ is its Lebesgue measure. Given an open set $A \subseteq \mathbb{R}$, we denote by $\mathcal{C}_0^\infty(A)$ the set of the infinitely differentiable real valued functions with compact support contained in A. The spaces $L^p(A;\mathbb{R}^n)$ and $W^{1,p}(A;\mathbb{R}^n)$, $(1 \leq p \leq +\infty$, A open subset of $\mathbb{R})$ will be simply denoted by $L^p(A)$ and $W^{1,p}(A)$, respectively.

We recall that the space $BV(A)$ of functions of bounded variation on an open subset A of $\mathbb{R}$ and with values in $\mathbb{R}^n$ can be defined as the space of those functions $u \in L^1(A)$ whose distributional derivative u' is a vector-valued Radon measure for which the total variation in A, denoted by $|u'|(A)$ or $\int_A |u'|$, is finite. We indicate by u'_a and u'_s the absolutely continuous and the singular part of the derivative u' with respect to the Lebesgue measure, and we set $u^+(t_0) = \lim_{t\to t_0^+} u(t)$, $u^-(t_0) = \lim_{t\to t_0^-} u(t)$.

Let $\phi\colon \mathbb{R}^n \to [0,+\infty)$ be a convex function such that there exists $\Lambda > 0$ with $\phi(\xi) \leq \Lambda(1+|\xi|)$ for every $\xi \in \mathbb{R}^n$; then the limit

$$\phi^\infty(\xi) = \lim_{t\to+\infty} \frac{\phi(t\xi)}{t}$$

exists and is finite for every $\xi \in \mathbb{R}^n$, thus defining the so-called *recession function* $\phi^\infty\colon \mathbb{R}^n \to [0,+\infty)$ of ϕ. Note that ϕ^∞ is positively homogeneous of degree 1. For every $A \in \mathcal{A}(\mathbb{R})$ and $u \in BV(A)$ we set

$$\int_A \phi(u') = \int_A \phi(u'_a)\, dt + \int_A \phi^\infty\Big(\frac{u'_s}{|u'_s|}\Big)|u'_s| \tag{2.1}$$

where $\dfrac{u'_s}{|u'_s|}$ is the Radon-Nikodym derivative of u'_s with respect to its total variation, and the last integral denotes the integration of $\phi^\infty(\frac{u'_s}{|u'_s|})$ with respect to $|u'_s|$. Formula (2.1) defines a "natural" extension of $\int_A \phi(u')\, dt$ from $W^{1,1}(A)$ to $BV(A)$; indeed, by [13], the right-hand side of (2.1) is the $L^1(A)$-lower semicontinuous envelope on $BV(A)$ of the functional which takes the value $\int_A \phi(u')\, dt$ on the functions $u \in W^{1,1}(A)$ and $+\infty$ on $BV(A) \setminus W^{1,1}(A)$.

Γ-convergence

Let X be a metric space endowed with a metric d. Let (F_h) be a sequence of functionals defined on X with values in $[-\infty,+\infty]$. For every $x \in X$ we set

$$\begin{aligned}
\big(\Gamma\text{-}\liminf_{h\to+\infty} F_h\big)(x) &= \inf\Big\{\liminf_{h\to+\infty} F_h(x_h) : d(x_h,x) \to 0\Big\}\\
\big(\Gamma\text{-}\limsup_{h\to+\infty} F_h\big)(x) &= \inf\Big\{\limsup_{h\to+\infty} F_h(x_h) : d(x_h,x) \to 0\Big\}.
\end{aligned}$$

We say that the sequence (F_h) Γ-converges to a functional $F\colon X \to [-\infty,+\infty]$ with respect to the topology induced by d, and we write $F_h \xrightarrow{\Gamma} F$ or $\Gamma\text{-}\lim_{h\to+\infty} F_h = F$, if

$$\big(\Gamma\text{-}\liminf_{h\to+\infty} F_h\big)(x) = \big(\Gamma\text{-}\limsup_{h\to+\infty} F_h\big)(x) = F(x)$$

for every $x \in X$. This means that for every $x \in X$: (a) if (x_h) converges to x then $F(x) \leq \liminf_{h\to+\infty} F_h(x_h)$ and (b) there exists a sequence (x_h) converging to x such that $F(x) = \lim_{h\to+\infty} F_h(x_h)$. It turns out that $\Gamma\text{-}\liminf_{h\to+\infty} F_h$, $\Gamma\text{-}\limsup_{h\to+\infty} F_h$, and $\Gamma\text{-}\lim_{h\to+\infty} F_h$ are lower semicontinuous on X.

If (F_h) is a constant sequence, i.e. $F_h = F$ for every h and for some functional F, then the Γ-limit exists and is given by the lower semicontinuous envelope of F.

Given a family $(F_\varepsilon)_{\varepsilon>0}$ of functionals on X, we say that it Γ-converges to a functional F for ε which tends to 0 if (F_{ε_h}) Γ-converges to F whenever (ε_h) is an infinitesimal sequence of positive numbers. We define (see [5, Proposition 3.3])

$$
\begin{aligned}
\big(\Gamma\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon\big)(x) &= \inf\{\big(\Gamma\text{-}\liminf_{h\to+\infty} F_{\varepsilon_h}\big)(x) : (\varepsilon_h) \text{ infinitesimal sequence}\}\\
\big(\Gamma\text{-}\limsup_{\varepsilon\to0^+} F_\varepsilon\big)(x) &= \sup\{\big(\Gamma\text{-}\limsup_{h\to+\infty} F_{\varepsilon_h}\big)(x) : (\varepsilon_h) \text{ infinitesimal sequence}\}\,.
\end{aligned}
$$

Both functionals are lower semicontinuous on X.

For the definitions and the main properties of Γ-convergence and relaxation we refer to [10, 11, 6, 7].

