Journal of Convex Analysis
Volume 5 (1998), No. 1, 1–17

Non-Coercive Variational Problems with Constraints on the Derivatives

Cristina Marcelli
Dipartimento di Matematica, Universitá di Perugia,
Via L. Vanvitelli 1, 06123 Perugia, Italy.
e-mail: marcelli@unipg.it

Received April 15, 1996
Revised manuscript received January 6, 1997

We establish a necessary and sufficient condition for the existence of the minimum of the functional $\int_0^1 f(t,v'(t))dt$ in the class $\mathcal{W}_d^p = \{v \in W^{1,p}([0,1]) : v(0) = 0, v(1) = d, v'(t) \geq \alpha\}$, in terms of a limitation of the slope d. Some applications to quasi-coercive and non-coercive integrands are also derived.

1. Introduction

In a recent paper [11] we considered one-dimensional free problems of the Calculus of Variations of the type

$$\text{minimize } \left\{\int_0^1 f(t,v'(t))dt\right\} \tag{P}$$

over the class $\tilde{\mathcal{W}}_d^p = \{v \in W^{1,p}([0,1]) : v(0) = 0, v(1) = d\}$, with $f(t,\cdot)$ convex but not necessarily coercive.

We proposed a necessary and sufficient condition expressed as a limitation on the width of the slope d, which improves an analogous result given by P. Brandi [6]. The key tool is the Euler equation, which in this setting provides to be a necessary and sufficient condition.

As particular cases, we discussed integrands of the type $f(t,z) = \phi(t)h(z)$, both quasi-coercive (i.e. h has a superlinear growth and $m = \min \phi(t) = 0$) and non-coercive (i.e. h has a linear growth).

In more detail, in the quasi-coercive case the result shows a strict link between the exponent p, the infinitesimal order of the function ϕ and the infinite order of the function h. The sufficient condition fits with the existence theorem given in a joint paper with A. Salvadori [13], regarding multiple integrals on $W^{1,1}$ with constraints on the gradient.

Whereas, in the non-coercive case, the condition depends on the the infititesimal orders of the functions $[\phi(t) - m]$ and $[h'(z) - \lim_{|\xi|\to+\infty} h'(\xi)]$.

The aim of the present paper is to discuss variational problems with contraints on the derivatives. More precisely, we deal with problem (P) over the class

$$\mathcal{W}_d^p = \{v \in W^{1,p}([0,1]) :\ v(0) = 0,\ v(1) = d,\ v'(t) \geq \alpha\}.$$

ISSN 0944-6532 / \$ 2.50

Recently, the Euler inclusion was proved in [12] as a necessary condition for the existence of the minimum of constrained problem (P). By using this result, we herein characterize the existence of the minimum in terms of a limitation of the slope d.

In particular, for integrands $f(t,z) = \phi(t)h(z)$, we show that the presence of a constraint on the derivatives has a regularizing effect on problem (P), since it widens the range of the slopes d for which the minimum exists.

More precisely, if $[h'(\alpha)\lim_{\xi\to+\infty} h'(\xi) > 0]$, then the conditions we obtain are the same to those of the free problem. Whereas, if $[h'(\alpha)\lim_{\xi\to+\infty} h'(\xi) \leq 0]$, the range of the slopes for which the constrained problem admits minimum is larger than that of the free problem. Infact, when $h'(\alpha) < 0$ we prove that the minimum exists for every $d \in [\alpha, \xi_0]$, with ξ_0 such that $h'(\xi_0) = 0$.

Finally, we wish to remark that the present result improves an analogous one proved by B. Botteron and B. Dacorogna in [5], where they gave a sufficient condition in the case $p = \infty$.

2. Preliminaries

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For every $X \subset \mathbb{R}$ we denote by X^0, $\mathrm{cl}(X)$ and $|X|$ the interior, the closure and the measure of X respectively.

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $f = f(t,z)$, be a given Carathéodory function, convex in the second argument for a.e. $t \in [0,1]$.

Moreover, we assume that a function $\lambda \in L^1([0,1])$ exists such that $f(t,z) \geq \lambda(t)$ for a.e. $t \in [0,1]$ and every $z \in \mathbb{R}$.

Let us now recall some properties of convex functions we will use in the following.

Let $g : \mathbb{R} \to \mathbb{R}$ be a given convex function. For every $\xi \in \mathbb{R}$ there exist (finite) the right and left derivatives g^+, g^-. These functions satisfy the following properties (see, e.g., [16]):

(i) they are monotone non decreasing;
(ii) $g^+(\xi) \geq g^-(\xi)$, $\xi \in \mathbb{R}$;
(iii) g^+ [respectively g^-] is right-continuous [left-continuous].

We will denote by $\partial g(\xi)$ the subdifferential of g at the point ξ, i.e. $\partial g(\xi) = [g^-(\xi), g^+(\xi)]$.

We can extend the functions g^-, g^+ to $\bar{\mathbb{R}}$ (with values in $\bar{\mathbb{R}}$) by putting

$$g^-(-\infty) = g^+(-\infty) = i = \lim_{\xi\to-\infty} g^-(\xi) = \lim_{\xi\to-\infty} g^+(\xi) ,$$

$$g^-(+\infty) = g^+(+\infty) = s = \lim_{\xi\to+\infty} g^-(\xi) = \lim_{\xi\to+\infty} g^+(\xi).$$

In this way the extended functions g^+, g^- again satisfy properties (i), (ii), (iii).

Recall that the conjugate function $g^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, defined by $g^*(\zeta) = \sup_{z\in\mathbb{R}}\{\zeta z - g(z)\}$, is a convex function satisfying the following property (see [16]):

$$y \in \partial g(\xi) \;\Leftrightarrow\; \xi \in \partial g^*(y). \tag{2.1}$$

Let $\mathrm{dom}[\partial g^*] = \{y : \partial g^*(y) \neq \emptyset\}$ and $\mathrm{range}[\partial g] = \bigcup_{\xi\in\mathbb{R}} \partial g(\xi)$. From (2.1) it follows that

$$]i, s[\subset \mathrm{range}[\partial g] = \mathrm{dom}[\partial g^*] \subset [i, s].$$

For every $y \in \mathrm{range}[\partial g]$ we put

$$g^{*+}(y) = \sup \partial g^*(y) \ , \quad g^{*-}(y) = \inf \partial g^*(y).$$

Note that for every $y \in]i, s[$ we have $g^{*+}(y) = \max \partial g^*(y)$ and $g^{*-}(y) = \min \partial g^*(y)$. Whereas if $i \in \mathrm{range}[\partial g]$ $[s \in \mathrm{range}[\partial g]]$, we have $g^{*-}(i) = -\infty$ $[g^{*+}(s) = +\infty]$.

