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Spectrum Stability of an Elliptic Operator to Domain Perturbations

Dorin Bucur
Equipe de Mathématiques UMR-CNRS 6623, Université de Franche-Comté, 16, route de Gray, 25030 Besançon, France.
e-mail: bucur@math.univ-fcomte.fr

Jean-Paul Zolésio
CNRS-INLN, 1361 Route des Lucioles, and Centre de Mathématiques Appliquées, EMP, 2004 Route des Lucioles, 06902 Sophia Antipolis Cedex, France.

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This paper raises the question of the stability of the full spectrum of an elliptic operator in divergence form (with homogeneous Dirichlet boundary conditions) with respect to domain perturbations which modify continuously a limited number of "small" eigenvalues. If the perturbations are semi-compact and modify the measure continuously, we prove that the stability of the first eigenvalue implies the stability of the full spectrum, under the hypothesis that the perturbed domain is connected.

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1. Introduction

Many mechanical problems raise the following question: do the small eigenvalues control the full spectrum of an operator with respect to its domain perturbations? We shall give a positive answer for the class of elliptic operators in divergence form for which the geometric domain perturbations are semi-compact and whose measures vary continuously. More precisely, we shall prove that if a connected open set is perturbed in this way, then the stability of the full spectrum depends only on the stability of the first eigenvalue. Moreover, we obtain the convergence in the sense of Mosco for the associated Sobolev spaces.

The mathematical reasons for which this result holds are the following. The semi-compact perturbations of the open set satisfy one of the two conditions for Mosco convergence. The second condition, which generally is the "difficult" one is deduced from the convergence of the first eigenvalue and the convergence of the measure which surprisingly can control all the cracks which may appear in the shape perturbation process. We first prove a lemma which asserts that the continuity of a sequence of eigenvectors corresponding to the first eigenvalue suffices to obtain the second Mosco condition. This follows from a result of [10] where it is established that the γ-convergence is equivalent to the stability of the solutions of the Laplace-Dirichlet problem with the right hand side equal to 1. In a second step, we prove that the shape continuity of the first eigenvalue yields the shape continuity of a sequence of eigenvectors. Finally, we make some remarks concerning the

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extension of this result to manifolds or to elliptic operators with non-smooth coefficients, and give some examples.

To make our framework more precise, let us consider an open set Ω and a sequential family of geometric perturbations denoted Ω_n. We are interested in the behavior of the eigenvalues for the Laplace operator with Dirichlet homogeneous boundary conditions on the variable domain Ω_n. Denote $\lambda_k(\Omega_n)$ the k-th eigenvalue of the Laplace-Dirichlet operator on Ω_n (counting multiplicities). Classic results prove that the γ-convergence of Ω_n to Ω yields the continuity of the full spectrum in the sense that $\lim_{n\to\infty} \lambda_k(\Omega_n) = \lambda_k(\Omega)$. Obviously, the converse implication is false if no geometric relation is imposed between Ω_n and Ω (consider for example $\Omega_n = B(0,1)$ the unit ball and $\Omega = B(1,1)$).

The purpose of this paper is to prove that in some specific situation one can obtain the γ-convergence as a consequence of the sequential continuity of the first eigenvalue. Therefore, to ensure the continuity of the full spectrum it suffices to control the first eigenvalue. Some final remarks point to the possibility of obtaining the same results while controlling instead of the first eigenvalue a more general shape functional. We shall give an example concerning the mappings $\Omega \to \lambda_1(\Omega) + \lambda_2(\Omega)$ and $\Omega \to \lambda_1(\Omega) \cdot \lambda_2(\Omega)$.

2. Some Preliminary Results

Let us introduce the main notations and recall some basic results. Suppose that B is a fixed ball of $\mathbb{R}^N$, which contains the set Ω and all its perturbations, generally denoted by $\{\Omega_n\}_{n\in\mathbb{N}}$.

Definition 2.1. Consider $\{\Omega_n\}_{n\in\mathbb{N}}, \Omega \subseteq B$. It is said that Ω_n γ-converges to Ω if $\forall f \in H^{-1}(B)$ the weak solution of the problem $-\Delta u = f$ in Ω_n , $u_{|\partial\Omega_n} = 0$ (denoted $u_{\Omega_n,f}$) converges to the solution of the same problem on Ω for $n \to \infty$ in the topology of $H_0^1(B)$.

The weak solutions $u_{\Omega_n,f}$ which belong to $H_0^1(\Omega_n)$ are assumed to have been extended by zero to elements of $H_0^1(B)$. Recall also the result of [10] which asserts that to obtain γ-convergence it suffices to prove continuity for $f \equiv 1$ only. This assertion is based on the maximum principle. Moreover, γ convergence is equivalent to the strong point wise convergence of the orthogonal projectors on the variable space $H_0^1(\Omega_n)$.