Setting of the problem and first results

Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ be a Borel function satisfying the following properties

$$
\begin{aligned}
&f(x,\cdot) \quad \text{is convex for every } x \in \mathbb{R}^n; && (2.2)\\
&f(\cdot,\xi) \quad \text{is } [0,1)^n\text{-periodic for every } \xi \in \mathbb{R}^n; && (2.3)\\
&\lambda|\xi| \leq f(x,\xi) \leq \Lambda(1+|\xi|) \quad \text{for every } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, && (2.4)
\end{aligned}
$$

where $0 < \lambda \leq \Lambda < +\infty$ are fixed constants. For every $\varepsilon > 0$ and $A \in \mathcal{A}(\mathbb{R})$ we define

$$
F_\varepsilon(u,A) = \begin{cases} \int_A f\left(\frac{u}{\varepsilon}, u'\right)\, dt & \text{if } u \in W^{1,1}(A)\\ +\infty & \text{if } u \in BV(A) \setminus W^{1,1}(A). \end{cases} \tag{2.5}
$$

Remark 2.1. For every $A \in \mathcal{A}(\mathbb{R})$ set $G(u,A) = \int_A |u'|\, dt$ if $u \in W^{1,1}(A)$, and $G(u,A) = +\infty$ if $u \in BV(A)\setminus W^{1,1}(A)$. By standard lower semicontinuity and approximation results for functions of bounded variations (see, e.g., [12]), it turns out that the $L^1(A)$-lower semicontinuous envelope of $G(\cdot,A)$ is given by

$$
\overline{G}(u,A) = \int_A |u'| \tag{2.6}
$$

for every $u \in BV(A)$. Therefore, estimates (2.4) on f imply that for every $A \in \mathcal{A}(\mathbb{R})$ the Γ-lower and the Γ-upper limits of any sequence $(F_{\varepsilon_h}(\cdot,A))$ with respect to the $L^1(A)$-topology are finite on $BV(A)$. In particular, if the sequence admits a Γ-limit $F(\cdot,A)$, then

$$
\lambda \int_A |u'| \leq F(u,A) \leq \Lambda\big(|A| + \int_A |u'|\big) \tag{2.7}
$$

for every $u \in BV(A)$.

Proposition 2.2. *Let (F_{ε_h}) be a sequence of functionals as defined in (2.5), with $\varepsilon_h \to 0$. Then there exists a subsequence $(F_{\varepsilon_{\sigma(h)}})$ of (F_{ε_h}) and a functional F defined on the pairs (u, A) with $A \in \mathcal{A}(\mathbb{R})$ and $u \in BV(A)$, such that*

$$F_{\varepsilon_{\sigma(h)}}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A) \qquad \text{on } BV(A) \text{ with respect to the } L^1(A)\text{-topology,}$$

for every $A \in \mathcal{A}(\mathbb{R})$. Moreover, for every such set A and $u \in BV(A)$ there exists a Borel measure on A which coincides with $F(u, \cdot)$ on the open subsets of A.

Remark 2.3. The result of this proposition clearly implies that

$$F_{\varepsilon_{\sigma(h)}}(\cdot, A) \xrightarrow{\Gamma} F(\cdot, A) \qquad \text{on } W^{1,1}(A) \text{ with respect to the } L^1(A)\text{-topology.}$$

The proof of Proposition 2.2 relies on the following technical lemma, which will be needed later on, too, and which can be obtained following the proof of Theorem 19.1 in [7].

Lemma 2.4. *For every $\eta > 0$ and for every $A', A'', B \in \mathcal{A}(\mathbb{R})$, with $A' \subset\subset A''$, there exists a constant $M > 0$ with the following property: for every $\varepsilon > 0$, $u \in W^{1,1}(A'')$ and $v \in W^{1,1}(B)$ there exists a function $\varphi \in C_0^\infty(A'')$, with $\varphi = 1$ in a neighbourhood of $\overline{A'}$ and $0 \le \varphi \le 1$ such that*

$$F_\varepsilon(\varphi u + (1-\varphi)v, A' \cup B) \le (1+\eta)[F_\varepsilon(u, A'') + F_\varepsilon(v, B)] + M\|u - v\|_{L^1(S)} + \eta,$$

where $S = (A'' \setminus A') \cap B$.

Proof (of Proposition 2.2). The compactness for the Γ-convergence (see Theorem 16.9 in [7]) guarantees the existence of a subsequence $(F_{\varepsilon_{\sigma(h)}})$ with the following property: if for every $A \in \mathcal{A}(\mathbb{R})$ we denote by $F^-(\cdot, A)$ and $F^+(\cdot, A)$ the Γ-lower and the Γ-upper limits, respectively, of the sequence $(F_{\varepsilon_{\sigma(h)}}(\cdot, A))$ on $BV(A)$, then the inner regular envelopes of F^+ and F^- have a common value $F(u, A)$ at every pair (u, A) with $A \in \mathcal{A}(\mathbb{R})$ and $u \in BV(A)$; i.e.

$$F(u, A) = \sup\{F^\pm(u, A') : \ A' \in \mathcal{A}(\mathbb{R}),\ A' \subset\subset A\}.$$

By means of Lemma 2.4 and standard techniques it is possible to see that $F^+(u, \cdot)$ is inner regular on A for every $u \in BV(A)$, so that $F \le F^- \le F^+ = F$. In other words, we have the convergence of $(F_{\varepsilon_{\sigma(h)}}(\cdot, A))$ to $F(\cdot, A)$ on $BV(A)$.

The measure property of $F(u, \cdot)$ follows from Theorem 18.5 in [7], taking into account Lemma 2.4. □

3. Homogenization in $W^{1,1}$

Here and in the sequel f is the function introduced in Section 2 satisfying conditions (2.2), (2.3) and (2.4), and (F_ε) is the family of functionals defined in (2.5). In this section we prove the following result.

Theorem 3.1. *For every bounded open subset A of $\mathbb{R}$ the limit $\Gamma\text{-}\lim_{\varepsilon \to 0^+} F_\varepsilon(\cdot, A)$ exists on $W^{1,1}(A)$ with respect to the $L^1(A)$-topology, and, for every $u \in W^{1,1}(A)$, takes the value*

$$F(u, A) = \int_A \phi(u')\, dt\,, \tag{3.1}$$

where $\phi : \mathbb{R}^n \to [0, +\infty)$ is a convex function such that

$$\lambda|\xi| \leq \phi(\xi) \leq \Lambda(1 + |\xi|), \qquad \textit{for every } \xi \in \mathbb{R}^n. \tag{3.2}$$

Moreover, ϕ is given by the following formula:

$$\phi(\xi) = \lim_{\varepsilon \to 0^+} \left[\inf \left\{ \int_0^1 f(\frac{u}{\varepsilon}, u') \ dt : u \in W^{1,1}(I),\ u(0) = 0,\ u(1) = \xi \right\} \right], \tag{3.3}$$

for every $\xi \in \mathbb{R}^n$. Finally, for every $A \in \mathcal{A}(\mathbb{R})$ and $u_0 \in W^{1,1}(A)$ the family $(F_\varepsilon(\cdot, A))$ Γ-converges to $F(\cdot, A)$ on $u_0 + W_0^{1,1}(A)$ with respect to the $L^1(A)$-topology.