In order to extend the domain of the functions g^{*+}, g^{*-} over $\bar{\mathbb{R}}$, if $i, s \notin \mathrm{range}[\partial g]$ we put

$$\begin{array}{l} g^{*+}(i) = g^{*-}(i) = -\infty = \lim\limits_{\xi\searrow i} g^{*-}(\xi) = \lim\limits_{\xi\searrow i} g^{*+}(\xi) \\ g^{*+}(s) = g^{*-}(s) = +\infty = \lim\limits_{\xi\nearrow s} g^{*-}(\xi) = \lim\limits_{\xi\nearrow s} g^{*+}(\xi) \end{array} ,$$

whereas, we put

$$\begin{array}{ll} g^{*+}(y) = g^{*-}(y) = -\infty & \text{for } \ y < i \\ g^{*+}(y) = g^{*-}(y) = +\infty & \text{for } \ y > s \end{array} .$$

In this way, the extended functions g^{*+}, g^{*-} satisfy the same properties (i), (ii), (iii) of the analogous functions g^+, g^-. Moreover, for every $y \in \mathbb{R}$ we have

$$g^+(\xi) < y \ \Leftrightarrow \ \xi < g^{*-}(y); \quad g^-(\xi) \leq y \ \Leftrightarrow \ \xi \leq g^{*+}(y).$$

Let $\alpha \in \mathbb{R}$ be fixed. In what follows we will consider the following elements of $\bar{\mathbb{R}}$:

$$l_\alpha = \operatorname*{ess\,inf}_{t\in[0,1]} f_z^-(t, \alpha), \quad l_s = \operatorname*{ess\,inf}_{t\in[0,1]} f_z^+(t, +\infty),$$

where $f_z^-(t, \cdot), f_z^+(t, \cdot)$ are the left and right derivatives of $f(t, \cdot)$ extended to $\bar{\mathbb{R}}$ as showed above.

For every $y \in \mathbb{R}$ we put

$$A_y = \{\ t \in [0, 1] \ : \ f_z^+(t, \alpha) < y\} = \{\ t \in [0, 1] \ : \ \alpha < f_\zeta^{*-}(t, y)\};$$

$$B_y = \{\ t \in [0, 1] \ : \ f_z^-(t, \alpha) \leq y\} = \{\ t \in [0, 1] \ : \ \alpha \leq f_\zeta^{*+}(t, y)\} \ ,$$

where $f_\zeta^{*-}(t, \cdot), f_\zeta^{*+}(t, \cdot)$ are the left and right derivatives of the conjugate function $f^*(t, \cdot)$.

Of course, we have $A_y \subset B_y$ and, if $y_1 < y_2$, we have $A_{y_1} \subset A_{y_2}$, $B_{y_1} \subset B_{y_2}$.

Let us now consider the following sets:

$$T_p^- = \{y \in [l_\alpha, l_s] \cap \mathbb{R}: \ f_\zeta^{*-}(\cdot, y) \in L^p(A_y)\},$$

$$T_p^+ = \{y \in [l_\alpha, l_s] \cap \mathbb{R}: \ f_\zeta^{*+}(\cdot, y) \in L^p(B_y)\},$$

where, if $|A_y| = 0$ $[|B_y| = 0]$, no condition is required for $f_\zeta^{*-}(\cdot, y)$ $[f_\zeta^{*+}(\cdot, y)]$.

Of course, the sets T_p^-, T_p^+ can be empty for some p, but the following result holds.

Lemma 2.1. *The sets T_p^-, T_p^+ are connected for every $p \in [1, +\infty]$. Moreover, $T_p^+ \subset T_p^-$ and $(T_p^+)^0 = (T_p^-)^0$.*

Proof. Let $y, y_0 \in [l_\alpha, l_s] \cap \mathbb{R}$ be fixed, with $y < y_0$. Note that

$$\alpha < f_\zeta^{**-}(t, y) \leq f_\zeta^{**+}(t, y) \leq f_\zeta^{**-}(t, y_0) \quad \text{ in } \ A_y, \tag{2.2}$$

$$\alpha \leq f_\zeta^{**+}(t, y) \leq f_\zeta^{**-}(t, y_0) \leq f_\zeta^{**+}(t, y_0) \quad \text{ in } \ B_y. \tag{2.3}$$

Therefore, since $A_y \subset B_y$, by (2.2) it immediatly follows that $T_p^+ \subset T_p^-$. Moreover, since $B_y \subset B_{y_0}$, by (2.3) we deduce that if $y_0 \in T_p^+$ then $[l_\alpha, y_0] \cap \mathbb{R} \subset T_p^+$. Analogously, we have $[l_\alpha, y_0] \cap \mathbb{R} \subset T_p^-$ for every $y_0 \in T_p^-$. Hence, the sets T_p^+, T_p^- are connected.

Finally, if $y_0 \in T_p^-$, by (2.3) we have $f_\zeta^{**+}(t, y) \in L^p(A_{y_0} \cap B_y)$. But for every $t \in B_y \setminus A_{y_0}$ we have $f_\zeta^{**+}(t, y) \leq f_\zeta^{**-}(t, y_0) \leq \alpha \leq f_\zeta^{**+}(t, y)$, i.e. $f_\zeta^{**+}(t, y) = \alpha$. Thus, we have $f_\zeta^{**+}(t, y) \in L^p(B_y)$ for every $y \in [l_\alpha, y_0] \cap \mathbb{R}$. Therefore, $(T_p^-)^0 \subset (T_p^+)^0$ and the proof is complete. □

Remark 2.2. In view of the proof of Lemma 2.1, when T_p^+, T_p^- are nonempty, we have

$$\inf T_p^- = \inf T_p^+ = l_\alpha \ , \qquad \sup T_p^+ = \sup T_p^-.$$

Moreover, if $l_\alpha \in \mathbb{R}$ then $\min T_p^- = \min T_p^+ = l_\alpha$. Therefore, the set $T_p^- \setminus T_p^+$ contains one point at the most, which is the least upper bound of both the sets.

Let $\psi^-, \psi^+ : [0, 1] \times [l_\alpha, l_s] \to \bar{\mathbb{R}}$ be the functions defined by

$$\psi^-(t, y) = \begin{cases} f_\zeta^{**-}(t, y) & t \in A_y \\ \alpha & \text{otherwise} \end{cases} \qquad \psi^+(t, y) = \begin{cases} f_\zeta^{**+}(t, y) & t \in B_y \\ \alpha & \text{otherwise} \end{cases}$$

and let $\Psi^-, \Psi^+ : [l_\alpha, l_s] \to \bar{\mathbb{R}}$ be the functions defined by

$$\Psi^-(y) = \int_0^1 \psi^-(t, y) dt \qquad \Psi^+(y) = \int_0^1 \psi^+(t, y) dt.$$

Lemma 2.3. *The function Ψ^+ [respectively Ψ^-] is monotone non decreasing and right-continuous in* $\mathrm{cl}(T_1^+)$ [*left-continuous in* $\mathrm{cl}(T_1^-)$].

Proof. As it is easy to check, the functions $\psi^+(t, \cdot)$, $\psi^-(t, \cdot)$ are monotone non decreasing; hence, also the functions Ψ^+, Ψ^- are non decreasing.

Let us now prove that the function $\psi^+(t, \cdot)$ is right-continuous in $\mathrm{cl}(T_1^+)$ for a.e. $t \in [0, 1]$. In order to do this, let $y_0 \in T_1^+$, $y_0 \neq \sup T_1^+$, be fixed.

For every $t \in B_{y_0}$ we have that $t \in B_y$ for $y > y_0$ and the assertion follows from the right-continuity of $f_\zeta^{**+}(t, \cdot)$.