The non-smooth perturbations considered in this paper are called semi-compact, i.e. $\forall K \subseteq\subseteq \Omega$ there exist some positive integer n_K such that $\forall n \geq n_K$ the set Ω_n contains K. Note that this condition represents half of compact convergence (see [8], [9]) which requires the same property also for the exterior of the variable domain. This relation means that the sets Ω_n cover each compact of the limit domain (this property is also satisfied by the Hausdorff complementary topology [4]). In fact, asking for semi compact convergence is a way to replace one of the two conditions for convergence in the sense of Mosco of the Sobolev spaces $H_0^1(\Omega_n)$.

The space $H_0^1(\Omega)$ is said to be the limit in the sense of Mosco (M-limit) of the sequence $\{H_0^1(\Omega_n)\}_{n\in\mathbb{N}}$ for $n \to \infty$ if

(M_1) $\quad \forall \varphi \in H_0^1(\Omega)$, $\exists \varphi_n \in H_0^1(\Omega_n)$, such that $\varphi_n \overset{H_0^1(B)}{\rightarrow} \varphi$

(M_2) $\quad \forall \varphi_{n_k} \in H_0^1(\Omega_{n_k})$, $\varphi_{n_k} \overset{H_0^1(B)}{\rightharpoonup} \varphi$ implies $\varphi \in H_0^1(\Omega)$

Lemma 2.2. *If Ω_n converges in the semi-compact sense to Ω then $\chi_\Omega \leq \liminf_{n\to\infty} \chi_{\Omega_n}$.*

Proof. For all $K \subseteq\subseteq \Omega$ we have $\chi_K \leq \chi_{\Omega_n}$ for $n \geq n_K$ and we choose a sequence of increasing compacts after letting $n \to \infty$. $\square$

Remark also that if Ω_n converges in the semi-compact sense to Ω and if $m(\Omega_n) \to m(\Omega)$ then $\chi_{\Omega_n} \xrightarrow{L^2} \chi_\Omega$ i.e. Ω_n converges in the sense of measures to Ω. Indeed, let us compute

$$\int_{\mathbb{R}^N} |\chi_\Omega - \chi_{\Omega_n}| dx = m(\Omega \setminus \Omega_n) + m(\Omega_n \setminus \Omega)$$

For all $\varepsilon > 0$ consider $K \subseteq\subseteq \Omega$ such that $m(\Omega \setminus K) < \varepsilon$. Then

$$m(\Omega \setminus \Omega_n) \leq \varepsilon + m(K \setminus \Omega_n)$$

and

$$m(\Omega_n \setminus \Omega) \leq m(\Omega_n \setminus K)$$

For n large enough such that $K \subseteq\subseteq \Omega_n$ we get $m(K \setminus \Omega_n) = 0$ and $m(\Omega_n \setminus K) = m(\Omega_n) - m(K) < m(\Omega_n) - m(\Omega) + \varepsilon$. Finally

$$\int_{\mathbb{R}^N} |\chi_\Omega - \chi_{\Omega_n}| dx \leq m(\Omega_n) - m(\Omega) + 2\varepsilon$$

From the convergence of the measures, letting $\varepsilon \to 0$ we obtain the convergence in measure of Ω_n to Ω.

In our paper, by $\lambda_k(\Omega)$ we denote the k^{th} eigenvalue counted with its multiplicity of the Laplace operator $(-\Delta)$ with Dirichlet boundary conditions on the open set Ω. This eigenvalue can be computed using the Max-Min formula (see [7])

$$\lambda_k(\Omega) = \max_{V_{k-1}} \min_{u \in H_0^1(\Omega) \setminus \{0\}, u \perp V_{k-1}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$$

where the minimum is taken for all subspaces of dimension $k-1$, $V_{k-1} \subseteq H_0^1(\Omega)$. Moreover the maximum is reached if V_{k-1} is generated by the first $k-1$ eigenvectors. This is the good way to denote the eigenvalues if one intends to get a continuity result. It is well known that a consequence of γ-convergence is the continuity of the k^{th} eigenvalue. The case where the eigenvalues are not counted with their multiplicities will be touched upon in the last paragraph.

Lemma 2.3. *If Ω_n converges in the semi-compact sense to Ω then*

$$\lambda_k(\Omega) \geq \limsup_{n \to \infty} \lambda_k(\Omega_n)$$

Proof. Choose a sequence of opens $\{A_i\}_{i \in \mathbb{N}}$ and $\overline{A}_i \subseteq A_{i+1} \subseteq \Omega$ such that A_i γ-converges to Ω. Hence the full spectrum converges and in particular we have $\lambda_k(\Omega) = \lim_{i \to \infty} \lambda_k(A_i)$. But for each $i \in \mathbb{N}$ there exists some $n_i \in \mathbb{N}$ such that $\overline{A}_i \subseteq \Omega_n$ for all $n \geq n_i$, implying by monotonicity $\lambda_k(\Omega_n) \leq \lambda_k(A_i)$. Letting first $n \to \infty$ and then $i \to \infty$ we get the upper semi-continuity of the k-th eigenvalue. $\square$

