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On Schur-Ostrowski Type Theorems for Group Majorizations

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In this paper the problem of the isotonicity of a function with respect to a group majorization is discussed. For a group induced cone ordering the isotonicity is characterized via some conditions of Schur-Ostrowski type. As an application, a characterization of the isotonicity of a quadratic form and a linear form is presented. In addition, it is shown that the conditions are necessary and sufficient for a group majorization to be a group induced cone ordering. In consequence, a finite reflection group is determined by the conditions. Finally, some results on the isotonicity of a vector-valued function are derived. In particular, a necessary and sufficient condition on the matrix majorization – Loewner ordering isotonicity is established for a matrix-valued function. The last extends directly the classical S-O condition.

Keywords: preordering, cone preordering, majorization, group majorization, group induced cone ordering, convex cone, dual cone, isotone function, Schur convex function, fundamental region, reflection group, Schur-Ostrowski's condition.

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1. Introduction

Because of many applications, the problem of the characterization of a isotone mapping defined on a preordered linear space is the object of interest for many mathematicians. Special attention in the literature is devoted to the isotonicity with respect to the *majorization preordering* and its generalizations (cf. [5, 10, 23, 24, 26, 28]).

A natural extension of the classical majorization concept is a *group majorization*, which is a vector preordering induced by a compact group of linear operators. The idea comes from Rado [25] and Mudholkar [19], while the development of the theory of a group majorization is due, among others, to Eaton and Perlman [10], Eaton [6, 7, 8, 9], Giovagnoli and Wynn [12], Giovagnoli and Romanazzi [11], Steerneman [29], and Miranda and Thompson [18].

An important class of the group majorization preorderings are *group induced cone orderings* (for short, GIC orderings) introduced by Eaton [7, 8], because many preorderings of practical interest are in this class. A basic feature of a GIC ordering is that on some set it is representable by a cone preordering. Thus some methods suitable for cone preorderings can be used to the study of GIC orderings.

Our purpose in this paper is to characterize isotone functions. We shall mainly focus on the isotonicity with respect to a GIC ordering. So, it is a good idea to employ a result by Marshall, Walkup, and Wets [16] characterizing a isotone real function w.r.t. a cone preordering (see Sec. 4) and an analogous result by Niezgoda and Otachel [22] for a vector-valued function (see Sec. 6).

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Thus for a real function we obtain a differential condition for the isotonicity in the form of some linear inequalities. In order to simplify the inequalities and to get so-called *Schur-Ostrowski type conditions*, in Section 3 we establish a generator of the cone related to a GIC ordering. This allows us to obtain a condition which implies results by Giovagnoli and Wynn [12, p. 119], and Eaton and Perlman [10, Th. 5.2].

As an application, we give a characterization of the isotonicity of a quadratic form and a linear form in the general case of a GIC ordering and we also present interpretations of the S-O type conditions in concreate cases of GIC orderings.

In Section 5, considering our problem in a wider context of a group majorization, we show that our S-O type condition is necessary and sufficient for a group majorization to be a GIC ordering. In consequence, a finite reflection group is determined by the condition.

Lastly, in Section 6 we derive some results on the isotonicity of a vector-valued function. In particular, we establish an equivalent condition on the matrix majorization – Loewner ordering isotonicity, which extends directly the classical S-O condition. Moreover, we generalize a result on the antiisotonicity of the Gibbsian states function.

2. Notation and preliminaries

Throughout the paper V is a finite-dimensional real linear space with inner product $\langle\cdot,\cdot\rangle$. As usual, by $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$ we denote the norm of $x \in V$. If $x, y \in V$ then $x \perp y$ means that x, y are orthogonal. The symbol I stand for the identity operator from V onto V.

For a set $A \subset V$ by $\operatorname{lin} A$ we mean the smallest linear subspace of V containing the set A. The symbols $\operatorname{int} A$ and $\operatorname{cl} A$ stand, respectively, for the interior and the closure of A. The *relative interior* of a set A is the set

$$\operatorname{ri} A = \{x \in \operatorname{aff} A : B(x,\varepsilon) \cap \operatorname{aff} A \subset A \ \text{ for some } \ \varepsilon > 0\},$$

where $\operatorname{aff} A$ denotes the smallest affine subset of V containing the set A, and $B(x,\varepsilon)$ is the open ball in V with the centre x and the radius ε. A set A is said to be *relatively open* if $\operatorname{ri} A = A$. It is known that if A is a nonempty convex subset of V then $\operatorname{ri} A$ and $\operatorname{cl} A$ are nonempty convex sets, and, in addition,

$$\operatorname{cl}\operatorname{ri} A = \operatorname{cl} A, \ \ \operatorname{ri}\operatorname{cl} A = \operatorname{ri} A, \ \text{ and } \ \operatorname{ri}\operatorname{ri} A = \operatorname{ri} A$$

(see [27, Th.6.2-6.3]).

For a function Ψ defined on V and a set $A \subset V$ the symbol $\Psi_{|A}$ denotes the restriction of Ψ to A.