Proposition 3.2. *The limit in* (3.3) *exists for every $\xi \in \mathbb{R}^n$ and defines a continuous function ϕ: $\mathbb{R}^n \to [0, +\infty)$ satisfying* (3.2).

Proof. The existence of the limit in (3.3) can be obtained as in Proposition III.8 in [1], the proof of which is independent of the fact that the growth condition for f is of order $p > 1$. Estimates (3.2) are an easy consequence of (2.4). Let us come to the proof of the continuity.

For every $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$ set

$$M_\varepsilon(\xi) = \inf \left\{ \int_0^1 f(\frac{u}{\varepsilon}, u') \ dt : u \in W^{1,1}(I),\ u(0) = 0,\ u(1) = \xi \right\}. \tag{3.4}$$

Fix $\xi, \zeta \in \mathbb{R}^n$ and $\varepsilon > 0$. Let $u \in W^{1,1}(I)$, with $u(0) = 0$ and $u(1) = \xi$. For any $\sigma \in (0, 1)$, define

$$v_\sigma(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1 - \sigma; \\ u(1 - \sigma) + \frac{t - (1 - \sigma)}{\sigma}[\zeta - u(1 - \sigma)] & \text{if } 1 - \sigma \leq t \leq 1. \end{cases}$$

Then, $v_\sigma \in W^{1,1}(I)$, $v_\sigma(0) = 0$ and $v_\sigma(1) = \zeta$. Hence, it follows that

$$\begin{aligned} M_\varepsilon(\zeta) &\leq \int_0^1 f\left(\frac{v_\sigma}{\varepsilon}, v_\sigma'\right) \ dt \leq \int_0^1 f\left(\frac{u}{\varepsilon}, u'\right) \ dt + \Lambda \int_{1-\sigma}^1 \left(1 + \frac{1}{\sigma}|\zeta - u(1 - \sigma)|\right) \ dt \\ &\leq \int_0^1 f\left(\frac{u}{\varepsilon}, u'\right) \ dt + \Lambda(\sigma + |\zeta - u(1 - \sigma)|). \end{aligned}$$

When σ tends to 0 we obtain

$$M_\varepsilon(\zeta) \leq \int_0^1 f\left(\frac{u}{\varepsilon}, u'\right) \ dt + \Lambda|\zeta - \xi|.$$

Taking the infimum with respect to u and hence passing to the limit, we have

$$\lim_{\varepsilon \to 0^+} M_\varepsilon(\zeta) \leq \lim_{\varepsilon \to 0^+} M_\varepsilon(\xi) + \Lambda|\zeta - \xi|.$$

The result follows by symmetry. □

Lemma 3.3. *Let (ε_h) be an infinitesimal sequence of positive numbers such that for every $A \in \mathcal{A}(\mathbb{R})$ the limit $F(\cdot, A) = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}(\cdot, A)$ exists on $W^{1,1}(A)$ or on $BV(A)$ (with respect to the $L^1(A)$-topology). Then for every $u \in W^{1,1}(A)$ or, respectively, $u \in BV(A)$, and for every $\zeta \in \mathbb{R}^n$ and $T \in \mathbb{R}$ we have*

$$F(u+\zeta, A) = F(u, A), \qquad F(u \circ \psi_T, A+T) = F(u, A),$$

where $\psi_T(t) = t - T$.

Proof. The proof is an easy consequence of the definition of Γ-convergence, taking into account that $\zeta \in \mathbb{R}^n$ can be approximated by a sequence $(\varepsilon_h m_h)$ with $m_h \in \mathbb{Z}^n$. $\square$

Proof (of Theorem 3.1). Assume that (ε_h) is an infinitesimal sequence of positive numbers such that for every $A \in \mathcal{A}(\mathbb{R})$ the limit $F(\cdot, A) = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}(\cdot, A)$ exists on $W^{1,1}(A)$ (with respect to the $L^1(A)$-topology). We set $u_\xi(t) = t\xi$ for $\xi \in \mathbb{R}^n$and $t \in \mathbb{R}$.

Step 1. We prove that there exists a convex function $\phi_0\colon \mathbb{R}^n \to [0, +\infty)$ such that

$$F(u, A) = \int_A \phi_0(u')\, dt$$

for every $A \in \mathcal{A}(\mathbb{R})$ and $u \in W^{1,1}(A)$. Let $A \in \mathcal{A}(\mathbb{R})$ be fixed. Then for every $u \in W^{1,1}(A)$ and E open subset of A

(a) $F(u, E) = F(v, E)$ if $u = v$ a.e. on E;
(b) there exists a Borel measure on A which coincides with $F(u, \cdot)$ on the open subsets of A;
(c) $F(u, E) \le \Lambda \int_E (1 + |u'|)\, dt$;
(d) $F(u+\zeta, E) = F(u, E)$ for every $\zeta \in \mathbb{R}^n$;
(e) $F(\cdot, E)$ is sequentially weakly lower semicontinuous on $W^{1,1}(A)$.

Indeed, property (a) is obvious, (b), (c) and (d) follow from Proposition 2.2, Remark 2.1 and Lemma 3.3, respectively, while (e) is an immediate consequence of the fact that F is a Γ- limit with respect to the L^1-topology.

Therefore we can apply Theorem 1.1 in [4] (see also [2]): the function

$$\phi_0(t, \xi) = \limsup_{\rho\to 0^+} \frac{F(u_\xi, (t-\rho, t+\rho))}{2\rho} \tag{3.5}$$

gives the integral representation

$$F(u, A) = \int_A \phi_0(t, u')\, dt$$

for every $u \in W^{1,1}(A)$. By (3.5) and Lemma 3.3 we obtain that $\phi_0(t, \xi)$ is constant with respect to the first variable; thus we can drop the dependence on t. Moreover, the function ϕ_0 is a convex function and it is not difficult to see that (2.7) and (3.5) implies (3.2) for ϕ_0; hence, the function ϕ_0 turns out to be continuous.

Step 2. $\phi(\xi) \le \phi_0(\xi)$ for every $\xi \in \mathbb{R}^n$.