Lemma 2.4. *There exists some constant $M = M(B)$ such that for all $f, g \in L^2(B)$, for all $\Omega \subseteq B$ we have $\|u_{\Omega,f} - u_{\Omega,g}\|_{H_0^1(B)} \leq M|f-g|_{L^2(B)}$.*

Lemma 2.5. *Suppose $\{\Omega_n\}_{n\in\mathbb{N}}, \Omega \subseteq B$ such that Ω_n converges in the semi-compact sense to Ω and $f, g \in L^2(B)$, $f \geq g \geq 0$. If $u_{\Omega_n,f} \overset{H_0^1(B)}{\rightarrow} u_{\Omega,f}$ then we also have $u_{\Omega_n,g} \overset{H_0^1(B)}{\rightarrow} u_{\Omega,g}$.*

The proof of this lemma is a simple consequence of the maximum principle (see [10], [2]).

3. The Stability Theorem

Our purpose is to derive the γ-convergence of a sequence of domains only from the shape continuity of the first eigenvalue. The proof of this result can be split in two steps. We first prove that under some assumptions the γ-convergence is equivalent to the shape continuity of an eigenvector corresponding to the first eigenvalue. Secondly we prove that the shape continuity of the first eigenvalue in this context gives the shape continuity of a normalized eigenvector.

In all the forthcoming statements we shall frequently use some of the following hypotheses:

(H_1) $\{\Omega_n\}_{n\in\mathbb{N}}$ converges in the semi compact sense to Ω
(H_2) $m(\Omega_n) \to m(\Omega)$
(H_3) Ω is connected

Let's give firstly the following lemma.

Lemma 3.1. *Under the hypotheses H_1, H_2, H_3 Ω_n γ-converges to Ω if and only if there exists a sequence of normalized eigenvectors corresponding to the first eigenvalue $\lambda_1(\Omega_n)$, denoted φ_{1,Ω_n} such that $\varphi_{1,\Omega_n} \overset{H_0^1(B)}{\longrightarrow} \varphi_{1,\Omega}$.*

Proof. Effectively, if Ω_n γ-converges to Ω then this assertion is obvious. Let's suppose the converse. We shall prove that the convergence of the eigenvectors gives the γ-convergence.

The idea of the proof is to establish the shape continuity of the solution of equation

$$-\Delta u_n = 1 \;\; in \;\; H_0^1(\Omega_n)$$

by proving successively the shape continuity of the solutions of some equations

$$-\Delta u_n^i = g_n^i \;\; in \;\; H_0^1(\Omega_n)$$

for an index $i = 0, .., 5$ and right hands g_n^i well chosen.

Since the first eigenvalues are also convergent the solutions of the equation

$$-\Delta y_n^0 = \lambda_1(\Omega_n)\varphi_{1,\Omega_n}, \;\; y_n^0 \in H_0^1(\Omega_n)$$

converge for $n \to \infty$ in $H_0^1(B)$ since $y_n^0 = \varphi_{1,\Omega_n}$. Let's denote by y_n^1 the solution of the equation

$$-\Delta y_n^1 = \lambda_1(\Omega)\varphi_{1,\Omega}, \;\; y_n^1 \in H_0^1(\Omega_n)$$

Then

$$\|y_n - y_n^1\|_{H_0^1(B)} \leq M|\lambda_1(\Omega_n)\varphi_{1,\Omega_n} - \lambda_1(\Omega)\varphi_{1,\Omega}|_{L^2(B)}$$

and so we get $y_n^1 \overset{H_0^1(B)}{\rightarrow} \varphi_{1,\Omega}$. The right hand side is fixed and equal to $\lambda_1(\Omega)\varphi_{1,\Omega}$. For any compact $K \subseteq\subseteq \Omega$ we get $\inf_{x\in K} \varphi_{1,\Omega}(x) = \varepsilon > 0$ since $\varphi_{1,\Omega}$ is of class C^∞ on Ω and is super harmonic. If the infimum of $\varphi_{1,\Omega}$ on K would be equal to zero, then the super harmonicity would give $\varphi_{1,\Omega} = 0$ on the connected component which contains the point, and from the connection hypothesis H_3 on the whole Ω which is in contradiction with the choice of $\varphi_{1,\Omega}$. Since the eigenvector is positive one can write

$$\lambda_1(\Omega)\varphi_{1,\Omega} \geq \lambda_1(\Omega)\varepsilon\chi_K \geq 0, \quad in \;\; B$$