Recall that a nonempty set $C \subset V$ is a *convex cone* if $\alpha C + \beta C \subset C$ for all scalars $\alpha, \beta \geq 0$. It is easy to check that $\operatorname{cl} C$ is a closed convex cone whenever C is a convex cone. For a set $T \subset V$ the symbol $\operatorname{cone} T$ denotes the convex cone of all nonnegative finite linear combinations of vectors in T. We shall call a subset T of a closed convex cone C a *generator* of C if $C = \operatorname{cl}\operatorname{cone} T$. The symbol $\operatorname{dual} C$ denotes the *dual cone* of a convex cone $C \subset V$ defined by

$$\operatorname{dual} C = \{y \in V : \langle x,y\rangle \geq 0, x \in C\}.$$

Note that $\operatorname{dual} C$ is closed and

$$\operatorname{dual}(\operatorname{dual} C) = \operatorname{cl} C.$$

A relation $\preceq$ on a nonempty set $A \subset V$ is called a *preordering* if

(i) $x \preceq x$ for all $x \in A$,
(ii) $x \preceq y$, $y \preceq z$ implies $x \preceq z$ for all $x, y, z \in A$.

The condition

$$y \preceq x, \ x \preceq y \ \text{ implies } \ y = x \ \text{ for all } \ x, y \in A$$

is not required.

A *cone preordering* on a set $A \subset V$ is a preordering $\preceq$ on A defined by

$$y \preceq x \ \text{ iff } \ x - y \in C \quad \text{for all } \ x, y \in A, \tag{2.1}$$

where $C \subset V$ is a convex cone. Then we also say that $\preceq$ is induced by C. It is clear that the cone preordering on A induced by C is induced by the convex cone $C \cap \operatorname{lin} A$, too.

Let $O(V)$ denote the orthogonal group acting on V and let G be a compact subgroup of $O(V)$. *Group majorization w.r.t. the group G*, abbreviated as *G-majorization*, is the preordering $\preceq$ on V defined by

$$y \preceq x \ \text{ iff } \ y \in C(x) \quad \text{for all } \ x, y \in V, \tag{2.2}$$

where $C(x)$ is the convex hull of the set $\operatorname{orb}(x) = \{gx : g \in G\}$. It is easily seen that G-majorization $\preceq$ is G-invariant in the sense that

$$y \preceq x \ \text{ iff } \ g_1 y \preceq g_2 x \ \text{ for all } \ x, y \in V \ \text{ and } \ g_1, g_2 \in G.$$

A function Ψ defined on V is said to be *group-invariant* w.r.t. the group G (for short, *G-invariant*) if

$$\Psi(gx) = \Psi(x) \ \text{ for all } \ x \in V \ \text{ and } \ g \in G.$$

Basic properties of a group majorization are collected in the following theorem, which employs the *support functions*

$$m(v, z) = \sup_{g \in G} \langle v, gz \rangle, \quad v \in V, \tag{2.3}$$

of the sets $C(z)$, $z \in V$ (cf. [27, Sec. 13]).

Theorem 2.1 ([7, 12]). *Let $\preceq$ be the group majorization induced on V by a compact group $G \subset O(V)$. Then for any $x, y \in V$ the following statements are equivalent:*

(i) $y \preceq x$,
(ii) $\Psi(y) \le \Psi(x)$ *for all G-invariant convex real functions Ψ defined on V,*
(iii) $m(v, y) \le m(v, x)$ *for all $v \in V$.*

It was noted in [10, Remarks 2.3 and 2.5] that minimal points w.r.t. G-majorization form the linear subspace

$$M_G = \{x \in V : gx = x \ \text{ for all } \ g \in G\} \tag{2.4}$$

and

$$C(x) \cap M_G = \{\bar{x}\}, \tag{2.5}$$

where $\bar{x}$ is the unique minimal point in $C(x)$. On the other hand $\bar{x} = Px$, where P denotes the orthoprojector from V onto M_G. In addition,

$$C(x) = C(\hat{x}) + \bar{x}, \tag{2.6}$$

where $\hat{x} = x - \bar{x} = Qx$ and $Q = I - P$ is the orthoprojector from V onto $M_G^{\perp}$, the last symbol means the orthocomplement of M_G to V. Thus, by (2.5) and (2.6), for any $x, y \in V$

$$y \preceq x \ \text{ iff } \ \bar{y} = \bar{x} \ \text{ and } \ \hat{y} \preceq \hat{x}. \tag{2.7}$$

Moreover, if μ denotes the normed Haar measure on G then

$$\bar{x} = \int_G gx\, d\mu(g) \tag{2.8}$$

(see [12, p. 113]).

3. Group induced cone orderings

The structure of a group majorization is particularly simple when this preordering is a *group induced cone ordering.*

Definition 3.1. A group majorization w.r.t. a compact group $G \subset O(V)$ is said to be a group induced cone ordering (for short, a GIC ordering) if there exists a nonempty closed convex cone $D \subset V$ such that

(A1) $\mathrm{orb}(x) \cap D$ is not empty for each $x \in V$,
(A2) $m(x, y) = \langle x, y\rangle$ for all $x, y \in D$.

Note that in the above definition the same role as D plays gD for any $g \in G$. This follows from G-invariance of the support function.

