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Convergence Theorems for a Pair of Nonexpansive Mappings

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1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then a mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping T of C into itself is called quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. For two mappings S, T of C into itself, Das and Debata [2] considered the following iteration scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. In this case of $S = T$, such an iteration scheme was considered by Ishikawa [5]; see also Mann [7]. Das and Debata [2] studied the strong convergence of the iterates $\{x_n\}$ defined by (1.1) in the case when E is strictly convex and S, T are quasi-nonexpansive mappings; see also Rhoades [10]. On the other hand, Tan and Xu [15] discussed the weak convergence of the iterates $\{x_n\}$ defined by (1.1) in the case when E is uniformly convex and S, T are nonexpansive mappings with $S = T$. Recently Takahashi and Kim [13] proved the following: Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let U be a nonexpansive mapping of C into itself. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1.1) in the case of $U = S = T$, where $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ or $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, converge weakly to a fixed point of U. Further, they obtained the following: Let C be a nonempty closed convex subset of a strictly convex Banach space E and let U be a nonexpansive mapping of C into itself such that $U(C)$ is contained in a compact subset of C. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1.1) in the case of $U = S = T$, converge strongly to a fixed point of U. $\{\alpha_n\}$ and $\{\beta_n\}$ in [13] are different from those in [2], [10] and [15].

In this paper, we study the iteration schemes defined by (1.1). We first consider the weak convergence of iterates $\{x_n\}$ defined by (1.1) in a uniformly convex Banach space which

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satisfies Opial's condition or whose norm is Fréchet differentiable. Further we discuss the strong convergence of iterates $\{x_n\}$ defined by (1.1) in a strictly convex Banach space. The convergence theorems are generalizations of [13].

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a Banach space and let I be the identity operator on E. Let C be a nonempty subset of E. Then, a mapping T of C into itself is said to be *nonexpansive* on C if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into itself. Then we denote by $F(T)$ the set of fixed points of T. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} \mid \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if

$$\delta(\varepsilon) > 0$$

for every $\varepsilon > 0$. If E is uniformly convex, then for each r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\| \frac{x + y}{2} \right\| \leq r \left(1 - \delta\left(\frac{\varepsilon}{r} \right) \right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x - y\| \geq \varepsilon$. A Banach space E is also said to be strictly convex if

$$\left\| \frac{x + y}{2} \right\| < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A uniformly convex Banach space is reflexive and strictly convex. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1 - \lambda)x + \lambda y\| \text{ for } x, y \in E \text{ and } \lambda \in (0, 1),$$

then $x = y$. Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals f on E. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. It is said to be Fréchet differentiable if for each x with $\|x\| = 1$, this limit is attained uniformly for y with $\|y\| = 1$. When $\{x_n\}$ is a sequence in E, $x_n \to x$ and $x_n \rightharpoonup x$ will symbolize strong and weak convergence, respectively. We also denote by $\overline{\text{co}}A$ the closure of the convex hull of A. A Banach space E is said to satisfy Opial's condition [8] if $x_n \rightharpoonup x$ and $x \neq y$ imply

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|.$$

With each $x \in E$, we associate the set

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

Then the multivalued operator $J : E \to E^*$ is called the duality mapping of E. If the norm of E is Gâteaux differentiable, the duality mapping is single-valued. The following lemma which was proved by Reich [9] is essential to prove the theorems in Section 3; see also [13].

Lemma 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{T_1, T_2, T_3, \cdots\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_1$ for all $n \geq 1$. Then, the set $\bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\{S_m x : m \geq n\} \cap F$ consists of at most one point, where $F = \bigcap_{n=1}^{\infty} F(T_n)$.*

We also know the following lemma proved by Schu [11].