Using Lemma 2.4 we get the shape continuity for the equations

$$-\Delta y_n^2 = \lambda_1(\Omega)\varepsilon\chi_K \;\; in \;\; H_0^1(\Omega_n)$$

or simply

$$-\Delta y_n^{3K} = \chi_K \;\; in \;\; H_0^1(\Omega_n)$$

Let's prove the shape continuity for the solutions of

$$-\Delta y_n^4 = \chi_\Omega \;\; in \;\; H_0^1(\Omega_n)$$

For any compact $K \subseteq\subseteq \Omega$ one can write

$$\|y_n^4 - y^4\| \leq \|y_n^4 - y_n^{3K}\| + \|y_n^{3K} - y^{3K}\| + \|y^{3K} - y^4\| \leq$$

$$\leq 2M|\chi_\Omega - \chi_K| + \|y_n^{3K} - y^{3K}\|$$

Choosing firstly a compact K such that $2M|\chi_\Omega - \chi_K| < \frac{\varepsilon}{2}$ and then n large enough such that $\|y_n^{3K} - y^{3K}\| < \frac{\varepsilon}{2}$ the conclusion follows.

In a similar way, we get the continuity of the solutions of

$$-\Delta y_n^5 = \chi_{\Omega_n} \;\; in \;\; H_0^1(\Omega_n)$$

since $\chi_{\Omega_n} \overset{L^2(B)}{\rightarrow} \chi_\Omega$. This last result proves that the solutions of

$$-\Delta y_n^5 = 1 \;\; in \;\; H_0^1(\Omega_n)$$

are convergent for $n \to \infty$ and we get from [10] the γ-convergence. □

Lemma 3.2. *Suppose that hypotheses H_1, H_2, H_3 hold. Then the following two assertions are equivalent*

(1) $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ *and there exists some constant* $c \in \mathbb{R}_+, c \neq \lambda_1(\Omega)$ *such that* $\lambda_2(\Omega_n) \to c$

(2) *there exists a sequence of normalized eigenvectors corresponding to the first eigenvalue which converges.*

Moreover, if one of the previous situations holds, then $c = \lambda_2(\Omega)$.

Proof. It is clear that 2. implies 1. We shall prove the converse, namely if $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ and $\lambda_2(\Omega_n) \to c \neq \lambda_1(\Omega)$ then there exists a sequence of normalized eigenvectors corresponding to the first eigenvalue which converges.

There exists a sequence of elements $u_n \in \mathcal{D}(\Omega) \cap H_0^1(\Omega_n)$ such that $\varphi_{1,\Omega} = \lim_{n\to\infty} u_n$ in $H_0^1(B)$. The existence of such a sequence is a consequence of the semi-compact convergence. Moreover, without loosing the generality one can suppose that the elements u_n are normalized, i.e. $|u_n|_{L^2(B)} = 1$.

One can decompose the vectors u_n in the space $H_0^1(\Omega_n)$ in the following way

$$u_n = a_n \varphi_{1,\Omega_n} + v_n$$

where $a_n \in \mathbb{R}_+$ and $v_n \in H_0^1(\Omega_n)$, $v_n \perp \varphi_{1,\Omega_n}$. The condition of normalization of u_n gives $|v_n|_{L^2(B)} = 1 - a_n^2$. By hypothesis we have $u_n \to \varphi_{1,\Omega}$ and we want to get $\varphi_{1,\Omega_n} \to \varphi_{1,\Omega}$. It suffices to prove the following convergence $\varphi_{1,\Omega_n} - u_n \to 0$ in $H_0^1(B)$ or equivalent

$$\int_B |\nabla\varphi_{1,\Omega_n} - \nabla u_n|^2 dx \to 0$$

or

$$(1-a_n)^2\lambda_1(\Omega_n) + \int_B |\nabla v_n|^2 dx \to 0$$

On the other side, since $\lambda_1(\Omega_n)$ converges to $\lambda_1(\Omega)$ we have

$$\int_B |\nabla u_n|^2 dx - \int_B |\nabla \varphi_{1,\Omega_n}|^2 dx \to 0$$

or still

$$a_n^2\lambda_1(\Omega_n) + \int_{\Omega_n} |\nabla v_n|^2 dx - \lambda_1(\Omega_n) \to 0$$

It suffices so to prove the following relation

$$(1-a_n)^2\lambda_1(\Omega_n) + \lambda_1(\Omega_n) - a_n^2\lambda_1(\Omega_n) \to 0$$

or equivalently $a_n \to 1$. Since the normality of the vectors u_n gives $a_n^2 = 1 - |v_n|^2_{L^2(B)}$ it suffices to have $|v_n|_{L^2(B)} \to 0$.