Let $\xi \in \mathbb{R}^n$. From the definition of Γ-limit there exists a sequence (u_h) in $W^{1,1}(I)$ converging to u_ξ in $L^1(I)$, such that

$$\lim_{h\to+\infty} F_{\varepsilon_h}(u_h, I) = F(u_\xi, I) = \phi_0(\xi).$$

By Lemma 2.4 for every $0 < \eta < 1/2$ there exists $M > 0$ such that for every $h \in \mathbb{N}$ we can find a function $w_h \in W^{1,1}(I)$ with $w_h(0) = 0$ and $w_h(1) = \xi$ such that

$$F_{\varepsilon_h}(w_h, I) \le (1+\eta)[F_{\varepsilon_h}(u_h, I) + F_{\varepsilon_h}(u_\xi, A_\eta)] + M\|u_h - u_\xi\|_{L^1(I)} + \eta,$$

where $A_\eta = (0,\eta) \cup (1-\eta, 1)$. Then by formula (3.3) for ϕ

$$\phi(\xi) \le \liminf_{h\to+\infty} F_{\varepsilon_h}(w_h, I) \le (1+\eta)\big(\phi_0(\xi) + 2\Lambda(1+|\xi|)\eta\big) + \eta.$$

We conclude letting η tend to 0.

Step 3. $\phi(\xi) \ge \phi_0(\xi)$ for every $\xi \in \mathbb{R}^n$.

Since ϕ and ϕ_0 are continuous (Proposition 3.2 and Step 1 above) we can assume $\xi \in \mathbb{Q}^n$. Let $M_\varepsilon(\xi)$ be as in (3.4). Fix $\sigma > 0$; then there exist $\eta > 0$ and $u \in W^{1,1}(I)$, with $u(0) = 0$, $u(1) = \xi$, such that

$$\int_0^1 f\left(\frac{u}{\eta}, u'\right)\, dt \le M_\eta(\xi) + \frac{\sigma}{2} \le \lim_{\varepsilon\to 0^+} M_\varepsilon(\xi) + \frac{\sigma}{2} + \frac{\sigma}{2} = \phi(\xi) + \sigma\,. \tag{3.6}$$

Set $v(t) = u(t) - u_\xi(t)$, when $t \in [0,1]$ and extend v by periodicity on the whole real line. For every $h \in \mathbb{N}$, let us define

$$w_h(t) = u_\xi(t) + \frac{\varepsilon_h}{\eta} v\left(\frac{t}{(\varepsilon_h/\eta)}\right) \qquad \text{for every } t \in \mathbb{R}.$$

Since v is bounded, (w_h) converges to u_ξ in the $L^1(I)$-topology. Moreover, since $\xi \in \mathbb{Q}^n$, we may assume that η is such that $\xi/\eta \in \mathbb{Z}^n$. Hence, we have

$$\begin{aligned}\int_0^1 f\left(\frac{w_h}{\varepsilon_h}, w_h'\right)\, dt &\le \sum_{j=0}^{[\frac{\eta}{\varepsilon_h}]} \int_{j\frac{\varepsilon_h}{\eta}}^{(j+1)\frac{\varepsilon_h}{\eta}} f\left(\frac{1}{\eta}u(\frac{t}{\varepsilon_h/\eta} - j) + j\frac{\xi}{\eta}, u'(\frac{t}{\varepsilon_h/\eta} - j)\right)\, dt \\ &\le \left(1 + \frac{\varepsilon_h}{\eta}\right)\int_0^1 f\left(\frac{u}{\eta}, u'\right)\, dt.\end{aligned}$$

Passing to the lower limits and taking into account (3.6), we obtain

$$\begin{aligned}F(u_\xi, I) &\le \liminf_{h\to+\infty} \int_0^1 f\left(\frac{w_h}{\varepsilon_h}, w_h'\right)\, dt \\ &\le \lim_{h\to+\infty}\left(1+\frac{\varepsilon_h}{\eta}\right)\int_0^1 f\left(\frac{u}{\eta}, u'\right)\, dt = \int_0^1 f\left(\frac{u}{\eta}, u'\right)\, dt \le \phi(\xi) + \sigma.\end{aligned}$$

Since, by Step 1, $F(u_\xi, I) = \phi_0(\xi)$, the result follows letting σ tend to 0.

Step 4. Up to now we have proved that all the Γ-convergent sequences (F_{ε_h}), with $\varepsilon_h \to 0$, have the same limit, given by (3.1). Taking into account Proposition 2.2, we obtain the Γ-convergence of (F_ε) by applying Proposition 8.3 in [7] (Urysohn property of Γ-convergence), which asserts that (F_{ε_h}) converges to F if and only if every subsequence of (F_{ε_h}) contains a further subsequence converging to F.

Step 5. Let $A \in \mathcal{A}(\mathbb{R})$ and $u_0 \in W^{1,1}(A)$. In order to prove the Γ-convergence on $u_0 + W_0^{1,1}(A)$ it is now sufficient to show that for every $u \in u_0 + W_0^{1,1}(A)$ we can find a sequence (w_h) in $u_0 + W_0^{1,1}(A)$ such that

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(w_h, A) \leq F(u, A).$$

From the convergence of $(F_{\varepsilon_h}(\cdot, A))$ on $W^{1,1}(A)$ we obtain a sequence (u_h) in $W^{1,1}(A)$ which converges to u in $L^1(A)$ and such that $\lim\limits_{h\to+\infty} F_{\varepsilon_h}(u_h, A) = F(u, A)$. Fix $\eta > 0$ and K compact subset of A. Apply Lemma 2.4 to join the functions u_h and u, with $A'' = A$, $A' \supseteq K$, $B = A \setminus K$. Then there exists a constant $M > 0$ and a sequence (w_h) in $u_0 + W_0^{1,1}(A)$ converging to u in $L^1(A)$, such that

$$F_{\varepsilon_h}(w_h, A) \leq (1+\eta)[F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(u, A \setminus K)] + M\|u_h - u\|_{L^1(A)} + \eta\,.$$

Since $F_{\varepsilon_h}(u, A \setminus K) \leq \Lambda \int_{A\setminus K}(1 + |u'|)\, dt$, we conclude by passing to the upper limit and taking the arbitrariness of η and K into account.

Since this holds for any sequence (F_{ε_h}), the theorem follows taking into account Proposition 8.3 in [7], as in Step 4. $\square$

4. Homogenization in $W^{1,1}$: the positively homogeneous case

In this section we assume that the integrand function f in (2.5) is positively homogeneous of degree 1 in the second variable, i.e.

$$f(x, \alpha\xi) = \alpha f(x, \xi) \qquad \text{for every } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \text{ and } \alpha > 0. \tag{4.1}$$

In Remark 4.3 we point out how, under this additional condition on f, Theorem 3.1 is equivalent to the analogous homogenization theorem proved in [1] for integrand functions satisfying a growth condition of order $p > 1$. We shall also see (Remark 4.4) how Theorem 3.1 can be obtained in a simpler way, if (4.1) holds.