Condition (A1) says

$$V = \bigcup_{g \in G} gD,$$

that is, for each $x \in V$ there exists a $g \in G$ satisfying $gx \in D$, while condition (A2) is equivalent to the rearrangement type inequalities

$$\langle x, gy\rangle \le \langle x, y\rangle \ \text{ for all } \ x, y \in D \text{ and } g \in G. \tag{3.1}$$

The last means

$$\|x - y\| \le \|x - gy\| \ \text{ for all } \ x, y \in D \text{ and } g \in G. \tag{3.2}$$

Both conditions (A1) and (A2) guarantee the existence and the uniqueness of the operator $(\cdot)^* : V \to D$ with the property that for each $x \in V$ the vector x^* is the unique element of the set $\mathrm{orb}(x) \cap D$ (see Lemma 3.3 below). This operator is a *maximal invariant* for G

(see [14, Chap. 6. Sec. 2.]). It is easy to see that $x^* = x$ for all $x \in D$, which gives that $(\cdot)^*$ is idempotent, and

$$\|x^* - y^*\| \leq \|x - y\|, \quad x, y \in V,$$

which implies the continuity of the operator.

One can deduce that a GIC ordering $\preceq$ restricted to its cone D is the cone preordering on D induced by dual D:

$$y \preceq x \text{ iff } \langle t, x - y \rangle \geq 0 \text{ for all } t \in T, \tag{3.3}$$

where $x, y \in D$ and T is a generator of D. To see this it is sufficient to employ Theorem 2.1 and (A2). Moreover, for any $x, y \in V$ connection (3.3) extends to

$$y \preceq x \text{ iff } y^* \preceq x^* \text{ iff } \langle t, x^* - y^* \rangle \geq 0 \text{ for all } t \in T. \tag{3.4}$$

In our considerations we shall use the following lemma.

Lemma 3.2. *Assume that the conditions* (A1) *and* (A2) *are met for a compact group* $G \subset O(V)$ *and a closed convex cone* $D \subset V$*. If* z *is an arbitrary point in* $\operatorname{ri} D$ *then the set* $\{(I - g)z : g \in G\}$ *is a generator of the convex cone* $\operatorname{dual} D$*.*

Proof. Fix $z \in \operatorname{ri} D$. By (3.1) it is clear that

$$\operatorname{cl}\operatorname{cone}\{(I - g)z : g \in G\} \subset \operatorname{dual} D.$$

Now we shall prove

$$\operatorname{dual} D \subset \operatorname{cl}\operatorname{cone}\{(I - g)z : g \in G\}.$$

To this end it is sufficient to show that

$$\operatorname{dual}\operatorname{cl}\operatorname{cone}\{(I - g)z : g \in G\} \subset D.$$

Let $x \in \operatorname{dual}\operatorname{cl}\operatorname{cone}\{(I - g)z : g \in G\}$. Then $\langle x, (I - g)z \rangle \geq 0$ for all $g \in G$. By (A1) there exist $x_0 \in D$ and $g_0 \in G$ such that $x = g_0 x_0$. Thus we have $\langle g_0 x_0, (I - g_0)z \rangle \geq 0$. This and (3.1) imply

$$\langle x_0, z \rangle \geq \langle g_0 x_0, z \rangle \geq \langle g_0 x_0, g_0 z \rangle = \langle x_0, z \rangle.$$

Hence $\langle (I - g_0)x_0, z \rangle = 0$.

The last leads to $(I - g_0)x_0 \perp x_0$. In fact, since $z \in \operatorname{ri} D$, so there exists $\varepsilon > 0$ such that $z + \lambda(x_0 - z) \in D$ for all real numbers λ satisfying $|\lambda| < \varepsilon$. In particular, $z \pm \frac{1}{2}\varepsilon(x_0 - z) \in D$. But by (3.1) we get $(I - g_0)x_0 \in \operatorname{dual} D$, so we have

$$\begin{aligned} 0 \leq \langle z \pm \frac{1}{2}\varepsilon(x_0 - z), (I - g_0)x_0 \rangle &= \pm\frac{1}{2}\varepsilon\langle x_0, (I - g_0)x_0 \rangle + (1 \mp \frac{1}{2}\varepsilon)\langle z, (I - g_0)x_0 \rangle \\ &= \pm\frac{1}{2}\varepsilon\langle x_0, (I - g_0)x_0 \rangle. \end{aligned}$$

Hence $\langle (I - g_0)x_0, x_0 \rangle = 0$, as claimed.

Therefore we obtain

$$\langle g_0 x_0, x_0 \rangle = \|x_0\|^2 = \|g_0 x_0\| \cdot \|x_0\|.$$

Thus, by Cauchy-Schwarz inequality, we get $g_0 x_0 = \rho x_0$ for some real number $\rho > 0$. Since g_0 is an isometry, so $g_0 x_0 = x_0$ and hence $x = x_0 \in D$. $\square$

The next lemma extends Lemma 4.3 of [10] and Lemma 4.1 of [29]. We cite it from [21].

Lemma 3.3. *Let G be a compact subgroup of $O(V)$ and let $D \subset V$ be a nonempty set such that condition* (A2) *holds. Then $D \cap gD = \{x \in D : gx = x\}$ and $\operatorname{ri} D \cap gD$ is the empty set for all $g \in G$ such that $g_{|D} \neq I_{|D}$. In particular, if $gx \in D$ for some $x \in \operatorname{ri} D$ and $g \in G$ then $g_{|D} = I_{|D}$. In addition, if $g_1 x = g_2 x$ for some $x \in \operatorname{ri} D$ and $g_1, g_2 \in G$ then $g_{1|D} = g_{2|D}$.*

An important notion in the context of a GIC ordering is *fundamental region.* A set $F \subset V$ is said to be a fundamental region for a group $G \subset O(V)$ if

(i) F is open,
(ii) $F \cap gF = \emptyset$ for all $g \in G$ such that $g \neq I$,
(iii) $V = \bigcup_{g \in G} g(\operatorname{cl} F)$.