Let's suppose the contrary, that is the existence of a subsequence still denoted with the same index and of a real number $\alpha > 0$ such that $|v_n|^2_{L^2(B)} > \alpha > 0$. Then we have

$$\frac{\int_{\Omega_n} |\nabla v_n|^2 dx}{\int_{\Omega_n} v_n^2 dx} - \lambda_1(\Omega_n) \to 0$$

Since $\frac{\int_{\Omega_n} |\nabla v_n| dx}{\int_{\Omega_n} v_n^2 dx} \geq \lambda_2(\Omega_n) \geq \lambda_1(\Omega_n)$ car $v_n \perp \varphi_{1,\Omega_n}$ we get

$$\lim_{n\to\infty} [\lambda_2(\Omega_n) - \lambda_1(\Omega_n)] = 0$$

The continuity of the first eigenvalue and the convergence hypothesis on the second, give $\lambda_1(\Omega) = c$, in contradiction with the choice of $c \neq \lambda_1(\Omega)$. $\square$

Theorem 3.3. *Suppose that Ω_n converges in the semi-compact sense to Ω, such that $m(\Omega_n) \to m(\Omega)$ and Ω is supposed to be connected. If $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ then Ω_n γ-converges to Ω.*

Proof. Suppose by contradiction that we do not have the γ-convergence. If $\lambda_2(\Omega_n) \to \lambda_2(\Omega)$ then the γ-convergence is a consequence of the previous lemma and hence we have $\lambda_2(\Omega_n) \not\to \lambda_2(\Omega)$. By compactness one can subtract a sequence $\lambda_2(\Omega_{n_s}) \to c \neq \lambda_2(\Omega)$. We shall prove that this situation is impossible. If $c \neq \lambda_1(\Omega)$ then from the previous lemma we have the γ convergence and as a consequence $\lambda_2(\Omega_n) \to \lambda_2(\Omega)$ in contradiction with our choice. Therefore $c = \lambda_1(\Omega)$. Again the compactness allows us to subtract a sub sequence of $\{\Omega_{n_s}\}$ such that $\lambda_3(\Omega_{n_{s_1 s_2}}) \to c_2$. We shall prove that $c_2 = \lambda_1(\Omega)$. We shall use an induction method and in a more general formulation suppose by absurd that there exists a subsequence still denoted with the normal index such that k is the first index with $\lambda_k(\Omega_n) \to c \neq \lambda_1(\Omega)$ and $\lambda_j(\Omega_n) \to \lambda_1(\Omega)$ for $j = \overline{1, k-1}$ et $k \geq 3$. We shall prove that this situation can not hold. There exists a normalized sequence $u_n \in H_0^1(\Omega_n)$ such that $u_n \overset{H_0^1(B)}{\to} \varphi_{1,\Omega}$. Decompose u_n in the following way

$$u_n = \sum_{j=1}^{k-1} a_{n,j}\varphi_{j,\Omega_n} + v_n$$

with $v_n \perp \varphi_{j,\Omega_n}$, $\forall j = \overline{1, k-1}$. The normality of u_n give

$$\sum_{j=1}^{k-1} a_{n,j}^2 + \int_B v_n^2 dx = 1$$

We shall prove that $v_n \overset{H_0^1(B)}{\to} 0$. As the following sequences have the same limit, we have

$$\int_B |\nabla u_n|^2 dx - \int_B |\nabla \varphi_{1,\Omega_n}|^2 dx \to 0$$

or equivalently

$$\sum_{j=1}^{k-1} a_{n,j}^2 \lambda_j(\Omega_n) + \int_B |\nabla v_n|^2 dx - \lambda_1(\Omega_n) \to 0$$

The normality of u and the fact that $\lambda_j(\Omega_n) \to \lambda_1(\Omega)$ give from the previous lemma

$$\int_B |\nabla v_n|^2 dx - \lambda_1(\Omega) \int_B v_n^2 dx \to 0 \tag{3.1}$$

If there exists a sub sequence still denoted with the same index such that $\int_B v_n^2 dx \not\to 0$ then

$$\frac{\int_B |\nabla v_n|^2 dx}{\int_B v_n^2 dx} - \lambda_1(\Omega) \to 0$$

which gives $\lambda_k(\Omega_n) - \lambda_1(\Omega) \to 0$, in contradiction with our supposition. So $\int_B v_n^2 dx \to 0$ which implies from (3.1) $\int_B |\nabla v_n|^2 dx \to 0$ and hence

$$\sum_{j=1}^{k-1} a_{n,j}\varphi_{j,\Omega_n} \overset{H_0^1(B)}{\to} \varphi_{1,\Omega}$$

Then the solutions of the equations

$$-\Delta y_n = \sum_{j=1}^{k-1} a_{n,j}\lambda_j(\Omega_n)\varphi_{j,\Omega_n} \ \ in \ \ \Omega_n$$

converge to the solution of the equation

$$-\Delta y = \lambda_1(\Omega)\varphi_{1,\Omega} \ \ in \ \ \Omega$$

Moreover, the right hand sides converge either

$$\sum_{j=1}^{k-1} a_{n,j}\lambda_j(\Omega_n)\varphi_{j,\Omega_n} \overset{H_0^1(B)}{\to} \lambda_1(\Omega)\varphi_{1,\Omega}$$

and so one can apply the arguments of the previous lemma obtaining that the solutions of the problem

$$-\Delta y_n^1 = \lambda_1\varphi_{1,\Omega} \ \ in \ \ \Omega_n$$

converge to the solution of the problem

$$-\Delta y = \lambda_1(\Omega)\varphi_{1,\Omega}$$

As in the previous lemma this fact gives the γ-convergence , in contradiction with our hypothesis on the convergence of $\lambda_{k-1}(\Omega_n)$ to $\lambda_1(\Omega)$ which is simple (and $k \geq 3$).