Let us now take $t \notin B_{y_0}$, i.e. $\alpha > f_\zeta^{**+}(t, y_0)$. Then we have $\alpha > f_\zeta^{**+}(t, \bar{y})$ for some $\bar{y} > y_0$. Hence, we deduce $\lim\limits_{y \searrow y_0} \psi^+(t, y) = \alpha = \psi^+(t, y_0)$.

The right-continuity of the function Ψ^+ can be deduced by using the monotone convergence theorem, taking the right-continuity of $\psi^+(t, \cdot)$ into account.

The proof regarding Ψ^- is analogous. □

Remark 2.4. Of course, if $f(t,\cdot)$ is strictly convex, we have $f_\zeta^{*+}(t,y) = f_\zeta^{*-}(t,y) = f_\zeta^{*}(t,y)$. Therefore, if $f(t,\cdot)$ is strictly convex for a.e. $t \in [0,1]$ we have $T_p^+ = T_p^- = T_p$ and $\Psi^+(y) = \Psi^-(y) = \Psi(y)$ is continuous in $\mathrm{cl}(T_1)$.

3. The general result

Let $\alpha \in \mathbb{R}$ be fixed. For every $d \in \mathbb{R}$ and every $p \in [1,+\infty]$ we put

$$\mathcal{W}_d^p = \{v \in W^{1,p}([0,1]) : v(0) = 0, v(1) = d, v'(t) \geq \alpha\}$$

and let $F : \mathcal{W}_d^p \to \bar{\mathbb{R}}$ be the functional defined by $F(v) = \displaystyle\int_0^1 f(t, v'(t))dt$.

In this paper we will consider the following variational problem with constraints on the derivatives

$$\text{minimize } \{F(v) \ : \ v \in \mathcal{W}_d^p\}. \tag{P}$$

In a recent paper [11], we obtained a necessary and sufficient condition for the solvability of the free problem, given in terms of a limitation on the slope d. The aim of this paper is to establish an analogous result for the constrained variational problem (P).

In what follows we will make use of the following result proved in [12].

Lemma 3.1 ([12]). *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, convex in the second argument, and let $u_0 \in \mathcal{W}_d^p$ be such that*

$$\int_0^1 f(t, u_0'(t))dt = \min_{v \in \mathcal{W}_d^p} \int_0^1 f(t, v'(t))dt.$$

Then, put $A^* = \{t \in [0,1] \ : \ u_0'(t) > \alpha\}$, there exists a constant $y_0 \in \mathbb{R}$ such that

$$y_0 \in \partial_z f(t, u_0'(t)) \quad \text{for a.e. } t \in A^*.$$

The following lemma completes the study of the necessary conditions we need in our main theorem.

Lemma 3.2. *Let $u_0 \in \mathcal{W}_d^p$ be a non linear minimizer for problem (P) and let y_0 be the constant given by the Euler inclusion (see* Lemma 3.1*).*

If $f(t,\cdot)$ is strictly convex and C^1 for a.e. $t \in [0,1]$, then we have

$$u_0'(t) > \alpha \qquad \text{for a.e. } \ t \in A_{y_0}.$$

Proof. Since u_0 is not linear, put $A^* = \{t : u_0'(t) > \alpha\}$ we have $|A^*| > 0$. Let us suppose, by contradiction, that $|A_{y_0} \setminus A^*| > 0$.

From now on we will divide the proof into steps.

Step 1. Let us now prove that a real value $\tilde{y} < y_0$ exists such that the set

$$C = \{t \in A_{y_0} \setminus A^* : \alpha < f_\zeta^*(t, \tilde{y}) < +\infty\}$$

has positive measure.

In order to do this, for every y put

$$F_y = \{t \in A_{y_0} \setminus A^* : f^*_\zeta(t, y) < +\infty\} \ .$$

We have two cases: $|F_{y_0}| > 0$ or $|F_{y_0}| = 0$.

If $|F_{y_0}| > 0$, let $(y_n)_n$ be an increasing sequence convergent to y_0. Put $C_n = \{t \in F_{y_0} : f^*_\zeta(t, y_n) > \alpha\}$, taking the continuity of $f^*_\zeta(t, \cdot)$ into account, we deduce that $F_{y_0} = \bigcup_{n\in\mathbb{N}} C_n$. Hence, an integer $\tilde{n}$ exists such that put $\tilde{y} = y_{\tilde{n}}$, the set

$$C_{\tilde{n}} \subset \{t \in A_{y_0} \setminus A^* : \alpha < f^*_\zeta(t, \tilde{y}) < +\infty\}$$

has positive measure.

In the other case, if $|F_{y_0}| = 0$, let

$$y' = \inf\{y \le y_0 : |F_y| = 0\} \ .$$

Of course, $y' \in \mathbb{R}$ and $|F_{y'}| = 0$.

Let $(q_n)_n$ be the sequence of all the rational numbers of $[y' - 1, y'[$. Put $H_n = \{t \in A_{y_0} \setminus A^* : \alpha < f^*_\zeta(t, q_n) < +\infty\}$, let $H = \bigcup_{n\in\mathbb{N}} H_n$. The assertion of Step 1 will be proved if we show that $|H| > 0$.

Assume, by contradiction, $|H| = 0$. Then, for a.e. $t \in A_{y_0} \setminus A^*$ and every $n \in \mathbb{N}$ we have

$$f^*_\zeta(t, q_n) \le \alpha \quad \text{ or } \quad f^*_\zeta(t, q_n) = +\infty \ . \tag{3.1}$$

Put $B = \{t \in A_{y_0} \setminus A^* : f^*_\zeta(t, q_n) \le \alpha \ \text{ for some } \ n \in \mathbb{N}\}$, since $|F_{q_n}| > 0$ we have $|B| > 0$. Let us now fix $t_0 \in B \setminus F_{y'}$. Put $s = \sup\{q_n : f^*_\zeta(t_0, q_n) \le \alpha\}$, we have $f^*_\zeta(t_0, s) \le \alpha$. But since $t_0 \notin F_{y'}$, we have $s < y'$ and by (3.1) we deduce $f^*_\zeta(t_0, y) = +\infty$ for every $y \in]s, y'[$, in contradiction with the continuity of $f^*_\zeta(t_0, \cdot)$.

Step 2. By virtue of the continuity of $f^*_\zeta(t, \cdot)$, we can also deduce that there exists a real $y^* \in]\tilde{y}, y_0[$ such that the set

$$D = \{t \in A^* : f^*_\zeta(t, y^*) > \alpha\}$$

has positive measure.

By virtue of Lemma 3.1 we have that $f^*_\zeta(t, y_0), f^*_\zeta(t, y^*) \in L^1(D)$. Let

$$I = \int_D [f^*_\zeta(t, y_0) - f^*_\zeta(t, y^*)] dt. \tag{3.2}$$

Since $f_z(t, \cdot)$ is differentiable, we have $f^*_\zeta(t, y^*) < f^*_\zeta(t, y_0)$ for a.e. $t \in [0, 1]$, hence $I > 0$.