For a finite group there always exists a convex fundamental region F (see [13, Chap. 3]). Giovagnoli and Wynn [12] proved that $\operatorname{cl} F$ is a closed convex cone under a suitable choice of a fundamental region F. Furthermore, for a finite group G, by Theorem 4.1 in [29], F is unique up to transformations by G iff the group majorization induces by G is the GIC ordering with the convex cone $D = \operatorname{cl} F$.

The following theorem completes the above facts.

Theorem 3.4. *Let $G \subset O(V)$ be a compact group. If there exists a fundamental region F for G then G is finite.*

Proof. Suppose G is an infinite group. Because G is a compact group so there exists an infinite sequence (g_m) of elements of G such that $g_i \neq g_j$ for $i \neq j$, and (g_m) is convergent to some element in G.

Because F is open so that there exists an open ball $B(z, \varepsilon) \subset F$ with the centre $z \neq 0$ and the radius $\varepsilon > 0$. Now, it can be shown that

$$\|g - \tilde{g}\| \geq \varepsilon \|z\|^{-1} \text{ for all } g, \tilde{g} \in G, g \neq \tilde{g}, \tag{3.5}$$

where

$$\|g - \tilde{g}\| = \sup_{\|x\|=1} \|(g - \tilde{g})x\|$$

is the norm of the operator $g - \tilde{g}$.

Suppose not. Then $\|g - \tilde{g}\| < \varepsilon \|z\|^{-1}$ for some $g, \tilde{g} \in G, g \neq \tilde{g}$. Therefore, by $G \subset O(V)$, one can write

$$\|z - g^{-1}\tilde{g}z\| = \|gz - \tilde{g}z\| \leq \|g - \tilde{g}\| \cdot \|z\| < \varepsilon.$$

Hence $g^{-1}\tilde{g}z \in B(z, \varepsilon) \subset F$, and consequently $z \in \tilde{g}^{-1}gF$. However, $z \in B(z, \varepsilon) \subset F$, so that $z \in F \cap \tilde{g}^{-1}gF$, where $\tilde{g}^{-1}g \neq I$. The last fact contradicts (ii) of the definition of a fundamental region completing the proof of (3.5).

Therefore in particular by (3.5)

$$\|g_i - g_j\| \geq \varepsilon \|z\|^{-1} > 0 \text{ for all } i \neq j.$$

But this is impossible by the convergence of the sequence (g_m). Thus G must be a finite group, as claimed. $\square$

In the sequel we shall call a set $F \subset V$ a *relatively open fundamental region* for a group $G \subset O(V)$ if

(i) F is relatively open,
(ii) $F \cap gF = \emptyset$ for all $g \in G$ such that $g_{|F} \neq I_{|F}$,
(iii) $V = \bigcup_{g \in G} g(\operatorname{cl} F)$.

The motivation is the fact that when G induces a GIC ordering with suitable convex cone D then the set $F = \operatorname{ri} D$ is a relatively open fundamental region for G in the sense of the above definition (see Lemma 3.3).

We conclude this section with three classical examples to illustrate the notions considered above. Another examples may be found in [1, 7, 8, 9, 10, 12, 18].

Example 3.5. Take V to be R^n, the n-dimensional Euclidean space of column vectors with the usual inner product $\langle x, y \rangle = x^T y$, where $(\cdot)^T$ denotes the transpose. The *classical majorization* is the preordering on R^n defined as follows (see [15, p. 7]). For any vectors $x, y \in R^n$ one writes

$$y \preceq x \ \text{ iff } \ \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i \ \text{ and } \ \sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]}\ , \ k = 1, 2, \ldots, n-1, \tag{3.6}$$

where $z_{[1]} \geq z_{[2]} \geq \ldots \geq z_{[n]}$ are the entries of a vector $z \in R^n$ in nonincreasing order.

It is well-known that this preordering is the group majorization induced via (2.2) by $G = \mathcal{P}_n$ being the permutation group acting on R^n. Elements of $\mathcal{P}_n$ can be represented as n-by-n orthogonal matrices whose each entry is 0 or 1.

Moreover, the preordering is a GIC ordering related to the closed convex cone

$$D = \{x \in R^n \ : \ x_1 \geq x_2 \geq \ldots \geq x_n\}.$$

A fundamental region for $\mathcal{P}_n$ is

$$F = \{x \in R^n \ : \ x_1 > x_2 > \ldots > x_n\}.$$

In this example

$$x^* = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})^T.$$

A generator of D is the set

$$T = \{t^{(1)}, t^{(2)}, \ldots, t^{(n)}, t^{(n+1)}\},$$

where for $k = 1, \ldots, n$

$$t^{(k)} = (1, \ldots, 1, 0, \ldots, 0)^T$$

is the vector whose the first k entries are 1 and the remaining ones are 0, and, in addition,

$$t^{(n+1)} = -t^{(n)}.$$

Thus (3.6) is implied by (3.4).