So $\lambda_k(\Omega_n) \to \lambda_1(\Omega)$, and since the limit is unique this means applying the same arguments for $k = 2, 3, ...$ that $\lambda_k(\Omega_{n_s}) \to \lambda_1(\Omega)$ for all $k \in \mathbb{N}$, the sequence $\{\Omega_{n_s}\}$ being that one chosen in the beginning. But this situation is impossible, since for the ball one knows that $\lambda_r(B) \to \infty$ for $r \to \infty$ (see [6]). Moreover, by inclusion and monotonicity $\lambda_r(\Omega_n) \geq \lambda_r(B)$. Then for some r enough large such that $\lambda_r(B) \geq \lambda_1(\Omega) + 1$ we can not have $\lambda_r(\Omega_{n_s}) \to \lambda_1(\Omega)$ and hence the proof is finished. □

Corollary 3.4. *Under hypotheses H_1, H_2, H_3 the following two assertions are equivalent*

(1) $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$

(2) $\lambda_i(\Omega_n) \to \lambda_i(\Omega)$ *for all* $i \in \mathbb{N}^*$

The previous results can be extended in the following way.

Corollary 3.5. *Suppose that $\Omega \to F(\Omega)$ is a γ-continuous function and the mapping $\Omega \to F(\Omega) - \lambda_1(\Omega)$ is decreasing on inclusions. Then the following assertions are equivalent if H_1, H_2, H_3 are satisfied.*

(1) $F(\Omega_n) \to F(\Omega)$

(2) Ω_n *γ-converges to* Ω

For the proof we see that the γ continuity of $F(\cdot)$ with the monotonicity of $F(\cdot) - \lambda_1(\cdot)$ gives from *1.* the continuity of the first eigenvalue and Theorem 3.3 can be applied. As an example the functional can be chosen as

$$F(\Omega) = \lambda_1(\Omega) + \lambda_2(\Omega)$$

Identically if the functional $F(\cdot)$ is chosen such that $\Omega \to \frac{F(\Omega)}{\lambda_1(\Omega)}$ is decreasing on inclusions then Corollary 3.5 is still true. For example the functional can be chosen as $F(\Omega) = \lambda_1(\Omega) \cdot \lambda_2(\Omega)$.

4. Mosco Conditions and γ-convergence

Theorem 3.3 can be modified as follows. If $\{\Omega_n\}_{n\in\mathbb{N}}$ is a perturbation of an open connected set such that the first Mosco condition is satisfied, we are interested under which supplementary conditions Ω_n γ-converges to Ω, or equivalently M_2 holds? The semi compact convergence is a way to satisfy the first Mosco condition and therefore one can replace hypothesis H_1 with the Mosco condition M_1 and reformulate Theorem 3.3:

Proposition 4.1. *If $\{\Omega_n\}_{n\in\mathbb{N}}$ is a perturbation of an open set and M_1, H_2, H_3 are satisfied, then Ω_n γ-converges to Ω if and only if* $\lim_{n\to\infty} \lambda_1(\Omega_n) = \lambda_1(\Omega)$.

The proof is similar with Theorem 3.3 using the following arguments. If M_1 and H_2 are satisfied, then $\chi_{\Omega_n} \overset{L^2(B)}{\rightarrow} \chi_\Omega$. Indeed, let's consider w_Ω the weak solution of the problem $-\Delta w_\Omega = 1$ in $H_0^1(\Omega)$. Then $w_\Omega > 0$ a.e. in Ω and from M_1 there exists a sequence $\varphi_n \in H_0^1(\Omega_n)$ such that $\varphi_n \overset{H_0^1(B)}{\rightarrow} w_\Omega$. Then a.e. we have the following two situations: if $\chi_\Omega(x) = 0$ then $0 = \chi_\Omega(x) \leq \liminf_{n\to\infty} \chi_{\Omega_n}(x)$. If $x \in \Omega$ then $w_\Omega(x) > 0$ and $\varphi_n(x) > 0$ for n large enough, which means $\chi_{\Omega_n}(x) = 1$ and hence $\chi_\Omega(x) \leq \liminf_{n\to\infty} \chi_{\Omega_n}(x) = 1$. This relation together with H_2 gives $\chi_{\Omega_n} \overset{L^2(B)}{\rightarrow} \chi_\Omega$. Another argument necessary for Proposition 4.1 is the result of Lemma 2.5 where condition H_1 is replaced by M_1. The proof follows as in [10] or [2].

A "converse" question can also be raised, namely in which conditions the γ-convergence can be derived if the second Mosco condition is satisfied? The fact that the second Mosco condition is satisfied can be seen through the frame of [1] and means that Ω contains each weak γ-limit point of the sequence $\{\Omega_n\}_{n\in\mathbb{N}}$. Let's recall firstly a classical result.