Lemma 2.2. *Let E be a uniformly convex Banach space, let $\{t_n\}$ be a sequence of real numbers such that $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = a$ for some $a \geq 0$. Then, $\lim \|x_n - y_n\| = 0$.*

3. Weak convergence theorems

In this section, we prove weak convergence theorems for a pair of nonexpansive mappings. Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself. For $n \geq 1$, $\alpha_n, \beta_n \in [0,1]$ and $x \in C$, we define a mapping T_n of C into itself by

$$T_n x = \alpha_n S[\beta_n T x + (1-\beta_n)x] + (1-\alpha_n)x$$

for every $x \in C$. Then, T_n is also nonexpansive; see [15]. Further we have the following: If $0 < \alpha_n \leq 1$, $0 < \beta_n < 1$ and $F(S) \cap F(T)$ is nonempty, then

$$F(T_n) = F(S) \cap F(T).$$

Further, in the case of $S = T$, if $0 < \alpha_n \leq 1$ and $0 \leq \beta_n < 1$, then

$$F(T_n) = F(T).$$

In fact, if $z \in F(S) \cap F(T)$, then it is obvious that $z \in F(T_n)$. Conversely, if $z \in F(T_n)$, we have

$$z = \alpha_n S[\beta_n T z + (1-\beta_n) z] + (1-\alpha_n) z$$

and hence $z = S[\beta_n T z + (1-\beta_n) z]$. Let $w \in F(S) \cap F(T)$. Then we obtain

$$\begin{aligned} \|z - w\| &= \|S[\beta_n T z + (1-\beta_n) z] - w\| \\ &\leq \|\beta_n T z + (1-\beta_n) z - w\| \\ &\leq \beta_n \|T z - w\| + (1-\beta_n) \|z - w\| \\ &\leq \|z - w\|. \end{aligned}$$

So we have $\|z - w\| = \|\beta_n T z + (1-\beta_n) z - w\| = \|T z - w\|$. Since E is strictly convex, we have $T z = z$ and hence $S z = z$. This implies $F(T_n) = F(S) \cap F(T)$. In the case of

$S = T$, the result has already been proved in [13]. We also know that the iterates $\{x_n\}$ defined by (1.1) can be written as

$$x_{n+1} = T_n T_{n-1} \cdots T_1 x_1$$

for all $n \geq 1$. The following theorem is used to discuss the weak convergence of the iterates defined by (1.1).

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$$

for all $n \geq 1$, where $\alpha_n, \beta_n \in [0,1]$. Then the following hold:

(i) *If $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \rightharpoonup y$ implies $y \in F(S)$;*
(ii) *if $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \rightharpoonup y$ implies $y \in F(T)$;*
(iii) *if α_n, $\beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \rightharpoonup y$ implies $y \in F(S) \cap F(T)$.*

Proof. Let $x_1 \in C$ and $w \in F(S) \cap F(T)$. Putting $r = \|x_1 - w\|$, then the set $D = \{y \in E : \|y - w\| \leq r\} \cap C$ is a nonempty bounded closed convex subset of C which is invariant under S and T. So we may assume, without loss of generality, that C is bounded. By the definition of $\{x_n\}$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \|\alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n - w\| \\
&\leq \alpha_n \|S[\beta_n T x_n + (1-\beta_n)x_n] - w\| + (1-\alpha_n)\|x_n - w\| \\
&\leq \alpha_n \|\beta_n T x_n + (1-\beta_n)x_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&\leq \alpha_n[\beta_n \|T x_n - w\| + (1-\beta_n)\|x_n - w\|] + (1-\alpha_n)\|x_n - w\| \\
&\leq \|x_n - w\|
\end{aligned}$$

and hence the limit of $\{\|x_n - w\|\}$ exists. Put $c = \lim_{n\to\infty} \|x_n - w\|$ and $y_n = \beta_n T x_n + (1-\beta_n)x_n$ for all $n \geq 1$. Since

$$\begin{aligned}
\|S y_n - w\| &\leq \|y_n - w\| \\
&= \|\beta_n T x_n + (1-\beta_n)x_n - w\| \\
&\leq \beta_n \|T x_n - w\| + (1-\beta_n)\|x_n - w\| \\
&\leq \|x_n - w\|,
\end{aligned}$$

we have

$$\begin{aligned}
\limsup_{n\to\infty} \|S y_n - w\| &\leq \limsup_{n\to\infty} \|y_n - w\| \\
&\leq \lim_{n\to\infty} \|x_n - w\| \\
&= c.
\end{aligned}$$