Let $E \subset C$ be a set of positive measure such that $f^*_\zeta(t,\tilde{y}) \in L^1(E)$ and

$$\int_E |f^*_\zeta(t,\tilde{y}) - \alpha| < I/2. \tag{3.3}$$

For every $y \in [y^*, y_0]$ let

$$G(y) = \int_E [f^*_\zeta(t,\tilde{y}) - \alpha]dt + \int_D [f^*_\zeta(t,y) - f^*_\zeta(t,y_0)]dt.$$

Note that G is continuous; moreover, $G(y_0) = \displaystyle\int_E [f^*_\zeta(t,\tilde{y}) - \alpha]dt > 0$, whereas by (3.2), (3.3) we have

$$G(y^*) = \int_E [f^*_\zeta(t,\tilde{y}) - \alpha]dt + \int_D [f^*_\zeta(t,y^*) - f^*_\zeta(t,y_0)]dt \le I/2 - I < 0.$$

Hence, there exists $\bar{y} \in]y^*, y_0[$ such that $G(\bar{y}) = 0$.

Let us now consider the function $w : [0,1] \to \mathbb{R}$ defined by

$$w(t) = \begin{cases} f^*_\zeta(t,\bar{y}) & \text{in } D \\ f^*_\zeta(t,\tilde{y}) & \text{in } E \\ u_0'(t) & \text{otherwise }, \end{cases}$$

and let $v(t) = \int_0^t w(\tau)d\tau$.

Note that $w(t) \ge \alpha$ for every $t \in [0,1]$. Moreover, we have

$$\int_0^1 w(t)dt = \int_D f^*_\zeta(t,\bar{y})dt + \int_E f^*_\zeta(t,\tilde{y})dt + \int_{A^*\setminus D} f^*_\zeta(t,y_0)dt + \alpha(1 - |A^*|) - \alpha|E| =$$

$$= \int_{A^*} f^*_\zeta(t,y_0)dt + \alpha(1 - |A^*|) + \int_D [f^*_\zeta(t,\bar{y}) - f^*_\zeta(t,y_0)]dt + \int_E [f^*_\zeta(t,\tilde{y}) - \alpha]dt =$$

$$= \int_0^1 u_0'(t)dt + G(\bar{y}) = d.$$

Therefore, we have $v \in \mathcal{W}^p_d$, but

$$F(u_0) - F(v) = \int_0^1 [f(t,u_0'(t)) - f(t,v'(t))]dt \ge$$

$$\ge \bar{y} \int_D [u_0'(t) - v'(t)]dt + \tilde{y} \int_E [u_0'(t) - v'(t)]dt + \int_{[0,1]\setminus(D\cup E)} f_z(t,v'(t))[u_0'(t) - v'(t)]dt =$$

$$= \bar{y} \int_{D\cup E} [u_0'(t) - v'(t)]dt + (\tilde{y} - \bar{y}) \int_E [u_0'(t) - v'(t)]dt = (\tilde{y} - \bar{y}) \int_E [\alpha - f^*_\zeta(t,\tilde{y})]dt > 0,$$

which is a contradiction. $\square$

We are now ready to state and prove our main result.

Theorem 3.3 (necessary and sufficient condition). *If one of the following conditions is satisfied*

(i) $d = \alpha$

(ii) $\alpha < d < \sup_{y \in T_p^+} \int_0^1 \psi^+(t,y)dt, \quad (T_p^+ \neq \emptyset)$

(iii) $d = \max_{y \in T_p^+} \int_0^1 \psi^+(t,y)dt, \quad (T_p^+ \neq \emptyset)$

then problem (P) *admits an optimal solution in* $\mathcal{W}_d^p$.

Conversely, if problem (P) *admits an optimal solution and* $f(t,\cdot)$ *is* C^1 *and strictly convex for a.e.* $t \in [0,1]$*, then one of the conditions* (i), (ii), (iii) *is satisfied.*

Proof of Theorem 3.3. (Sufficient condition) Let us first prove that

$$\alpha = \inf_{y \in T_p^-} \int_0^1 \psi^-(t,y)dt.$$

By virtue of what observed in Remark 2.2, if $l_\alpha \in \mathbb{R}$ we have $l_\alpha = \min T_p^-$ and $\psi^-(t,l_\alpha) \equiv \alpha$, hence $\alpha = \min_{y \in T_p^-} \int_0^1 \psi^-(t,y)dt$. Whereas, if $-\infty = l_\alpha = \inf T_p^-$, since $\lim_{y \to -\infty} \psi^-(t,y) = \alpha$ for every $t \in [0,1]$, we have that $\inf_{y \in T_p^-} \int_0^1 \psi^-(t,y)dt = \lim_{y \to -\infty} \int_0^1 \psi^-(t,y)dt = \alpha$.

If $d = \alpha$ then $\mathcal{W}_d^p = \{u_0\}$ where $u_0(t) = dt$. Hence, of course, u_0 is the optimal solution for problem (P_1).

Assume now that (ii) holds. Since Ψ^+ is right-continuous and Ψ^- is left-continuous, we have that a constant $y_0 \in T_p^+$ exists such that $\Psi^-(y_0) \leq d \leq \Psi^+(y_0)$.

For every $s \in [0,1]$ let us now consider the function $v_s : [0,1] \to \tilde{\mathbb{R}}$ defined by

$$v_s(t) = \begin{cases} \psi^+(t,y_0) & \text{if} \quad 0 \leq t \leq s \\ \psi^-(t,y_0) & \text{if} \quad s < t \leq 1 \end{cases}$$

and let $V : [0,1] \to \mathbb{R}$ be the continuous function defined by $V(s) = \int_0^1 v_s(t)dt$. Since $V(0) = \Psi^-(y_0) \leq d \leq \Psi^+(y_0) = V(1)$, a constant $s_0 \in [0,1]$ exists such that $V(s_0) = \int_0^1 v_{s_0}(t)dt = d$.

Put $u_0(t) = \int_a^t v_{s_0}(t)dt$, let us prove that u_0 is a minimizer for the problem (P).

Of course, $u_0 \in \mathcal{W}_d^p$. Let us fix $v \in \mathcal{W}_d^p$ and prove that $F(v) \geq F(u_0)$.

Let $E = [0,s_0]$ and $F = [s_0,1]$. By virtue of the convexity of $f(t,\cdot)$, we have that $f(t,\xi') - f(t,\xi) \geq y(\xi' - \xi)$ for every $y \in \partial f_z(t,\xi)$, hence we have

$$F(v) - F(u_0) = \int_E [f(t,v'(t)) - f(t,u_0'(t))]dt + \int_F [f(t,v'(t)) - f(t,u_0'(t))]dt \geq$$

$$\geq \int_{E\cap B_{y_0}} y_0[v'(t)-u_0'(t)]dt + \int_{E\setminus B_{y_0}} f_z^-(t,\alpha)[v'(t)-\alpha]dt+$$

$$+\int_{F\cap A_{y_0}} y_0[v'(t)-u_0'(t)]dt + \int_{F\setminus A_{y_0}} f_z^+(t,\alpha)[v'(t)-\alpha]dt =$$

$$= y_0\int_0^1 [v'(t)-u_0'(t)]dt + \int_{E\setminus B_{y_0}} [f_z^-(t,\alpha)-y_0][v'(t)-\alpha]dt+$$

$$+\int_{F\setminus A_{y_0}} [f_z^+(t,\alpha)-y_0][v'(t)-\alpha]dt \geq 0.$$

Finally, in the case (iii) is satisfied, there exists a constant $y \in T_p^+$ such that $\Psi^+(y) = d$. Then, put $u_0(t) = \int_0^1 \psi^+(t,y)dt$, we can analogously prove that u_0 is a minimizer for the problem (P).