The key fact is that the positive 1-homogeneity makes the functional $F_\varepsilon(\cdot, J)$ (J open interval) invariant under reparametrizations of curves in $W^{1,1}(J)$; more precisely, if $u, v \in W^{1,1}(J)$ and $v = u \circ \tau$ with τ an absolutely continuous, increasing function from J onto J, then $F_\varepsilon(u, J) = F_\varepsilon(v, J)$.

Lemma 4.1. *Let J be a bounded open interval of $\mathbb{R}$, and $u_0 \in W^{1,1}(J)$. Let (u_h) be a sequence in $u_0 + W_0^{1,1}(J)$, bounded in $W^{1,1}(J)$. Then, for every $h \in \mathbb{N}$, there exists a Lipschitz function τ_h from J onto J with strictly positive derivative a.e. and such that $(u_h \circ \tau_h)$ is a sequence of Lipschitz functions which is bounded in $W^{1,\infty}(J)$ and belongs to $u_0 + W_0^{1,1}(J)$.*

Proof. It is not restrictive to assume $J = (0, 1)$. For every $h \in \mathbb{N}$ set $L_h = \int_0^1 |u_h'|\, dt$, and let $\sigma_h : [0, 1] \to [0, 1]$ be the function defined by

$$\sigma_h(t) = \int_0^t \frac{1 + |u_h'|}{1 + L_h}\, d\tau.$$

Clearly, $\sigma_h(0) = 0$, $\sigma_h(1) = 1$ and $\sigma_h' > 0$ a.e.. Set $\tau_h = \sigma_h^{-1}$ and $v_h = u_h \circ \tau_h$; it is easy to see that τ_h and v_h are absolutely continuous and

$$v_h'(s) = u_h'(\tau_h(s)) \frac{1 + L_h}{1 + |u_h'(\tau_h(s))|} \qquad \text{for a.e. } s \in J\,. \tag{4.2}$$

By assumption (L_h) has an upper bound, say C; then (v_h) is a sequence of Lipschitz functions whose Lipschitz constants are bounded by $1 + C$. □

The following proposition is a bit more general than what needed in the subsequent remark, which relates the homogenization results for 1-homogeneous and p-homogeneous integrand functions.

Proposition 4.2. *Let $p > 1$. For every $\varepsilon > 0$ let $f_\varepsilon : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the following properties*

$$f_\varepsilon(x, \cdot) \quad \textit{is positively homogeneous of degree } 1 \textit{ for every } x \in \mathbb{R}^n$$
$$\lambda|\xi| \leq f_\varepsilon(x, \xi) \leq \Lambda(1 + |\xi|) \quad \textit{for every } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $0 < \lambda \leq \Lambda < +\infty$ are fixed constants. For every $A \in \mathcal{A}(\mathbb{R})$, $u \in W^{1,1}(A)$ and $v \in W^{1,p}(A)$ define

$$F_\varepsilon(u, A) = \int_A f_\varepsilon(u(t), u'(t))\, dt\,, \qquad G_\varepsilon(v, A) = \int_A \big(f_\varepsilon(v(t), v'(t))\big)^p\, dt.$$

Then $(F_\varepsilon(\cdot, A))$ Γ-converges on $W^{1,1}(A)$ with respect to the $L^1(A)$-topology for every $A \in \mathcal{A}(\mathbb{R})$ if and only if $(G_\varepsilon(\cdot, A))$ Γ-converges on $W^{1,p}(A)$ with respect to the $L^p(A)$-topology for every $A \in \mathcal{A}(\mathbb{R})$. If $F(\cdot, A)$ and $G(\cdot, A)$ are their respective Γ-limits, then there exists a Borel function $f\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ such that

$$F(u, A) = \int_A f(u, u')\, dt\,, \qquad G(v, A) = \int_A \big(f(v, v')\big)^p\, dt$$

whenever $u \in W^{1,1}(A)$ and $v \in W^{1,p}(A)$.

Proof. *Step 1.* Let (ε_h) be an infinitesimal sequence such that for every $A \in \mathcal{A}(\mathbb{R})$ the sequences $(F_{\varepsilon_h}(\cdot, A))$ and $(G_{\varepsilon_h}(\cdot, A))$ Γ-converge on $W^{1,1}(A)$ in the $L^1(A)$-topology and, respectively, on $W^{1,p}(A)$ in the $L^p(A)$-topology; let $F(\cdot, A)$ and $G(\cdot, A)$ be their respective limits. Then there exist Borel functions $f, g\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ such that

$$F(u, A) = \int_A f(u, u')\, dt\,, \qquad G(v, A) = \int_A g(v, v')\, dt$$

for every $A \in \mathcal{A}(\mathbb{R})$, $u \in W^{1,1}(A)$ and $v \in W^{1,p}(A)$. Moreover,

$$f(x, \xi) = \liminf_{\eta \to 0^+} \frac{1}{\eta} \inf\{F(u, (0, \eta)) : u \in W^{1,1}(0, \eta),\ u(0) = x,\ u(\eta) = x + \eta\xi\}\,, \tag{4.3}$$

$$g(x, \xi) = \liminf_{\eta \to 0^+} \frac{1}{\eta} \inf\{G(v, (0, \eta)) : v \in W^{1,p}(0, \eta),\ v(0) = x,\ v(\eta) = x + \eta\xi\}\,, \tag{4.4}$$

whenever $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

These facts are an easy consequence of Theorem 7.1 in [3] and Theorem 6.1 in [2] (growth condition of order $p>1$ or $p=1$ respectively). We want to show that $g=f^p$.