Further, we have

$$\begin{aligned}\lim_{n\to\infty} \|\alpha_n(Sy_n - w) + (1-\alpha_n)(x_n - w)\| &= \lim_{n\to\infty} \|x_{n+1} - w\| \\ &= c.\end{aligned}$$

If $0 < a \le \alpha_n \le b < 1$, by Lemma 2.2, we have

$$\lim_{n\to\infty} (Sy_n - x_n) = 0.$$

We have also

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Sy_n\| + \|Sy_n - x_n\| \\ &\le \|x_n - y_n\| + \|Sy_n - x_n\| \\ &= \beta_n \|Tx_n - x_n\| + \|Sy_n - x_n\| .\end{aligned} \tag{3.1}$$

On the other hand, if $0 < a \le \alpha_n \le 1$, we have, for $n \ge 1$,

$$\begin{aligned}\|x_{n+1} - w\| &\le \alpha_n \|Sy_n - w\| + (1-\alpha_n) \|x_n - w\| \\ &\le \alpha_n \|y_n - w\| + (1-\alpha_n) \|x_n - w\|\end{aligned}$$

and hence

$$\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \le \|y_n - w\| - \|x_n - w\| .$$

So, we have

$$c \le \liminf_{n\to\infty} \|y_n - w\| .$$

Since $\limsup_{n\to\infty} \|y_n - w\| \le c$, we have

$$\begin{aligned}c &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|\beta_n(Tx_n - w) + (1-\beta_n)(x_n - w)\| .\end{aligned} \tag{3.2}$$

Now we show (i). Assume $x_{n_i} \rightharpoonup y$. Then since $0 \le \beta_n \le b < 1$, we have $\liminf_{i\to\infty} \beta_{n_i} = 0$ or $\liminf_{i\to\infty} \beta_{n_i} > 0$. If $\liminf_{i\to\infty} \beta_{n_i} > 0$, from (3.2) and Lemma 2.2, we have

$$\lim_{i\to\infty} [Tx_{n_i} - x_{n_i}] = 0.$$

So, from (3.1), we have

$$\lim_{i\to\infty} [Sx_{n_i} - x_{n_i}] = 0.$$

Since $I - S$ is demiclosed [1], we have $y \in F(S)$. If $\liminf_{i\to\infty} \beta_{n_i} = 0$, then since $\{x_n\}$ is bounded, by (3.1) we have a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} [Sx_{n_{i_j}} - x_{n_{i_j}}] = 0.$$

Since $I - S$ is demiclosed, we have $y \in F(S)$. This completes the proof of (i). Next we show (ii). Since $0 < a \le \alpha_n \le 1$, we have (3.2). By $0 < a \le \beta_n \le b < 1$ and Lemma 2.2, we have

$$\lim_{n\to\infty} [Tx_n - x_n] = 0.$$

Since $x_{n_i} \rightharpoonup y$ and $I - T$ is demiclosed, we have $y \in F(T)$. This completes the proof of (ii). (iii) is obvious from (i) and (ii). $\square$

Using Theorem 3.1, we can prove the following theorem which was obtained by Takahashi and Kim [13].