(Necessary condition) Let $u_0 \in \mathcal{W}_d^p$ be an optimal solution for problem (P).

Note that $d \geq \alpha$ is a necessary admissibility condition. Then, assume now $d > \alpha$ and let us prove that (ii) or (iii) holds.

Put $A^* = \{t \in [0,1] \ : \ u_0'(t) > \alpha\}$, since $d > \alpha$ we have $|A^*| > 0$. Then, since $f(t,\cdot)$ is C^1 and strictly convex, by applying Lemma 3.1 we have that a constant $y_0 \in \mathbb{R}$ exists such that

$$y_0 = f_z(t, u_0'(t)) \quad \text{ a.e. in } A^*. \tag{3.4}$$

The result will be proved if we show that $y_0 \in T_p$ and $\psi(t,y_0) = u_0'(t)$ for a.e. $t \in [0,1]$.

Since $|A^*| > 0$, by (3.4) we have $l_\alpha \leq y_0$. Moreover, by Lemma 3.2 we deduce that for a.e. $t \in A_{y_0}$ we have $y_0 = f_z(t,u_0'(t))$, whereas, if $t \notin A_{y_0}$ we have $y_0 \leq f_z(t,\alpha)$. Therefore, we deduce $y_0 \leq f_z(t,+\infty)$ for a.e. $t \in [0,1]$, i.e. $y \leq l_s$.

Furthermore, for a.e. $t \in A_{y_0}$ we have $y_0 = f_z(t,u_0'(t))$, i.e. $f_\zeta^*(t,y_0) = u_0'(t)$. Hence, we have $f_\zeta^*(\cdot,y_0) \in L^p(A_{y_0})$ and then $y_0 \in T_p$.

Finally, note that for a.e. $t \in A^*$ by (3.2) we have $y_0 \geq f_z(t,\alpha)$, i.e. $t \in B_{y_0}$. Hence, $\psi(t,y_0) = f_\zeta^*(t,y_0) = u_0'(t)$. Whereas, by Lemma 3.2 we deduce that $|A_{y_0} \setminus A^*| = 0$, then we have $\psi(t,y_0) = \alpha = u_0'(t)$ for a.e. $t \notin A^*$, and the proof is complete. □

Remark 3.4. Note that if $\sup T_p^+ = +\infty$, we have

$$\sup_{y\in T_p^+} \int_0^1 \psi^+(t,y)dt = +\infty. \tag{3.5}$$

Infact, in this case we have $l_s = +\infty$, i.e. $f_z^-(t,+\infty) = +\infty$ for a.e. $t \in [0,1]$. Hence, $\lim_{y\to+\infty} f_\zeta^{*+}(t,y) = \lim_{y\to+\infty} \psi^+(t,y) = +\infty$ a.e. t, and by virtue of the monotone convergence theorem we deduce (3.5).

Remark 3.5. If $f(t,\cdot)$ is strictly convex for a.e. $t \in [0,1]$, by virtue of the continuity of the function Ψ in $\mathrm{cl}(T_1)$ (see Lemma 2.3 and Remark 2.4), the necessary and sufficient condition of the previous theorem for $p = 1$ becomes: $T_1 \neq \emptyset$ and

$$\alpha \leq d \leq \sup_{y \in T_1} \int_0^1 \psi(t,y)dt.$$

Moreover, when $(T_p)^0 \neq \emptyset$, put $s = \sup T_p$, we have

$$\sup_{y \in T_p} \int_0^1 \psi(t,y)dt = \lim_{y \nearrow s} \int_0^1 \psi(t,y)dt = \int_{B_s} f_\zeta^*(t,s)dt + \alpha(1 - |B_s|),$$

where $B_s = \{t \in [0,1] : f_z(t,\alpha) \leq s\}$.

We conclude this section with two examples.

Example 3.6. (non existence for any $d \neq \alpha$). Let $\alpha = 0$ and

$$f(t,z) = \begin{cases} \exp(z - 1/t) & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}$$

Of course $f_z(t,\xi) = f(t,\xi)$ and $l_s = \operatorname{ess\,inf}_{t\in[0,1]} f_z(t,+\infty) = +\infty$, whereas $l_0 = \operatorname{ess\,inf}_{t\in[0,1]} f_z(t,0)$ $= 0$. Moreover, $f_\zeta^*(t,y) = \ln y + 1/t$ for $y > 0$ and $f_\zeta^*(t,0) = -\infty$. Therefore, $|B_0| = 0$, $B_y = [0, 1/|\ln y|]$ for $0 < y < 1$, whereas $B_y = [0,1]$ for $y \geq 1$. Hence, $f_\zeta^*(\cdot,y) \notin L^1(B_y)$ for any $y > 0$ and then $T_p = \{0\}$ for every $p \in [1,+\infty]$. Therefore, by applying Theorem 3.3, we deduce that the minimum exists if and only if $d = 0$.

Example 3.7. (existence for every $d \geq \alpha$). Let $\alpha = 0$ and $f(t,z) = \exp(-z - 1/t)$. We have $l_0 = -1/e$, $l_s = 0$, $B_y = [-1/\ln|y|, 1]$ for every $y \in [-1/e, 0[$. Then, $f_\zeta^*(t,y) = -1/t - \ln|y| \in L^1(B_y)$ for every $y \in [-1/e, 0[$, and

$$\lim_{y \to 0} \int_{B_y} f_\zeta^*(t,y)dt = +\infty,$$

i.e. there exists the minimum for every $d \geq 0$.

This example was already considered by Botteron and Dacorogna in [3], where they gave a sufficient condition for the existence of the minimum (see Remark 5.10 for the details).

4. Applications: quasi-coercive case

In the following we consider integrands of the type

$$f(t,z) = \phi(t)h(z)\ ,$$

with $\phi(t) > 0$ for a.e. $t \in [0,1]$ and $h \in C^1(\mathbb{R})$ strictly convex. In this framework, denoted the inverse function of h' by g, we have that $f_\zeta^*(t,y) = g(\dfrac{y}{\phi(t)})$ for a.e. $t \in [0,1]$.

In the present and in the next section we assume $d > \alpha$, in order to exclude trivial cases of existence ($d = \alpha$) or of non-existence ($d < \alpha$).

We now discuss the "quasi-coercive" case, i.e. integrands which are coercive, with the exception at the most of a finite number of straight lines orthogonal to the t-axis, where it may happen that the function f does not grow.