Let $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$ and $\eta>0$ be fixed. Define

$$I_h=\inf\{F_{\varepsilon_h}(u,(0,\eta)):u\in W^{1,1}(0,\eta),\ u(0)=x,\ u(\eta)=x+\eta\xi\} \tag{4.5}$$

$$J_h=\inf\{G_{\varepsilon_h}(v,(0,\eta)):v\in W^{1,p}(0,\eta),\ v(0)=x,\ v(\eta)=x+\eta\xi\}\,. \tag{4.6}$$

We show that

$$J_h=\eta^{1-p}(I_h)^p. \tag{4.7}$$

Let v be an admissible function for problem (4.6). By Jensen's inequality we have

$$G_{\varepsilon_h}(v,(0,\eta))=\int_0^\eta\big(f_{\varepsilon_h}(v,v')\big)^p\,dt\ge\eta^{1-p}\Big(\int_0^\eta f_{\varepsilon_h}(v,v')\,dt\Big)^p\ge\eta^{1-p}(I_h)^p\,.$$

We prove that this lower bound is actually the infimum J_h. For every $\delta>0$ let u_δ be an admissible function for problem (4.5) such that $F_{\varepsilon_h}(u_\delta,(0,\eta))\le I_h+\delta$. Consider the function $\sigma_\delta\colon[0,\eta]\to[0,\eta]$ defined by

$$\sigma_\delta(t)=\gamma_\delta^{-1}\int_0^t(f_{\varepsilon_h}(u_\delta,u'_\delta)+\delta)\,ds\,,$$

where $\gamma_\delta=\frac{1}{\eta}\int_0^\eta(f_{\varepsilon_h}(u_\delta,u'_\delta)+\delta)\,ds$. Thus, σ_δ is an absolutely continuous function with strictly positive derivative a.e. and $\sigma_\delta(0)=0$, $\sigma_\delta(\eta)=\eta$; then its inverse τ_δ is an absolutely continuous function from $[0,\eta]$ onto $[0,\eta]$ with strictly positive derivative a.e., and $v_\delta=u_\delta\circ\tau_\delta$ is absolutely continuous. From the growth conditions of f_{ε_h} it follows that v'_δ is bounded; in particular $v_\delta\in W^{1,p}(0,\eta)$. Moreover, we have

$$\begin{aligned}\int_0^\eta(f_{\varepsilon_h}(v_\delta,v'_\delta))^p\,ds&=\gamma_\delta^p\int_0^\eta\big(f_{\varepsilon_h}(u_\delta(\tau_\delta(s)),u'_\delta(\tau_\delta(s)))\big)^p\big(f_{\varepsilon_h}(u_\delta(\tau_\delta(s)),u'_\delta(\tau_\delta(s))+\delta)\big)^{-p}\,ds\\&\le\eta\gamma_\delta^p\le\eta\Big(\frac{1}{\eta}(I_h+\delta)+\delta\Big)^p\,.\end{aligned}$$

When δ tends to 0 the last term tends to $\eta^{1-p}(I_h)^p$, hence $J_h\le\eta^{1-p}(I_h)^p$. This proves (4.7).

Let now (δ_h) be an infinitesimal sequence of positive numbers; for every $h\in\mathbb{N}$ let u_h be an admissible function for problem (4.5) such that $F_{\varepsilon_h}(u_h,(0,\eta))\le I_h+\delta_h$. In view of the invariance of $F_{\varepsilon_h}(\cdot,(0,\eta))$ with respect to reparametrizations, we can replace (u_h) by the sequence obtained from Lemma 4.1 applied to (u_h). It can be easily verified that Lemma 2.4 holds without change for the family (F_ε) now under consideration. Thus, as in Step 5 in the proof of Theorem 3.1, we deduce the Γ-convergence of $(F_{\varepsilon_h}(\cdot,(0,\eta))$ on $\{u\in W^{1,1}(0,\eta):\ u(0)=x,\ u(\eta)=x+\eta\xi\}$. By Corollary 7.20 in [7] it follows that

$$\inf\{F(u,(0,\eta)):u\in W^{1,1}(0,\eta),\ u(0)=x,\ u(\eta)=x+\eta\xi\}=\lim_{h\to+\infty}I_h.$$

The same result for the sequence $(G_{\varepsilon_h}(\cdot,(0,\eta)))$ can be obtained by analogous techniques: see Theorems 21.1 and 7.8 in [7]. Thus

$$\inf\{G(v,(0,\eta)):v\in W^{1,p}(0,\eta),\ v(0)=x,\ v(\eta)=x+\eta\xi\}=\lim_{h\to+\infty}J_h\,.$$

Therefore, in view of (4.7),

$$g(x,\xi) = \liminf_{\eta\to 0^+} \frac{1}{\eta} \lim_{h\to+\infty} \eta^{1-p}(I_h)^p = \Big(\liminf_{\eta\to 0^+} \frac{1}{\eta} \lim_{h\to+\infty} I_h\Big)^p = (f(x,\xi))^p\,.$$

Step 2. As remarked above, Lemma 2.4 still holds for the sequence (F_ε) we deal with at present. As a consequence, we obtain for (F_ε) the compactness property stated in Proposition 2.2. The analogous result for the family (G_ε) follows from Theorem 19.6 in [7]. This, together with Step 1, allows to conclude the proof of the Proposition, taking into account the Urysohn property of Γ-convergence (see [7, Proposition 8.3]). ☐

Remark 4.3. [1] deals with the homogenization of functionals

$$G_\varepsilon(u,A) = \int_A g(\frac{u}{\varepsilon}, u')\, dt\,,$$

where $A \in \mathcal{A}(\mathbb{R})$, $u \in W^{1,p}(A)$, $p > 1$, and $g(x,\xi)$ is a non-negative Borel function periodic in x, convex in ξ and such that $\lambda|\xi|^p \le g(x,\xi) \le \Lambda(1+|\xi|^p)$ $(0 < \lambda \le \Lambda < +\infty)$ for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Theorem III.1 in [1] proves that for every $A \in \mathcal{A}(\mathbb{R})$ the limit $\Gamma\text{-}\lim_{\varepsilon\to 0^+} G_\varepsilon(\cdot, A)$ exists on $W^{1,p}(A)$ with respect to the $L^p(A)$-topology, and can be represented in the form

$$\int_A \varphi(u')\, dt\,,$$

where $\varphi : \mathbb{R}^n \to [0,+\infty)$ is a convex function satisfying

$$\lambda|\xi|^p \le \varphi(\xi) \le \Lambda(1+|\xi|^p), \qquad \text{for every } \xi \in \mathbb{R}^n.$$

By means of Proposition 4.2 this result implies Theorem 3.1 in case f (the integrand function in the definition of F_ε) is positively 1-homogeneous in the second variable (take $g = f^p$). On the contrary, if g is positively p-homogeneous in the second variable, then the homogenization result of Theorem III.1 in [1] follows from Theorem 3.1.