Theorem 3.2 ([13]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping of C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ or $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some $a,b \in \mathbb{R}$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let z be a fixed point of T. Then, as in the proof of Theorem 3.1, $\lim_{n\to\infty} \|x_n - z\|$ exists. Let z_1 and z_2 be two weak subsequential limits of the sequence $\{x_n\}$, that is, $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$. Then we know $z_1, z_2 \in F(T)$ by Theorem 3.1. We claim $z_1 = z_2$. If not, by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{j\to\infty} \left\|x_{n_j} - z_2\right\| \\
&< \lim_{j\to\infty} \left\|x_{n_j} - z_1\right\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This is a contradiction. So, we have $z_1 = z_2$. We now assume that E has a Fréchet differentiable norm. As in the proof of Theorem 3.1, we may assume that C is bounded. So, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in C$. Then by Theorem 3.1, we obtain $z \in F(T)$. From Lemma 2.1, we have

$$\{z\} = \cap_{n=1}^{\infty} \overline{\mathrm{co}}\{x_m : m \geq n\} \cap F(T).$$

Hence $\{x_n\}$ converges weakly to a fixed point T. □

The following is a weak convergence theorem for a pair of nonexpansive mappings in a Banach space.

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where α_n, $\beta_n \in [a,b]$ for some $a,b \in \mathbb{R}$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Note that if $T_n x = \alpha_n S[\beta_n T x + (1-\beta_n)x] + (1-\alpha_n)x$ for every $x \in C$, then $F(T_n) = F(S) \cap F(T)$. Then as in the proof of Theorem 3.2, we can prove Theorem 3.3. □

4. Strong convergence theorems

In this section, we first prove the following theorem which is used to discuss the strong convergence of iterates defined by (1.1).

Theorem 4.1. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let T, S be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(T) \cap F(S)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Then the following hold:*

(i) *If $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \to z$ implies $z \in F(S)$;*

(ii) *if $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \to z$ implies $z \in F(T)$;*

(iii) *if $\alpha_n,\ \beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$, then $x_{n_i} \to z$ implies $z \in F(S) \cap F(T)$.*

Proof. By Mazur's theorem [3], $D = \overline{\text{co}}\{S(C) \cup T(C) \cup \{x_1\}\}$ is a compact subset of C which contains the sequence$\{x_n\}$. To prove (i), let $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$ and $x_{n_i} \to z$. Assume $Sz \neq z$ and let $w \in F(S) \cap F(T)$. Then, as in the proof of Theorem 3.1, we have that $\lim \|x_n - w\|$ exists. Let $c = \lim \|x_n - w\|$. Since $x_{n_i} \to z$, we have $\|z - w\| = c$. From $Sz \neq z$, we have $c > 0$. Further we have $S[\beta Tz + (1-\beta)z] \neq z$ for all $\beta \in [0,b]$. In fact, if $z = S[\beta Tz + (1-\beta)z]$ for $\beta = 0$, we have $Sz = z$. This is a contradiction. If $z = S[\beta Tz + (1-\beta)z]$ for some $\beta \in (0,b]$, we have

$$\begin{aligned}
\|z - w\| &= \|S[\beta Tz + (1-\beta)z] - w\| \\
&\leq \|\beta Tz + (1-\beta)z - w\| \\
&\leq \beta \|Tz - w\| + (1-\beta)\|z - w\| \\
&\leq \|z - w\|.
\end{aligned}$$

Since E is strictly convex , we have $Tz = z$. So, we have

$$z = S[\beta Tz + (1-\beta)z] = Sz.$$

This is also a contradiction. Therefore, we have $S[\beta Tz + (1-\beta)z] \neq z$ for all $\beta \in [0,b]$. We also know that $\|z - w\| = c$ and $\|S[\beta Tz + (1-\beta)z] - w\| \leq \|z - w\| = c$. Since E is strictly convex, we have, for any $\alpha \in [a,b]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)z - w\| < c. \tag{4.1}$$