More precisely, we assume that

$$\lim_{z\to+\infty} h(z)/z = +\infty \ , \quad \text{but} \quad m = \operatorname*{ess\,inf}_{t\in[0,1]} \phi(t) = 0.$$

Moreover, assume that there exists a finite number of points $t_1, \cdots, t_k$ such that $\lim\limits_{t\to t_i}\phi(t) = 0$, $i = 1, \cdots, k$, but $\liminf\limits_{t\to\tau} \phi(t) > 0$ for $\tau \in [0,1] \setminus \{t_1, \cdots, t_k\}$.

In what follows, we denote by $o^*(\beta)$ (for $t \to t_0$) the family of the functions which are infinitesimal for $t \to t_0$ of order $\geq \beta$, and we denote by $O^*(\beta)$ the family of the functions which are infinitesimal of order $\leq \beta$.

Theorem 4.1.

(i) *Let $p < +\infty$. Assume that two positive real numbers β, γ exist such that $\phi(t) \in O^*(\beta)$ when $t \to t_i$, $i = 1, \cdots, k$, and $\dfrac{1}{h'(z)} \in o^*(\gamma)$ when $z \to +\infty$.*
Then, if $\beta p < \gamma$ the functional F admits minimum in $\mathcal{W}^p_d$ for every d.

(ii) *Let $p < +\infty$. Assume that $\phi(t) \in o^*(\beta)$, $\dfrac{1}{h'(z)} \in O^*(\gamma)$, with $\beta p \geq \gamma$.*
Then,
- *if $h'(\alpha) \geq 0$ the minimum does not exist;*
- *if $h'(\alpha) < 0$ the minimum exists if and only if $d \leq \xi_0$, where $\xi_0 > \alpha$ is such that $h'(\xi_0) = 0$.*

(iii) *Let $p = +\infty$. Then, the same conclusion of part* (ii) *holds.*

Proof. The proof of part (i) is analogous to that of part (i) of Theorem 5 in [11].

(ii) Since $\lim\limits_{z\to+\infty} h'(z) = +\infty$, we have $l_s = +\infty$. Moreover, analogously to what done in the proof of part (ii) of Theorem 5 in [11], we deduce $\lim\limits_{t\to t_i}|t - t_i|^{\frac{p\beta}{\gamma}}|g^p(\dfrac{y}{\phi(t)})| \neq 0$. Therefore, since $\frac{p\beta}{\gamma} \geq 1$, we have $g(\frac{y}{\phi(t)}) \notin L^p(B_y)$ for every $y > 0$.

Note that if $h'(\alpha) \geq 0$ we have $l_\alpha = 0$ and as it is easy to see, we have $T_p = \{0\}$ and $\Psi(0) = \alpha$. Hence, since $d > \alpha$, the minimum does not exist.

Whereas, if $h'(\alpha) < 0$ we have $l_\alpha = \operatorname*{ess\,inf}\limits_{t\in[0,1]} \phi(t)h'(\alpha) = h'(\alpha) \operatorname*{ess\,sup}\limits_{t\in[0,1]} \phi(t) < 0$.

Moreover, for $y \in]l_\alpha, 0]$ we have $h'(\alpha) \leq \frac{y}{\phi(t)} \leq 0$ for a.e. $t \in B_y$. Then, by the monotonicity of g we have $\alpha \leq g(\frac{y}{\phi(t)}) \leq g(0) = \xi_0$ in B_y, hence $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ for every $y \in]l_\alpha, 0]$. Therefore, $\max T_p = 0$ and

$$\max_{y\in T_p} \int_0^1 \psi(t,y)dt = g(0) = \xi_0.$$

Thus, the minimum exists if and only if $d \leq g(0) = \xi_0$.

(iii) Since $\lim_{t\to t_i} g(\dfrac{y}{\phi(t)}) = +\infty$ for every $y \neq 0$, in the case $h'(\alpha) \geq 0$ we have that $g(\frac{y}{\phi(t)}) \notin L^\infty(B_y)$ for every $y > 0$, then $T_\infty = \{0\}$ and the assertion follows.

Whereas, if $h'(\alpha) < 0$, by virtue of what proved above we have that $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ for every $y \in]l_\alpha, 0]$ and $\max_{y\in T_\infty} \int_0^1 \psi(t,y)dt = g(0) = \xi_0$. □

Remark 4.2. The assertion is similar to that obtained in [11] (Theorem 5) for the free problem, with the exception of (ii) and (iii) when $h'(\alpha) < 0$. Infact, in these cases the free problem admits a solution only in the trivial case $h'(d) = 0$, whereas the constrained problem admits a solution for every $d \in [\alpha, \xi_0]$.

In a joint paper with A. Salvadori [13], we have considered the problem of the existence of optimal solutions for the multiple integral of the calculus of variations

$$F[v] = \int_G f(t, v(t), Dv(t))dt$$

over a class $\Omega \subset W^{1,1}(G, \mathbb{R}^n)$, $G \subset \mathbb{R}^m$, under constraints of the type

$$(t, v(t)) \in A, \quad Dv(t) \in Q(t, v(t)).$$

We proposed a precise comparison among various growth conditions on the function f, and we achieved (via direct method of the calculus of variations) an existence result under the assumption that one of these conditions holds. In particular, we cosidered a generalization of a Tonelli's local growth condition (γ_4) (see [13]), which essentially requires that

$$f(t,x,z) \geq \phi(t)h(\|z\|)$$

where $\phi \in O^*(\beta)$, $1/h \in o^*(1+\gamma)$, with $\beta < \gamma$.

The present Theorem 3.3 shows that in our setting if $h'(\alpha) \geq 0$ the assumption $\beta < \gamma$ in condition (γ_4) is optimal, and gives a necessary and sufficient condition for the regularity of the minimizer.

5. Applications: non-coercive case

Also in this section we discuss the case of integrands of the type

$$f(t,z) = \phi(t)h(z)\ ,$$

with $\phi(t)$ measurable, nonnegative, and $h \in C^1(\mathbb{R})$ strictly convex, but with linear growth, at the most, for $z \to +\infty$. In more detail, we now assume

$$\lim_{z\to+\infty} h'(z) = \tilde{h} \in \mathbb{R}\,;$$

then, $f_z(t, +\infty) = \tilde{h}\ \phi(t)$.

As before, we assume $d > \alpha$ in order to exclude trivial cases.

We put $m = \operatorname{ess\,inf}_{t\in[0,1]} \phi(t)$ and $M = \operatorname{ess\,sup}_{t\in[0,1]} \phi(t) \in \mathbb{R} \cup \{+\infty\}$. Moreover, when $0 \in h'(\mathbb{R})$, let $\xi_0 \in \mathbb{R}$ be such that $h'(\xi_0) = 0$.

We divide the treatment into three cases: $\tilde{h} > 0$, $\tilde{h} < 0$, $\tilde{h} = 0$.

5.1. Case $\tilde{h} > 0$

Theorem 5.1. *Let $\tilde{h} > 0$.*

(a) *If $m = 0$ and $h'(\alpha) \geq 0$ then the minimum does not exist.*
(b) *If $m = 0$ and $h'(\alpha) < 0$ then the minimum exists if and only if $d \leq \xi_0$.*
(c) *If $m > 0$, put $B = \{t \in [0,1] \ : \ h'(\alpha)\phi(t) < \tilde{h}m\}$ and*

$$S = \int_B g(\frac{\tilde{h}\ m}{\phi(t)})dt + \alpha(1 - |B|),$$

if $d < S$ the minimum exists for every $p \in [1, +\infty]$; conversely, if the minimum exists for some $p \in [1, +\infty]$, then $d \leq S$.