Example 3.6. Put V to be the linear space $\mathcal{M}_n$ of all n-by-n real matrices. The inner product of $x, y \in \mathcal{M}_n$ is the trace tr xy^T. Let G be the group of linear operators of the form

$$x \to g_1 x g_2^T, \quad x \in \mathcal{M}_n,$$

with g_1, g_2 running over all n-by-n orthogonal matrices. It may be found in [7, pp. 17-18] that the group majorization $\preceq$ induced by this group is the GIC ordering under the cone

$$D = \{x \in \mathcal{M}_n : x_{11} \geq x_{22} \geq \ldots \geq x_{nn} \geq 0, \text{ and } x_{ij} = 0, i \neq j\}.$$

A generator of D is the set T consisting of the diagonal matrices

$$t^{(k)} = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)^T, \quad k = 1, 2, \ldots, n,$$

whose the first k diagonal entries are 1 and the remaining ones are 0. Moreover, for any $x \in V$

$$x^* = \operatorname{diag} s(x)$$

is the diagonal matrix with the n-vector $s(x)$ of the singular values of x on the principal diagonal. The dual cone of D is the set

$$\operatorname{dual} D = \{x \in \mathcal{M}_n \ : \ \sum_{i=1}^{k} x_{ii} \geq 0, \ k = 1, 2, \ldots, n\}.$$

The set

$$F = \{x \in \mathcal{M}_n : x_{11} > x_{22} > \ldots > x_{nn} > 0, \text{ and } x_{ij} = 0, i \neq j\}$$

is a relatively open fundamental region for infinite group G.

Thus (3.4) implies for any matrices x, y

$$y \preceq x \text{ iff } \sum_{i=1}^{k} s_i(y) \leq \sum_{i=1}^{k} s_i(x), \ k = 1, 2, \ldots, n. \tag{3.7}$$

The above means that $y \preceq x$ iff so-called *weak majorization* relation holds between the vectors $s(y)$ and $s(x)$ of the singular values of y and x, respectively.

Example 3.7. Following Eaton [9, p. 168] we present an example of a group which does not induce a GIC ordering. Let V be R^n with the usual inner product and let $G = \mathcal{P}_n \cup -\mathcal{P}_n$, where as in Example 3.5 $\mathcal{P}_n$ is the group of all n-by-n permutation matrices. A fundamental region for G is

$$F = \{x \in R^n \ : \ x_1 > x_2 > \ldots > x_n, \ \sum_{i=1}^{n} x_i > 0\}.$$

Then (A1) is met for $D = \operatorname{cl} F$, but (A2) fails.

4. Differential characterization of G-isotonicity for real functions

Let Ψ be a real function defined on a subset A of V. Suppose $\preceq$ is a preordering on V. We call Ψ *isotone* on A with respect to $\preceq$ iff

$$y \preceq x \text{ implies } \Psi(y) \leq \Psi(x) \text{ for all } x, y \in A.$$

Ψ is *antiisotone* on A whenever $-\Psi$ is isotone on this set.

If V is provided with a group majorization structure induced by a compact group $G \subset O(V)$ then *G-isotonicity* means isotonicity w.r.t. G-majorization.

It is understood that G-isotonicity implies G-invariance. Moreover, it is readily seen from Theorem 2.1 that all G-invariant convex real functions defined on V are G-isotone on V. An another class of G-isotone functions is formed by all G-invariant functions Ψ such that for any real number λ the set

$$\{x \in V : \Psi(x) \leq \lambda\}$$

is convex.

In addition, if G-majorization is a GIC ordering with a convex cone D as in Definition 3.1 then G-isotonicity on V of a function Ψ is equivalent to its G-invariance and G-isotonicity on D of the function $\Psi_{|D}$.

For the case of the classical majorization, i.e. when G is the permutation group $\mathcal{P}_n$ acting on $V = R^n$, we have the following differential characterization of G-isotone real functions (also called in this case *Schur convex functions*).

Theorem 4.1 ([28, 23]). *Assume that Ψ is symmetric real function having a differential on R^n. Then a necessary and sufficient condition that Ψ be a Schur convex function on R^n is*

$$(x_i - x_j)\left(\frac{\partial \Psi}{\partial x_i}(x) - \frac{\partial \Psi}{\partial x_j}(x)\right) \geq 0 \ \textit{for all} \ x \in R^n \ \textit{and} \ 1 \leq i, j \leq n. \tag{4.1}$$

Many authors cite the above *Schur-Ostrowski's condition* in the form

$$\frac{\partial \Psi}{\partial x_1}(x) \geq \frac{\partial \Psi}{\partial x_2}(x) \geq \ldots \geq \frac{\partial \Psi}{\partial x_n}(x) \ \text{ for all } x \in R^n \text{ such that } x_1 \geq x_2 \geq \ldots \geq x_n. \tag{4.2}$$

Much general result is due to Eaton and Perlman [10]. It concerns reflection groups. Recall that a closed group $G \subset O(V)$ is a *reflection group* if G is the closure of a subgroup of $O(V)$ generated by some set of reflections S_r in the form

$$S_r x = x - 2\langle r, x\rangle r, \quad x \in V,$$

where $r \in V$, $\|r\| = 1$.