Remark 4.4. We point out that if f is positively 1-homogeneous with respect to the second variable, then Step 3 in the proof of Theorem 3.1 can be obtained in the following way. Let $M_\varepsilon(\xi)$ be as in (3.4). For any $h \in \mathbb{N}$ and $\sigma > 0$, there exists $u_h^\sigma \in W^{1,1}(I)$ such that $u_h^\sigma(0) = 0$, $u_h^\sigma(1) = \xi$ and

$$F_{\varepsilon_h}(u_h^\sigma, I) = \int_0^1 f\left(\frac{u_h^\sigma}{\varepsilon_h}, (u_h^\sigma)'\right)\, dt \le M_{\varepsilon_h}(\xi) + \sigma\,.$$

In particular

$$F_{\varepsilon_h}(u_h^\sigma, I) \le \int_0^1 f\left(\frac{t\xi}{\varepsilon_h}, \xi\right)\, dt + \sigma \le \Lambda(1+|\xi|) + \sigma\,;$$

by the growth condition (2.4), it follows that $\|(u_h^\sigma)'\|_{L^1(I)}$ is bounded indipendently of h.

Now, let $\tau_h^\sigma : [0,1] \to [0,1]$ and $v_h^\sigma = u_h^\sigma \circ \tau_h^\sigma$ be as in Lemma 4.1; then, there exists $v^\sigma \in W^{1,1}(I)$, with $v^\sigma(0) = 0$ and $v^\sigma(1) = \xi$, such that, up to a subsequence, (v_h^σ) converges to v^σ uniformly on $[0,1]$. Thanks to the homogeneity of f, we have $F_{\varepsilon_h}(v_h^\sigma, I) = F_{\varepsilon_h}(u_h^\sigma, I)$.

Passing to the lower limits, using Jensen's inequality and taking into account Proposition 3.2, we have

$$\phi_0(\xi) = \phi_0\left(\int_0^1 (v^\sigma)'\,ds\right) \leq \int_0^1 \phi_0((v^\sigma)')\,ds = F(v^\sigma, I) \leq \liminf_{h\to+\infty} F_{\varepsilon_h}(v_h^\sigma, I)$$
$$= \liminf_{h\to+\infty} F_{\varepsilon_h}(u_h^\sigma, I) \leq \lim_{h\to+\infty} M_{\varepsilon_h}(\xi) + \sigma = \phi(\xi) + \sigma.$$

We conclude letting σ tend to 0.

Remark 4.5. We observe that the functional $F(\cdot, I)$ is lower semicontinuous on $W^{1,1}(I)$ with respect to the $L^1(I)$-topology, since it is a Γ-limit. Therefore, an application of Lemma 4.1 guarantees the existence of a solution to the problem of minimizing $F(\cdot, I)$ on the space of curves belonging to $W^{1,1}(I)$, whereas in the general case with linear growth, the lack of compactness of the minimizing sequences leads to introduce the space BV, in order to obtain a solution for the minimum problem. This latter case will be treated in the next section.

5. Homogenization in BV

We now study the homogenization in the space BV. To this purpose, we recall the following result from [8, Theorem 3.5].

Theorem 5.1. *Let F be a functional with values in $[0, +\infty)$ defined on the pairs (u, A) with $A \in \mathcal{A}(\mathbb{R})$ and $u \in BV(A)$, and satisfying the following properties: for every $A \in \mathcal{A}(\mathbb{R})$, $u, v \in BV(A)$ we have*

(i) $F(u + \zeta, A) = F(u, A)$ *for every* $\zeta \in \mathbb{R}^n$*;*
(ii) $F(u \circ \psi_T, A + T) = F(u, A)$ *for every* $T \in \mathbb{R}$*, where* $(\psi_T \circ u)(t) = u(t - T)$*;*
(iii) $F(u, A) \leq \Lambda(|A| + |u'|(A))$*;*
(iv) *there exists a Borel measure μ on A such that $\mu(E) = F(u, E)$ for every open subset E of A;*
(v) *$F(\cdot, A)$ is sequentially $L^1(A)$-lower semicontinuous;*
(vi) *$F(u, A) = F(v, A)$ when $u = v$ a.e. on A;*
(vii) $\inf_{\tau>0} R_\tau F(\cdot, I_0)$ *is sequentially $L^1(I_0)$-lower semicontinuous at u_ξ for every $\xi \in \mathbb{R}^n$ and for every bounded open interval I_0 with $0 \in I_0$ and $|I_0| = 1$;*

here $R_\tau F(w, I_0) = \frac{1}{\tau} F(O_{\frac{1}{\tau}} w, \tau I_0)$, $O_{\frac{1}{\tau}} w(t) = \tau u(\frac{t}{\tau})$, and $u_\xi(t) = t\xi$. Then there exists a Borel function $\phi_0 \colon \mathbb{R}^n \to [0, +\infty)$ such that

$$F(u, A) = \int_A \phi_0(u_a')\,dt + \int_A \phi_0^\infty\Big(\frac{u_s'}{|u_s'|}\Big)\,|u_s'| \quad \textit{for every } A \in \mathcal{A}(\mathbb{R}) \textit{ and } u \in BV(A). \tag{5.1}$$

Theorem 5.2. *For every bounded open subset A of $\mathbb{R}$, the limit $\Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon(\cdot, A)$ exists on $BV(A)$ with respect to the $L^1(A)$-topology, and, for every $u \in BV(A)$, takes the value*

$$F(u, A) = \int_A \phi(u_a')\,dt + \int_A \phi^\infty\Big(\frac{u_s'}{|u_s'|}\Big)\,|u_s'|, \tag{5.2}$$

where ϕ is given in (3.3).

Proof. Let (ε_h) be an infinitesimal sequence of positive numbers such that for every $A \in \mathcal{A}(\mathbb{R})$ the limit $F(\cdot, A) = \Gamma\text{-}\lim_{h\to+\infty} F_{\varepsilon_h}(\cdot, A)$ exists on $BV(A)$, according to Proposition 2.2.

By Lemma 3.3, we obtain that F satisfies (i), and (ii) of Theorem 5.1 and by (2.7) and Proposition 2.2 it follows that F satisfies (iii) and (iv) of Theorem 5.1, respectively. Finally, (v) and (vi) are an immediate consequence of the fact that $F(\cdot, A)$ is a Γ-limit with respect to the $L^1(A)$-topology. Hence, in order to represent F in integral form, it is enough to prove that property (vii) of Theorem 5.1 holds, too.