Now, consider a real valued function g on $[0,1] \times [0,1]$ given by

$$g(\alpha, \beta) = \|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)z - w\|$$

for $\alpha, \beta \in [0,1] \times [0,1]$. Then g is continuous. From (4.1) and compactness of $[a,b] \times [0,b]$, we have

$$\max\{g(\alpha, \beta) : (\alpha, \beta) \in [a,b] \times [0,b]\} < c.$$

Choose a positive number r such that

$$\max\{g(\alpha,\beta) : (\alpha,\beta) \in [a,b] \times [0,b]\} < c - r.$$

Then from $x_{n_i} \to z$, we obtain an integer $m \geq 1$ such that $\|x_m - z\| < r$. Hence we have

$$\begin{aligned}
c &\leq \|x_{m+1} - w\| \\
&\leq \|x_{m+1} - \alpha_m S[\beta_m Tz + (1-\beta_m)z] - (1-\alpha_m)z\| \\
&\quad + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)z - w\| \\
&\leq \alpha_m \|\beta_m (Tx_m - Tz) + (1-\beta_m)(x_m - z)\| + (1-\alpha_m)\|x_m - z\| \\
&\quad + \|\alpha_m S[\beta_m Tz + (1-\beta_m)z] + (1-\alpha_m)z - w\| \\
&\leq \alpha_m \|\beta_m (Tx_m - Tz) + (1-\beta_m)(x_m - z)\| + (1-\alpha_m)\|x_m - z\| + c - r \\
&\leq \alpha_m (\beta_m \|x_m - z\| + (1-\beta_m)\|x_m - z\|) \\
&\quad + (1-\alpha_m)\|x_m - z\| + c - r \\
&< c.
\end{aligned}$$

This is a contradiction. So, we obtain $z = Sz$. This completes the proof of (i). To prove (ii), let $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some $a,b \in \mathbb{R}$ with $0 < a \leq b < 1$ and $x_{n_i} \to z$. Assume $Tz \neq z$ and let $w \in F(S) \cap F(T)$. Then, putting $c = \lim_{n\to\infty} \|x_n - w\|$, as in the proof of (i), we have $\|z - w\| = c > 0$ and $S[\beta Tz + (1-\beta)z] \neq z$ for all $\beta \in [a,b]$. Further we have that for any $\alpha \in [a,1]$,

$$\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)z - w\| < c. \tag{4.2}$$

In fact, if $\alpha \in [a,1)$, from strict convexity of E, we have the inequality. If $\alpha = 1$ and $\|\alpha S[\beta Tz + (1-\beta)z] + (1-\alpha)z - w\| = c$, we have

$$\begin{aligned}
c &= \|S[\beta Tz + (1-\beta)z] - w\| \\
&\leq \|\beta Tz + (1-\beta)z - w\| \\
&\leq \beta \|Tz - w\| + (1-\beta)\|z - w\| \\
&\leq \|z - w\|.
\end{aligned}$$

So, using strict convexity of E again, we have $z = Tz$. This is a contradiction. So, we have (4.2). Defining a real valued function g on $[0,1] \times [0,1]$ as in the proof of (i), we also have

$$\max\{g(\alpha,\beta) : (\alpha,\beta) \in [a,1] \times [a,b]\} < c.$$

Choose a positive number r such that

$$\max\{g(\alpha,\beta) : (\alpha,\beta) \in [a,1] \times [a,b]\} < c - r.$$

Then as in the proof of (i), we obtain $c \leq \|x_{m+1} - w\| < c$. This is a contradiction. Therefore we have $Tz = z$. (iii) is obvious from (i) and (ii). □

Using Theorem 4.1, we can prove the following theorem which was obtained by Takahashi and Kim [13].

Theorem 4.2. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping of C into itself such that $T(C)$ is contained in a compact subset of C. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We first show that $F(T)$ is nonempty. Fix $x_0 \in C$. For each $n \in \mathbb{N}$, consider a contraction mapping T_n given by

$$T_n x = \frac{1}{n}x_0 + (1-\frac{1}{n})Tx$$

for every $x \in C$. Then T_n has a unique fixed point u_n in C. Since the closure of $T(C)$ is compact, there exists a subsequence $\{Tu_{n_i}\}$ of the sequence $\{Tu_n\}$ such that $\{Tu_{n_i}\}$ converges strongly to v. Since $T(C)$ is bounded and