Theorem 4.2 ([10]). *Let G be a reflection group acting on R^n. Assume that Ψ is a G-invariant real function possesing a differential on R^n. Then a necessary and sufficient condition that Ψ be G-isotone on R^n is*

$$\langle r, x\rangle \cdot \langle r, \nabla\Psi(x)\rangle \geq 0 \ \textit{for all} \ x \in R^n \ \textit{and} \ r \ \textit{such that} \ S_r \in G. \tag{4.3}$$

An important example of a reflection group is the permutation group $\mathcal{P}_n$. So Theorem 4.1 is a corollary from Theorem 4.2, because in the case of this group each vector r has the form $r = \frac{1}{\sqrt{2}}(e^{(i)} - e^{(j)})$ with some $1 \leq i, j \leq n$, where $e^{(k)}$ denotes the kth coordinate vector in R^n (see [10, p. 841]). For another examples of reflection groups and interpretation of (4.3) for these groups see also [8].

In [8] Eaton posed the question if the condition

$$\langle (I-g)x, \nabla\Psi(x)\rangle \geq 0 \quad \text{for all } g \in G \text{ and } x \in V, \tag{4.4}$$

being in the same line as (4.1) and (4.3), implies G-isotonicity on V of a G-invariant differentiable function $\Psi : V \to R$ in the general case of a G-majorization. It is known that (4.4) is necessary for the G-isotonicity (see [10, Prop. 2.2]).

The sufficiency of this condition for the case of a GIC ordering was noted in [12, p. 120] in the context of their Theorem 2. However, it seems that their remark regards finite groups only, because for infinite ones the assumptions of that theorem are not valid (see our Th. 3.4 in Sec. 3).

For the completeness, in this section we present a theorem like the mentioned remark in [12] for GIC orderings with arbitrary (finite or infinite) groups as well as some examples and applications. Furthermore, we discuss the problem of reduction of the assumptions. On account of forthcoming applications in Section 5, we prefer a two-variable version of (4.4).

As we already saw in Section 3, a GIC ordering on appropriate convex cone D is generated by the cone dual D. So, we need a tool to explore isotonicity w.r.t. a cone preordering. The following theorem is very useful to our aim.

Theorem 4.3 ([16]). *Let Φ be a real function defined on an open convex subset A of a finite-dimensional inner product space W. Suppose $\preceq$ is a preordering on A induced by a closed convex cone $C \subset W$ and T is a generator of the cone. If the gradient $\nabla\Phi(x)$ exists at each point x in A then Φ is isotone on A with respect to $\preceq$ iff $0 \leq \langle t, \nabla\Phi(x)\rangle$ for all $x \in A$ and $t \in T$.*

The essence of the assertion of the above theorem is the fact that for each $x \in A$ the gradient $\nabla\Phi(x)$ belongs to dual C.

Now we are ready to give a characterization of the isotonicity for GIC orderings. Remind that a G-isotone function must be G-invariant. Therefore we restrict ourselves to G-invariant functions. For such a function Ψ the existence of the gradient on V is equivalent to the existence of the gradient on a subset $D \subset V$ satisfying $V = \bigcup_{g\in G} gD$. Moreover, the formula

$$\nabla\Psi(gx) = g\nabla\Psi(x)$$

holds for all $x \in D$ and $g \in G$.

Theorem 4.4. *Let G be a compact subgroup of the orthogonal group $O(V)$. Assume $\preceq$ is the group majorization induced by G via* (2.2) *which satisfies conditions* (A1) *and* (A2) *with some closed convex cone $D \subset V$. Suppose Ψ is a G-invariant real function defined on V with gradient $\nabla\Psi(x)$ at each point x in D. Then Ψ is G-isotone on V iff*

$$\langle (I-g)z, \nabla\Psi(x)\rangle \geq 0 \text{ for all } g \in G \text{ and } x, z \in \operatorname{ri} D. \tag{4.5}$$

Proof. If G is finite, i.e. if $\operatorname{int} D \neq \emptyset$ (see Th. 4.1 in [29] and our Th. 3.4), then the theorem is a consequence of Theorem 4.3 and Lemma 3.2. In the general case, i.e. when G is finite or infinite, some improvements are needed as follows.

Necessity. Suppose Ψ is G-isotone. First we shall show the inequality

$$x \preceq x + \lambda(I-g)z \quad \text{for all} \quad g \in G, \ x, z \in D \quad \text{and} \quad \lambda \geq 0. \tag{4.6}$$

Namely, for any $v \in D$ by (A2) and (3.1) we get

$$m(v,x) = \langle v, x\rangle \leq \langle v, x\rangle + \langle v, \lambda(I-g)z\rangle \leq m(v, x+\lambda(I-g)z).$$

Now, by (A1) and the G-invariance of the function $m(\cdot, u)$ (with a fixed u) we obtain

$$m(v,x) \leq m(v, x+\lambda(I-g)z)$$

for all $v \in V$, which together with Theorem 2.1 yields (4.6).

Now, by (4.6) we have

$$\Psi(x) \leq \Psi(x+\lambda(I-g)z)$$

for all $g \in G$, $x, z \in D$ and $\lambda \geq 0$. Thus the right-hand directional derivative of Ψ at the point x and in the direction $(I-g)z$, in symbol

$$\nabla_{(I-g)z}\Psi(x) = \lim_{\lambda\to 0^+} \frac{1}{\lambda}\left(\Psi(x+\lambda(I-g)z) - \Psi(x)\right),$$

is nonnegative. Therefore

$$\langle (I-g)z, \nabla\Psi(x)\rangle = \nabla_{(I-g)z}\Psi(x) \geq 0$$

for all $g \in G$ and $x, z \in D$, which implies (4.5).