Let $u \in BV(I)$, $\tau > 0$, and (v_h) a sequence in $W^{1,1}(\tau I_0)$ converging to $O_{\frac{1}{\tau}} u$ and such that

$$F(O_{\frac{1}{\tau}} u, \tau I_0) = \lim_{h\to+\infty} F_{\varepsilon_h}(v_h, \tau I_0).$$

For every $h \in \mathbb{N}$, let $u_h(t) = \frac{1}{\tau} v_h(\tau t)$; clearly, $u_h \in W^{1,1}(I_0)$ and (u_h) converges to u in $L^1(I_0)$. Then, it follows that

$$\begin{aligned} R_\tau F(u, I_0) &= \frac{1}{\tau} F(O_{\frac{1}{\tau}} u, \tau I_0) \\ &= \lim_{h\to+\infty} \frac{1}{\tau} F_{\varepsilon_h}(v_h, \tau I_0) = \lim_{h\to+\infty} \int_{I_0} f\Big(\frac{u_h}{\varepsilon_h/\tau}, u_h'\Big)\, dt \ge G(u), \end{aligned}$$

where $G = \Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon(\cdot, I_0)$. Passing to the infimum when $\tau > 0$, we have

$$\inf_{\tau>0} R_\tau F(u, I_0) \ge G(u) \qquad \text{for every } u \in BV(I_0). \tag{5.3}$$

Let (u_h) be a sequence in $BV(I_0)$ converging to u_ξ in $L^1(I_0)$; recalling the lower semicontinuity of G, by (5.3) we have

$$\begin{aligned} \liminf_{h\to+\infty} \inf_{\tau>0} R_\tau F(u_h, I_0) &\ge \liminf_{h\to+\infty} G(u_h) \\ &\ge G(u_\xi) = \inf_{\tau>0} R_\tau F(u_\xi, I_0). \end{aligned}$$

Indeed, since $u_\xi \in W^{1,1}(I_0)$, by Theorem 3.1, $G(u_\xi) = F(u_\xi, I_0) = \int_{I_0} \phi(\xi)\, dt = \inf_{\tau>0} R_\tau F(u_\xi, I_0)$. This proves property (vii).

By Theorem 5.1 we conclude that F can be represented as in (5.1). In view of Theorem 3.1, $\phi_0 = \phi$.

Since we have proved that all the Γ-convergent sequences (F_{ε_h}) have the same limit, we can conclude as in Step 4 of the proof of Theorem 3.1. □

6. Homogenization in BV with boundary data

Let $a, b \in \mathbb{R}^n$ be fixed. For every $\varepsilon > 0$ and $u \in BV(I)$ define

$$F_\varepsilon^0(u) = \begin{cases} \int_I f(\frac{u}{\varepsilon}, u')\, dt & \text{if } u \in W^{1,1}(I), u(0) = a, u(1) = b, \\ +\infty & \text{otherwise on } BV(I). \end{cases}$$

Theorem 6.1. *The limit* $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F^0_\varepsilon$ *exists on* $BV(I)$ *with respect to the* $L^1(I)$*-topology, and, for every* $u \in BV(I)$*, takes the value*

$$F^0(u) = \int_I \phi(u'_a)\ dt + \int_I \phi^\infty(\frac{u'_s}{|u'_s|})\ |u'_s| + \phi^\infty(u^+(0) - a) + \phi^\infty(b - u^-(1)), \tag{6.1}$$

where ϕ *is defined in* (3.3).

Proof. Denote by $\Phi(u)$ the right-hand side in (6.1). Let (ε_h) be an infinitesimal sequence of positive numbers.

Step 1. We prove that $\Phi \leq \Gamma - \liminf_{h\to+\infty} F^0_{\varepsilon_h}$ on $BV(I)$.

Let (u_h) be a sequence in $W^{1,1}(I)$ which converges in $L^1(I)$ to a function $u \in BV(I)$ and such that $u_h(0) = a$ and $u_h(1) = b$. For every $\sigma > 0$ let $I_\sigma = (-\sigma, 1+\sigma)$ and define $\widetilde{u}_h$ and $\widetilde{u}$ as the extensions of u_h and u, respectively, to I_σ with value a on $(-\sigma, 0)$ and b on $(1, 1+\sigma)$. Then by Theorem 5.2

$$\begin{aligned}\Phi(u) &\leq \int_{I_\sigma} \phi(\widetilde{u}'_a)\ dt + \int_{I_\sigma} \phi^\infty(\frac{\widetilde{u}'_s}{|\widetilde{u}'_s|})| \\ &\leq \liminf_{h\to+\infty} F_{\varepsilon_h}(\widetilde{u}_h, I_\sigma) \leq \liminf_{h\to+\infty} F^0_{\varepsilon_h}(u_h) + 2\Lambda\sigma|\widetilde{u}'_s|.\end{aligned}$$

Let now σ tend to 0.

Step 2. We prove that $\Gamma - \limsup_{h\to+\infty} F^0_{\varepsilon_h} \leq \Phi$ on $BV(I)$.

Let $u \in BV(I)$ and $\sigma > 0$. Define $J_\sigma = (\sigma, 1-\sigma)$ and

$$u_\sigma = \begin{cases} a & \text{on } (0,\sigma), \\ u & \text{on } J_\sigma, \\ b & \text{on } (1-\sigma, 1). \end{cases}$$

By Theorem 5.2 there exists a sequence (u_h) in $W^{1,1}(I)$ which converges to u_σ in $L^1(I)$ and

$$\lim_{h\to+\infty} F_{\varepsilon_h}(u_h, I) = \int_I \phi(u'_\sigma)\,.$$

Apply Lemma 2.4 with $A'' = I \supset\supset A' \supset\supset J_\sigma$, $B = I \setminus \overline{J}_\sigma$, $u = u_h$ and $v = u_\sigma$. Then for every $\eta > 0$ we can find a sequence (w_h) in $W^{1,1}(I)$, with $w_h(0) = a$ and $w_h(1) = b$, which converges to u_σ in $L^1(I)$ and

$$\limsup_{h\to+\infty} F_{\varepsilon_h}(w_h, I) \leq (1+\eta)[\int_I \phi(u'_\sigma) + 2\Lambda\sigma] + \eta\,.$$

Thus

$$(\Gamma - \limsup_{h\to+\infty} F^0_{\varepsilon_h})(u_\sigma) \leq (1+\eta)[\int_I \phi(u'_\sigma) + 2\Lambda\sigma] + \eta\,.$$

Now let η and σ tend to 0; taking into account the lower semicontinuity of the upper Γ-limit we have

$$\begin{aligned}(\Gamma\text{-}\limsup_{h\to+\infty} F^0_{\varepsilon_h})(u) &\leq \liminf_{\sigma\to 0^+}\int_I \phi(u'_\sigma)\\ &\leq \liminf_{\sigma\to 0^+}[\int_I \phi(u') + \phi^\infty(u^+(\sigma)-a) + \phi^\infty(b-u^-(1-\sigma)) + 2\Lambda\sigma]\\ &= \Phi(u)\,.\end{aligned}$$

□

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