Lemma 4.2. *Let consider $\{\varphi_n\}_{n\in\mathbb{N}} \in H_0^1(B)$ and $\varphi_n \overset{H_0^1(B)}{\rightarrow} \varphi$. Then for any $\psi \in H_0^1(B)$*

$$\min\{\varphi_n, \psi\} \overset{H_0^1(B)}{\rightarrow} \min\{\varphi_n, \psi\}$$

Proof. From the following relations

$$\min\{\varphi_n, \psi\} = \frac{\varphi_n + \psi - |\varphi_n - \psi|}{2} \quad \text{and} \quad |\varphi_n - \psi| = (\varphi_n - \psi)^+ + (\varphi_n - \psi)^-$$

it suffices to prove $\varphi_n^+ \overset{H_0^1(B)}{\rightarrow} \varphi^+$.

There exists some constant $M > 0$ such that $\forall n \in \mathbb{N} \|\varphi_n\|_{H_0^1(B)} \leq M$ and from the classical convergence $\varphi_n^+ \overset{L^2(B)}{\rightarrow} \varphi^+$ we obtain $\varphi_n^+ \overset{H_0^1(B)}{\rightharpoonup} \varphi^+$. To derive the strong convergence it remains to obtain

$$\int_B |\nabla \varphi_n^+|^2 dx \rightarrow \int_B |\nabla \varphi^+|^2 dx$$

or by a simple computation

$$\int_B \chi_{\{\varphi_n \geq 0\}} |\nabla \varphi_n|^2 dx \rightarrow \int_B \chi_{\{\varphi \geq 0\}} |\nabla \varphi|^2 dx$$

which is derived from the Lebesgue dominated convergence theorem. □

One can formulate the following

Proposition 4.3. *Let Ω be an open connected set, and $\{\Omega_n\}_{n\in\mathbb{N}}$ a sequential perturbation in B such that the second Mosco condition is satisfied. Then Ω_n γ-converges to Ω if and only if* $\lim_{n\to\infty} \lambda_1(\Omega_n) \rightarrow \lambda_1(\Omega)$.

Proof. It is clear the if Ω_n γ-converges to Ω we get the continuity of the first eigenvalue. Let's suppose the converse, namely $\lim_{n\to\infty} \lambda_1(\Omega_n) = \lambda_1(\Omega)$. Let consider $\varphi_{1,\Omega_n} \geq 0$ an L^2 normed eigenvector corresponding to $\lambda_1(\Omega_n)$. Then there exists some $M > 0$ such that $\forall n \in \mathbb{N} \|\varphi_{1,\Omega_n}\|_{H_0^1(B)} \leq M$ and for a subsequence one can write

$$\varphi_{1,\Omega_{n_k}} \overset{H_0^1(B)}{\rightharpoonup} \varphi$$

From the second Mosco condition M_2 we get $\varphi \in H_0^1(\Omega)$ and

$$\lambda_1(\Omega) \leq \int_B |\nabla \varphi|^2 dx \leq \liminf_{n\to\infty} \int_B |\nabla \varphi_{1,\Omega_{n_k}}|^2 dx = \liminf_{n\to\infty} \lambda_1(\Omega_{n_k}) = \lambda_1(\Omega)$$

So we deduce $\varphi_{1,\Omega_{n_k}} \overset{H_0^1(B)}{\rightarrow} \varphi$ and since φ is unique ($\varphi \geq 0, ; |\varphi|_{L^2} = 1$ and Ω connected) we get $\varphi_{1,\Omega_n} \overset{H_0^1(B)}{\rightharpoonup} \varphi$. Let's prove now the first Mosco condition.

Consider an element $\theta \in \mathcal{D}(\Omega)$. Then *supp* $\theta = K \subseteq\subseteq \Omega$ and there exists $\varepsilon > 0$ such that $\varphi \geq \varepsilon\theta^+ \geq 0$ and $\varphi \geq \varepsilon\theta^- \geq 0$. Then since $\varphi_{1,\Omega_n} \overset{H_0^1(B)}{\rightarrow} \varphi$ we have

$$\min\{\varphi_n, \varepsilon\theta^+\} \overset{H_0^1(B)}{\rightarrow} \min\{\varphi, \varepsilon\theta^+\} = \varepsilon\theta^+$$

$$\min\{\varphi_n, \varepsilon\theta^-\} \overset{H_0^1(B)}{\rightarrow} \min\{\varphi, \varepsilon\theta^-\} = \varepsilon\theta^-$$

But $\min\{\varphi_n, \varepsilon\theta^-\}, \min\{\varphi_n, \varepsilon\theta^+\} \in H_0^1(\Omega_n)$ and hence we found a sequence of $u_n \in H_0^1(\Omega_n)$ which strongly converges to θ, namely

$$u_n = \frac{1}{\varepsilon}[\min\{\varphi_n, \varepsilon\theta^+\} - \min\{\varphi_n, \varepsilon\theta^-\}]$$

Hence M_1 is also satisfied and the γ-convergence holds. □

Remark that for this situation no convergence of the measures was necessary.