Sufficiency. Suppose $\langle (I-g)z, \nabla\Psi(x)\rangle \geq 0$ for all $g \in G$ and $x, z \in \operatorname{ri} D$. Hence, by Lemma 3.2 and the property

$$\operatorname{dual}(\operatorname{dual} D) = D,$$

we have $\nabla\Psi(x) \in D$ for all $x \in \operatorname{ri} D$. Denote $W = \operatorname{lin} D$ and

$$C = \operatorname{dual}_W D = \{w \in W : \langle w, v\rangle \geq 0 \ \text{ for all } \ v \in D\}.$$

Consider the function $\Phi = \Psi_{|\operatorname{ri} D}$. It is clear that there exists gradient of Φ on $\operatorname{ri} D$. In general, $\nabla\Phi(x) = K\nabla\Psi(x)$, where K is orthoprojector from V onto W. But $\nabla\Psi(x) \in D$, $x \in \operatorname{ri} D$, so $\nabla\Phi(x) = \nabla\Psi(x)$. In consequence, $\nabla\Phi(x) \in D$, and therefore $\langle c, \nabla\Phi(x)\rangle \geq 0$ for all $c \in C$ and $x \in \operatorname{ri} D$. Observe that $x \in \operatorname{ri} D$ being an element of W belongs to the interior of D treated as a subset of W. So, an application of Theorem 4.3 with $T = C$ and $A = \operatorname{ri} D$ gives the isotonicity of Φ on $\operatorname{ri} D$ with respect to the cone preordering induced by $\operatorname{dual}_W D = \operatorname{dual} D \cap W$. This means that Ψ is isotone on $\operatorname{ri} D$ with respect to the cone preordering induced by $\operatorname{dual} D$. Therefore, by (3.3), Ψ is G-isotone on $\operatorname{ri} D$.

However, the latter implies the G-isotonicity of Ψ on D. In fact, for any $x, y \in D$ such that $y \preceq x$ there exist sequences $x_n, y_n \in \operatorname{ri} D$ satisfying $x_n \to x$, $y_n \to y$ under $n \to \infty$, and additionally $y_n \preceq x_n$. Namely, it is sufficient to put

$$x_n = x + \frac{1}{n}(u_0 - x) \quad \text{and} \quad y_n = y + \frac{1}{n}(u_0 - y),$$

where u_0 is a fixed point in $\mathrm{ri}\, D$ (see [27, Th. 6.1]). So

$$\Psi(y_n) \leq \Psi(x_n).$$

Now, the gradient assumption for Ψ leads to

$$\Psi(a+\lambda b) \to \Psi(a) \text{ under } \lambda \to 0$$

for any $a, b \in V$. Thus

$$\Psi(x_n) \to \Psi(x) \text{ and } \Psi(y_n) \to \Psi(y) \text{ under } n \to \infty.$$

Finally, we have $\Psi(y) \leq \Psi(x)$, as claimed. Thus Ψ is G-isotone on D.

In addition, because the function Ψ is G-invariant, so it must be G-isotone on V. □

Observe, by the above proof, that the last theorem remains valid if we put D instead of $\mathrm{ri}\, D$ in (4.5).

Corollary 4.5 ([12]). *Let G and Ψ be as in* Theorem 4.4. *Then Ψ is G-isotone on V iff*

$$\langle (I-g)x, \nabla\Psi(x)\rangle \geq 0 \ \text{ for all } g \in G \text{ and } x \in V. \tag{4.7}$$

Proof. The sufficiency of (4.7) for the G-isotonicity follows easily from Theorem 4.4 and Lemma 3.2.

Necessity. In a similar way as in the necessity part of the proof of Theorem 4.4 one can show that

$$\langle (I-g)x, \nabla\Psi(x)\rangle \geq 0 \text{ for any } g \in G, x \in D.$$

This and the fact $G \subset O(V)$ lead to

$$\langle \tilde{g}\,(I-g)x, \tilde{g}\nabla\Psi(x)\rangle \geq 0$$

for all $g, \tilde{g} \in G$, $x \in D$. Hence

$$\langle (I-\tilde{g}g\tilde{g}^{-1})\tilde{g}x, \tilde{g}\nabla\Psi(x)\rangle \geq 0$$

with any $g, \tilde{g} \in G$, $x \in D$. Next, the formula $\tilde{g}\nabla\Psi(x) = \nabla\Psi(\tilde{g}x)$ and the arbitrariness of $g \in G$ give

$$\langle (I-g)\tilde{g}x, \nabla\Psi(\tilde{g}x)\rangle \geq 0$$

for all $g, \tilde{g} \in G$ and $x \in D$. This and condition (A1) guarantee (4.7). □

Example 4.6. In this example we apply Theorem 4.4 to obtain a sufficient and necessary condition on the isotonicity w.r.t. a GIC ordering of a quadratic form.

Let $G \subset O(V)$ be any compact group. Consider a quadratic form $\Psi(x) = \langle x, Lx\rangle$, $x \in V$, where $L = L^T$ is a symmetric linear operator from V into V.

Note that Ψ is G-invariant iff L is commutative with each member of G.

If L is non-negative definite on V, i.e. $\langle x, Lx\rangle \geq 0$ for all $x \in V$, then the function $x \to \langle x, Lx\rangle^{\frac{1}{2}}$, $x \in V$, is a seminorm on V, so it is convex on V.