$$\begin{aligned}\|u_n - Tu_n\| &= \|\frac{1}{n}x_0 + (1-\frac{1}{n})Tu_n - Tu_n\| \\ &= \frac{1}{n}\|x_0 - Tu_n\|,\end{aligned}$$

we have $u_n - Tu_n \to 0$ as $n \to \infty$. So, from

$$\begin{aligned}\|v - Tv\| &\leq \|v - Tu_{n_i}\| + \|Tu_{n_i} - TTu_{n_i}\| + \|TTu_{n_i} - Tv\| \\ &\leq 2\|v - Tu_{n_i}\| + \|u_{n_i} - Tu_{n_i}\|,\end{aligned}$$

we have $v = Tv$. By Mazur's theorem [3], note that $\overline{\text{co}}(\{x_1\} \cup T(C))$ is a compact subset of C which contains the sequence $\{x_n\}$. Then there exist a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ and a point $z \in C$ such that $x_{n_i} \to z$. By Theorem 4.1, we have $Tz = z$ and hence $\lim_{n\to\infty} \|x_n - z\| = 0$. □

The following is a strong convergence theorem for a pair of nonexpansive mappings in a Banach space.

Theorem 4.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space E and let S, T be nonexpansive mappings of C into itself such that $S(C) \cup T(C)$ is contained in a compact subset of C and $F(T) \cap F(S)$ is nonempty. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where α_n, $\beta_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. For any $w \in F(S) \cap F(T)$, we have that $\lim_{n\to\infty} \|x_n - w\|$ exists. Further the sequence $\{x_n\}$ is contained in a compact subset of C. So, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to an element $z \in C$. By Theorem 4.1, we have $z \in F(S) \cap F(T)$. This implies $\lim_{n\to\infty} \|x_n - z\| = 0$. □

Finally, we prove a strong convergence theorem which is connected with results of [6] and [14].

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, and let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Let P be the metric projection of C onto $F(S) \cap F(T)$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\alpha_n, \beta_n \in [0,1]$. Then $\{Px_n\}$ converges strongly to an element of $F(S) \cap F(T)$.*

Proof. Let $y_n = \beta_n T x_n + (1-\beta_n)x_n$ for each n. Then

$$\begin{aligned}
\|Px_{n+1} - x_{n+1}\| &\leq \|Px_n - x_{n+1}\| \\
&= \|Px_n - \alpha_n S y_n - (1-\alpha_n)x_n\| \\
&\leq \alpha_n \|Px_n - y_n\| + (1-\alpha_n)\|Px_n - x_n\| \\
&\leq \alpha_n(\beta_n \|Px_n - Tx_n\| + (1-\beta_n)\|Px_n - x_n\|) + (1-\alpha_n)\|Px_n - x_n\| \\
&\leq \|Px_n - x_n\|.
\end{aligned} \tag{4.3}$$

So, we have that the limit of $\{\|Px_n - x_n\|\}$ exists. We denote the limit of $\{\|Px_n - x_n\|\}$ by r. Next, we show that for each $n, k \in \mathbb{N}$,

$$\|Px_n - x_{n+k}\| \leq \|Px_n - x_n\| \tag{4.4}$$

by mathematical induction. For $k = 1$, we have

$$\|Px_n - x_{n+1}\| \leq \|Px_n - x_n\|$$

by (4.3). We assume that for $k = l$,

$$\|Px_n - x_{n+l}\| \leq \|Px_n - x_n\|.$$

Then, we get

$$\begin{aligned}
\|Px_n - x_{n+l+1}\| &= \|Px_n - \alpha_{n+l} S y_{n+l} - (1-\alpha_{n+l})x_{n+l}\| \\
&\leq \alpha_{n+l}\|Px_n - y_{n+l}\| + (1-\alpha_{n+l})\|Px_n - x_{n+l}\| \\
&\leq \alpha_{n+l}(\beta_{n+l}\|Px_n - Tx_{n+l}\| + (1-\beta_{n+l})\|Px_n - x_{n+l}\|) \\
&\quad +(1-\alpha_{n+l})\|Px_n - x_{n+l}\| \\
&\leq \|Px_n - x_{n+l}\| \\
&\leq \|Px_n - x_n\|.
\end{aligned}$$