Proof. (a) In this case we have $l_\alpha = l_s = 0$. Put $B_0 = \{t \in [0,1] \ : \ \phi(t)h'(\alpha) \leq 0\}$, if $h'(\alpha) > 0$ we have $|B_0| = 0$ and since $d > \alpha$ the minimum does not exist; whereas, if $h'(\alpha) = 0$, then $B_0 = [0,1]$ and $\Psi(0) = g(0) = \alpha$. Then, again the minimum does not exist.

(b) Now we have $l_\alpha = Mh'(\alpha) < l_s = 0$. Moreover, for every $y < 0$ and every $t \in B_y$ we have $h'(\alpha) \leq \frac{y}{\phi(t)} < 0$, then

$$\alpha \leq g(\frac{y}{\phi(t)}) \leq g(0) = \xi_0 < +\infty \quad \text{ for every } t \in B_y,$$

then $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ for every $y \in]l_\alpha, 0]$ and

$$\max_{y \in T_p} \int_0^1 \psi(t,y)dt = g(0) = \xi_0.$$

(c) Note that $l_\alpha < l_s = m\ \tilde{h}$. Moreover, for every $y \in [l_\alpha, l_s[$ and every $t \in B_y = \{t \in [0,1] \ : \ \frac{y}{\phi(t)} \geq h'(\alpha)\}$ we have $\alpha \leq g(\frac{y}{\phi(t)}) \leq g(y/m) < +\infty$, then $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ for every $y \in [l_\alpha, l_s[$. Hence,

$$\sup_{y \in T_p} \int_0^1 \psi(t,y)dt = \int_B g(\tilde{h}\ m/\phi(t))dt + \alpha(1 - |B|),$$

and the assertion is proved. $\square$

The next result gives a sufficient condition to have $S = +\infty$, in such a way that the minimum exists for every $d \geq \alpha$.

Theorem 5.2. *Let $\tilde{h} > 0$ and $m > 0$. Assume that there exists a finite number of points $t_1, \cdots, t_k \in [0,1]$ with $\lim_{t \to t_i} \phi(t) = m$ for $i = 1, \cdots, k$, but $\liminf_{t \to \tau} \phi(t) > m$ for $\tau \in [0,1] \setminus \{t_1, \cdots, t_k\}$.*

Moreover suppose that there exist two constants $\gamma \geq \beta > 0$ exist such that

$$[\phi(t) - m] \in o^*(\gamma) \quad when \quad t \to t_i; \quad for\ some \quad i = 1, \cdots, k \tag{5.1}$$

$$[h'(z) - \tilde{h}] \in O^*(\beta) \quad when \quad z \to +\infty. \tag{5.2}$$

Then, the functional F admits minimum in $\mathcal{W}_d^p$ for every $d \geq \alpha$ and every $p \in [1, +\infty]$.

The proof is analogous to that of Theorem 6 in [11].

Remark 5.3. The previous theorem provides a result on existence of the minimum for every $d \in \mathbb{R}$, and it is based on conditions which guarantee that the upper bound S is $+\infty$. When these conditions are not satisfied, the previous bound may be finite. In this case, Theorem 5.1 asserts that for every $d < S$ the minimizer is in $W^{1,\infty}$, but it does not provide information in the case $d = S$. However, taking account of Remark 3.5, we have that if $d = S$ there exists the minimum for $p = 1$, but we do not know anything about the case $p > 1$.

The following theorem discusses the regularity of the minimizer when $d = S$. The proof is analogous to that of Theorem 7 in [11].

Theorem 5.4. *Suppose that the assumptions of* Theorem 5.2 *are satisfied, with* (5.1), (5.2) *replaced by the following conditions*

$(9')$ $\quad [\phi(t) - m] \in O^*(\gamma)$ *when* $t \to t_i,\ i = 1, \cdots, k$

$(10')$ $\quad [h'(z) - \tilde{h}] \in o^*(\beta)$ *when* $z \to +\infty$,

with $\gamma < \beta$.

Then, $S = \displaystyle\int_B g(\frac{m\tilde{h}}{\phi(t)})dt + \alpha(1 - |B|)$ *is finite. Moreover, if* $d = S$ *there exists the minimum in* $\mathcal{W}_d^p$ *for every* $p < \beta/\gamma$.

As a first application of Theorems 5.2, 5.4 it is immediate to prove the following result.

Corollary 5.5. *Let* $f(t, z) = \phi(t)\sqrt{1 + z^2}$, *with* $\phi(t)$ *satisfying the assumptions of* Theorem 5.2.

(i) *If* $[\phi(t) - m] \in o^*(\beta)$, *with* $\beta \geq 2$, *when* $t \to t_i$, *for some* $i = 1, \cdots, k$, *then the minimum exists for every* $d \geq \alpha$ *and every* $p \in [1, +\infty]$.

(ii) *If* $[\phi(t) - m] \in O^*(\beta)$, *with* $\beta < 2$, *when* $t \to t_i$, $i = 1, \cdots, k$, *and*

$$\alpha \leq d \leq \int_0^1 [\phi^2(t) - m^2]^{-1/2} dt$$

then there exists the minimum in $\mathcal{W}_d^p$ *for every* $p < 2/\beta$.

5.2. Case $\tilde{h} < 0$

Theorem 5.6. *Let* $\tilde{h} < 0$.

(a) *If* $M = +\infty$ *the minimum does not exist.*

(b) *If* $M < +\infty$, *then put* $B^* = \{t \in [0, 1] \ :\ h'(\alpha)\phi(t) < \tilde{h}M\}$ *and*

$$S^* = \int_{B^*} g(\frac{\tilde{h}M}{\phi(t)})dt + \alpha(|1 - |B^*|),$$

if $d < S^*$ *the minimum exists for every* $p \in [1, +\infty]$, *conversely, if the minimum exists for some* $p \in [1, +\infty]$, *then* $d \leq S^*$.

Proof. (a) In this case we have $l_\alpha = l_s = -\infty$, then the minimum does not exist.

(b) Now we have $l_\alpha = Mh'(\alpha) < l_s = M\tilde{h} < 0$. For every $y \in [l_\alpha, l_s[$ and every $t \in B_y$ we have

$$\alpha \le g(\frac{y}{\phi(t)}) \le g(y/M) < +\infty.$$

Then $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ and

$$\sup_{y\in T_p} \int_{B_y} \psi(t,y)dt = \int_{B^*} g(\tilde{h}M/\phi(t))dt + \alpha(1-|B^*|),$$

and the assertion is proved. □

Analogously to what done in the case $\tilde{h} > 0$, the following result gives a sufficient condition to have $S^* = +\infty$, in such a way that the minimum exists for every $d \ge \alpha$.