5. Final Remarks

The continuity of the first eigenvalue of the Laplace-Dirichlet operator in the previous context gives the continuity of the full spectrum of an elliptic operator in divergence form with L^∞ coefficients. This assertion is a consequence of the convergence in the sense of Mosco of the spaces $H_0^1(\Omega_n)$.

Let's give some examples which underline the necessity of each hypothesis H_1, H_2, H_3.

Example 5.1. Suppose that we do not ask the convergence of the measure. In 2-D we consider $\Omega_n = B(0,2) \cup B(4,1) \setminus \{x_1, ..., x_n\}$ where $x_1, ..., x_n$ are the first n points of rational coordinates of the ball $B(4,1)$. Then $\Omega_n \to B(0,2)$ in the semi-compact sense and $\varphi_{1,\Omega_n} = \varphi_{1,\Omega} = \varphi(B(0,2))$ but the γ-convergence does not take place.

Example 5.2. Suppose that we do not ask the connection for the limit domain. We choose $\Omega_n = B(0,2) \cup B(4,1) \setminus \{(x_1,0), ..., (x_n,0)\}$ where $x_1, ..., x_n$ are the first n points of rational coordinates of $[3,5] \times \{0\}$. Then $\Omega_n \to \Omega$ where $\Omega = B(0,2) \cup B(4,1) \setminus \{(x,0) | x \in [3,5] \times \{0\}\}$ in the semi compact sense and H_2 is satisfied. We have $\varphi_{1,\Omega_n} = \varphi_{1,\Omega} = \varphi(B(0,2))$, but there is no γ-convergence.

Generally, hypothesis H_1 does not imply H_2, only if some more assumptions are satisfied, like for example a uniform density perimeter condition in relation with the Hausdorff convergence (see [3]).

If one intends to replace the functional $F(\Omega) = \lambda_1(\Omega)$ by another functional, he can give the following type of arguments. Let us suppose that $F(\cdot)$ is strictly decreasing on inclusion and is γ continuous (two domains are identified if the capacity of their symmetric difference is equal to zero). If Ω_n converges in the semi compact sense to Ω then as in Lemma 2.3 one get the upper semi-continuity of $F(.)$. Following [5], [1] there exists a quasi open set A such that

$$\chi_A \leq \liminf_{n\to\infty} \chi_{\Omega_n} \quad and \quad F(A) \leq \liminf_{n\to\infty} F(\Omega_n)$$

Moreover, one can see that Ω satisfies the condition M_1 and A the condition M_2 of the Mosco convergence, and $\Omega \subseteq A$ (up to a set of zero capacity). From the monotonicity of $F(\cdot)$ we get $F(A) \leq F(\Omega)$. If by hypothesis we ask $F(\Omega) = F(A)$ then the strictly monotonicity gives $A = \Omega$ (up to a set of zero capacity), and from the Mosco convergence that Ω_n γ-converges to A. The difficulty is to establish situations when $F(A) = F(\Omega)$. This happens for example if we fix ourselfs in the context of Theorem 3.3 with $F(\Omega) = \lambda_1(\Omega)$, the limit set Ω connected and $m(\Omega_n) \to m(\Omega)$.

Some of the previous results, namely those which are not connected to the maximum principle, can also be reformulated under weaker hypotheses. For example Lemma 3.2 can be reformulated in a more general frame of elliptic operators, while the semi compact convergence is replaced with a condition corresponding to the first condition M_1 of the convergence in the sense of Mosco.

This result can also be extended on manifolds for operators of Laplace-Beltrami type.

If the eigenvalues are not counted with their multiplicities the result must receive another interpretation. For fixed Ω let's denote by

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) < ... < \lambda_k(\Omega) < ...$$

the eigenvalues of the Dirichlet Laplacian increasingly ordered, but without counting the multiplicities. Then Theorem 3.3 can be reformulated in the following way:

Theorem 5.3. *Suppose that Ω_n converges in the semi-compact sense to Ω, such that $m(\Omega_n) \to m(\Omega)$ and Ω is supposed to be connected. If $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ then for all $k \in \mathbb{N}$ for all neighborhoods $V_1, ..., V_k$ of $\lambda_1(\Omega), ..., \lambda_k(\Omega)$ there exists n_k large enough, such that for $n \geq n_k$ and $j = 1, .., k$ we have $\lambda_j(\Omega_n) \in V_j$.*

Another possible extension of some results of the paper is to consider finely open sets instead of open sets and to replace the Laplace operator by the p-Laplacian (for $1 < p < \infty$). In this case, the stability of the first eigenvalue of the p-Laplacian gives the γ_p-convergence.

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