This is complete the proof of (4.4). Now we show that $\{Px_n\}$ is a Cauchy sequence. If $r = \lim_{n\to\infty} \|Px_n - x_n\| = 0$, for an arbitrary positive number ε, there exists a positive integer n_0 such that $\|Px_n - x_n\| < \varepsilon$ for all $n \geq n_0$. By (4.4), we have for $m, n \in \mathbb{N}$ with $m > n \geq n_0$,

$$\begin{aligned}
\|Px_n - Px_m\| &\leq \|Px_n - Px_{n_0}\| + \|Px_{n_0} - Px_m\| \\
&\leq \|Px_n - x_n\| + \|x_n - Px_{n_0}\| + \|Px_{n_0} - x_m\| + \|x_m - Px_m\| \\
&\leq \|Px_n - x_n\| + \|x_{n_0} - Px_{n_0}\| + \|Px_{n_0} - x_{n_0}\| + \|x_m - Px_m\| \\
&< 4\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have that $\{Px_n\}$ is a Cauchy sequence. Next, we assume that $r > 0$ and $\{Px_n\}$ is not a Cauchy sequence. Then there exist a positive number ε and two subsequences $\{Px_{n_i}\}$, $\{Px_{m_i}\}$ of $\{Px_n\}$ such that $\|Px_{n_i} - Px_{m_i}\| \geq \varepsilon$ for all $i \in \mathbb{N}$. Also, there exists a positive number d such that $(r+d)\left(1-\delta(\frac{\varepsilon}{r+d})\right) < r$. By the definition of r, there exists a positive integer n_0 such that

$$r \leq \|Px_n - x_n\| < r + d$$

for all $n \geq n_0$. Let $n_i, m_i \geq n_0$ and $l \geq n_i, m_i$. By (4.4), we have

$$\|Px_{n_i} - x_l\| \leq \|Px_{n_i} - x_{n_i}\| < r + d$$

and

$$\|Px_{m_i} - x_l\| \leq \|Px_{m_i} - x_{m_i}\| < r + d.$$

By uniform convexity of E, we get

$$r \leq \|Px_l - x_l\| \leq \left\| \frac{Px_{n_i} + Px_{m_i}}{2} - x_l \right\| \leq (r+d)\left(1 - \delta(\frac{\varepsilon}{r+d})\right) < r.$$

This is a contradiction. This complete the proof. □

Using Theorem 4.4, we can prove the following result.

Theorem 4.5. *Let C be a nonempty closed convex subset of a Hilbert space H, and let S, T be nonexpansive mappings of C into itself such that $F(S) \cap F(T)$ is nonempty. Let P be the metric projection of C onto $F(S) \cap F(T)$. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n S[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\alpha_n, \beta_n \in [a, b]$ for some a,b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges weakly to an element z of $F(S) \cap F(T)$, where $z = \lim_{n\to\infty} Px_n$.*

Proof. By Theorem 3.3, $\{x_n\}$ converges weakly to an element z of $F(S) \cap F(T)$. By Theorem 4.4, $\{Px_n\}$ converges strongly to an element u of $F(S) \cap F(T)$. Since P is the metric projection of H onto $F(S) \cap F(T)$, we also know that $(x_n - Px_n, Px_n - y) \geq 0$ for all $y \in F(S) \cap F(T)$, where $(\cdot,\cdot)$ denotes the inner product of H. So, we have $(z-u, u-y) \geq 0$ for all $y \in F(S) \cap F(T)$. Putting $y = z$, we obtain $-\|z-u\|^2 \geq 0$ and hence $z = u$. This completes the proof. □

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