Theorem 5.7. *Let $M < +\infty$. Assume that there exists a finite number of points $\tau_1, \cdots, \tau_s \in [0,1]$, such that $\lim_{t\to\tau_j} \phi(t) = M$, for $j = 1, \cdots, s$; but $\limsup_{t\to\tau_0} \phi(t) < M$ for $\tau_0 \in [0,1] \setminus \{\tau_1, \cdots, \tau_s\}$.*

Moreover assume that there exist two constants $\gamma \ge \beta > 0$ such that

$$[\phi(t) - M] \in o^*(\gamma) \quad \text{when} \quad t \to \tau_j, \quad \text{for some} \quad j = 1, \cdots, s \tag{5.3}$$

$$[h'(z) - \tilde{h}] \in O^*(\beta) \quad \text{when} \quad z \to +\infty. \tag{5.4}$$

Then, the functional F admits minimum in $\mathcal{W}^p_d$ for every $d \ge \alpha$ and every $p \in [1, +\infty]$.

5.3. Case $\tilde{h} = 0$

Theorem 5.8. *If $\tilde{h} = 0$ and $M < +\infty$, then the minimum exists for every $d \ge \alpha$.*

Proof. Note that $l_\alpha = Mh'(\alpha) < 0 = l_s$. For every $y \in [l_\alpha, 0[$ and every $t \in B_y$, we have $\alpha \le g(\frac{y}{\phi(t)}) \le g(y/M) < +\infty$, i.e. $g(\frac{y}{\phi(t)}) \in L^\infty(B_y)$ for every $y \in [l_\alpha, 0[$. Then, $\max T_p = 0$ for every $p \in [1, +\infty]$. Hence, we have

$$\sup_{y\in T_p} \int_0^1 \psi(t,y)dt = g(0) = +\infty,$$

i.e. the minimum exists for every $d \ge \alpha$. □

Note that the existence of the minimum for the functional of Example 3.7 can be deduced also by the previous theorem.

In the next result we discuss the case $\tilde{h} = 0$ and $M = +\infty$. In what follows we assume that there exists a finite number of points $\tau_1, \cdots, \tau_s \in [0,1]$, such that $\lim_{t\to\tau_j} \phi(t) = +\infty$, for $j = 1, \cdots, s$; but $\limsup_{t\to\tau_0} \phi(t) < +\infty$ for $\tau_0 \in [0,1] \setminus \{\tau_1, \cdots, \tau_s\}$.

We omit the proof of the following theorem since analogous to that of Theorem 4.1.

Theorem 5.9. *Let $\tilde{h} = 0$ and $M = +\infty$.*

(a) *Assume that there exist two constants $\gamma, \beta > 0$ such that*

$$[1/\phi(t)] \in O^*(\beta) \quad when \quad t \to \tau_j, \quad j = 1, \cdots, s \tag{5.5}$$

$$[h'(z)] \in o^*(\gamma) \quad when \quad z \to +\infty. \tag{5.6}$$

Then, if $p\beta < \gamma$ the functional F admits minimum in $\mathcal{W}_d^p$ for every $d \geq \alpha$ and every $p \in [1, +\infty]$.

(b) *Assume that there exist two constants $\gamma, \beta > 0$ such that*

$$[1/\phi(t)] \in o^*(\beta) \quad when \quad t \to \tau_j, \quad for\ some \quad j = 1, \cdots, s \tag{5.7}$$

$$[h'(z)] \in O^*(\gamma) \quad when \quad z \to +\infty. \tag{5.8}$$

Then, if $p\beta \geq \gamma$ the functional F does not admit minimum in $\mathcal{W}_d^p$.

Remark 5.10. In [3], B. Botteron and B. Dacorogna gave a sufficient condition for the existence of the minimum in the case $p = +\infty$, $f \in C^1([0,1] \times \mathbb{R})$. In more detail, they proved that if

$$\sup_{t\in[0,1]} f_z(t,d) \leq \inf_{t\in[0,1]} f_z(t,+\infty), \tag{5.9}$$

then the minimum exists.

Note that this condition is only sufficient. For example, let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by $f(t,z) = (t^2+1)\sqrt{1+z^2}$. The condition (5.9) is satisfied if and only if $0 \leq d \leq \sqrt{1/3}$, whereas, by virtue of Corollary 5.5 we have that the minimum exists for any $d \geq 0$.

Finally, we wish to remark the recent paper by F. Weissbaum [17] where the Author gives a necessary and sufficient condition for the existence of the minimum for problem (P) whose proof is based on the Kuhn-Tucker theory (see Lemma 2.4 in [17]). But note that the cone determined in L^p by the constraints has empty interior, and then such a theory can not be applied.

Acknowledgements. The author is very grateful to Professor Paolo Marcellini for his useful advice and suggestions.

References

[1] J. M. Ball: Loss of the constraint in convex variational problems, Analyse Math. et Appl. (1988) 39–53.

[2] V. Barbu, T. Precupanu: Convexity and Optimization in Banach Spaces, Editura Academia, Bucarest, 1978.

[3] B. Botteron, B. Dacorogna: Existence of solutions for a variational problem associated to models in optimal foraging theory, J. Math. Anal. Appl. 147 (1990) 263–276.

[4] B. Botteron, B. Dacorogna: Existence and non-existence results for non-coercive variational problems and applications in ecology, J. Diff. Eqs. 85 (1990) 214–235.

[5] B. Botteron, P. Marcellini: A general approach to the existence of minimizers of one-dimensional non-coercive integrals of the calculus of variations, Ann. Inst. Henri Poincaré 8(2) (1991) 197–223.

[6] P. Brandi: Sul problema libero unidimensionale del calcolo delle variazioni, Atti Sem. Mat. Fis. Univ. Modena 28 (1979) 15–32.

[7] L. Cesari: Optimization Theory and Applications, Springer-Verlag, New York et al., 1983.

[8] L. Cesari, P. Brandi, A. Salvadori: Existence theorems concerning simple integrals of the calculus of variations for discontinuous solutions, Arch. Rat. Mech. Anal. 98 (1987) 307–328.

[9] B. Dacorogna: Direct Methods in the Calculus of Variations, Springer-Verlag, New York et al., 1989.

[10] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, Stud. in Math. and its Appl. (1), North-Holland Amer. Els., Amsterdam et al., 1976.

[11] C. Marcelli: One-dimensional non-coercive problems of the calculus of variations, Ann. Mat. Pura Appl., to appear.

[12] C. Marcelli, E. Outkine, M. Sychev: Remarks on necessary conditions for minimizers of one-dimensional variational problems, to appear.

[13] C. Marcelli, A. Salvadori: Remarks on growth conditions in calculus of variations, Atti Sem. Mat. Fis. Univ. Modena XL (1992) 199–214.

[14] P. Marcellini: Alcune osservazioni sull'esistenza del minimo di integrali del calcolo delle variazioni senza ipotesi di convessitá, Rend. Mat. 13(2) (1980) 271–281.

[15] P. Marcellini: Regularity of minimizers of integrands of the calculus of variations with non standard growth conditions, Arch. Rational. Mech. Anal. 105 (1989) 267–284.

[16] R. T. Rockafellar: Convex Analysis, Princeton University Press, 1970.

[17] F. Weissbaum: Some results on non-coercive variational problems and applications, Zeitschrift für Analysis und ihre Anwendungen 14(2) (1995) 285–325.