In consequence, by Theorem 2.1, when the operator commutates with the group and it is non-negative definite on V then the form is G-isotone on V.

The converse does not hold, i.e. there are G-isotone on V quadratic forms which are not non-negative definite on V. For instance, suppose for a moment that $V = R^2$ and $G = \{I, g\}$, where

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $y \preceq x$ means $|y_1| \leq |x_1|$ and $y_2 = x_2$. Put

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

That L commutates with G is evident. Also it is easy to see that

$$\Psi(x) = \langle x, Lx \rangle = x_1^2 - x_2^2$$

and Ψ is G-isotone on V. However, as we see, Ψ is not non-negative definite on the space V.

Return to the general situation considered in this example. Assume additionally that G induces a group majorization which is a GIC ordering with some closed convex cone $D \subset V$. Now we characterize G-isotonicity of quadratic forms as follows.

The quadratic form

$$\Psi(x) = \langle x, Lx \rangle, \quad x \in V,$$

is G-isotone on V iff L is commutative with G and L is non-negative definite on the space $M_G^\perp$ i.e. $\langle x, Lx \rangle \geq 0$ for all $x \in M_G^\perp$, where $M_G^\perp$ is the orthocomplement of the space $M_G = \{v \in V : gv = v, g \in G\}$ to V.

For the proof, first suppose G-isotonicity of Ψ on V. The G-invariance of Ψ implies the commutativity of L with G, which gives $Lx \in M_G$ for all $x \in M_G$. Indeed,

$$gLx = Lgx = Lx$$

for all $x \in M_G$ and $g \in G$. Moreover, $Lx \in M_G^\perp$ for all $x \in M_G^\perp$, because for any $y \in M_G$ we obtain

$$\langle Lx, y \rangle = \langle x, Ly \rangle = 0,$$

the last follows from the fact that $Ly \in M_G$. In other words, both spaces M_G and $M_G^\perp$ are L-invariant subspaces of V.

By Theorem 4.4, the G-isotonicity of Ψ yields

$$\langle (I - g)z, \nabla\Psi(x) \rangle \geq 0$$

for all $g \in G$ and $x, z \in \operatorname{ri} D$. But $\nabla\Psi(x) = 2Lx$, so on account of Lemma 3.2 we get $Lx \in D$ for all $x \in \operatorname{ri} D$. Next, by the continuity of L, we obtain $Lx \in D$ for all $x \in D$. Recall that $Lx \in M_G^\perp$ for $x \in M_G^\perp$. Therefore $Lx \in D \cap M_G^\perp$ whenever $x \in D \cap M_G^\perp$. Now applying (3.1) we have

$$\langle x, Lx \rangle \geq \langle x, gLx \rangle$$

for any $x \in D \cap M_G^{\perp}$ and $g \in G$. Hence, if μ denotes the normed Haar measure on G, then

$$\int_G \langle x, Lx \rangle \, d\mu(g) \geq \int_G \langle x, gLx \rangle \, d\mu(g),$$

and further

$$\langle x, Lx \rangle \geq \langle x, \int_G gLx \, d\mu(g) \rangle.$$

For $x \in D \cap M_G^{\perp}$ the vector $\int_G gLx \, d\mu(g)$ belongs to $M_G^{\perp}$, because $Lx \in M_G^{\perp}$ and $M_G^{\perp}$ is a G-invariant space. On the other hand, properties of the Haar measure imply that $\int_G gLx \, d\mu(g)$ is a common fix point for each member of G, i.e.

$$\int_G gLx \, d\mu(g) \in M_G.$$

In consequence,

$$\int_G gLx \, d\mu(g) \in M_G \cap M_G^{\perp} = \{0\}$$

for any $x \in D \cap M_G^{\perp}$. Therefore $\langle x, Lx \rangle \geq 0$ for any $x \in D \cap M_G^{\perp}$. Now, via (A1), G-invariance of $M_G^{\perp}$, and the commutativity of L with G, we get $\langle x, Lx \rangle \geq 0$ for all $x \in M_G^{\perp}$, as claimed.

Now suppose L commutates with G and it is non-negative definite on $M_G^{\perp}$. It remains to prove that the form $\Psi(x) = \langle x, Lx \rangle$, $x \in V$, is G-isotone on V.

First, consider the function $x \to \langle x, Lx \rangle^{\frac{1}{2}}$ for $x \in M_G^{\perp}$. It is a seminorm on $M_G^{\perp}$, so it is convex. In addition, because of the commutativity of L with G, this seminorm is G-invariant. All of this gives G-isotonicity of the seminorm on $M_G^{\perp}$. Hence Ψ is G-isotone on $M_G^{\perp}$.

Now we shall show that Ψ is G-isotone on whole space V. Fix any $x, y \in V$ such that $y \preceq x$. Write

$$x = \bar{x} + \hat{x} \text{ and } y = \bar{y} + \hat{y},$$

where $\bar{x}, \bar{y} \in M_G$ and $\hat{x}, \hat{y} \in M_G^{\perp}$ are defined as in Section 2. Then, by (2.7), $\hat{y} \preceq \hat{x}$ and $\bar{y} = \bar{x}$. Since L commutates with G, the spaces M_G and $M_G^{\perp}$ are L-invariant. Then a direct calculation shows

$$\langle x, Lx \rangle = \langle \bar{x}, L\bar{x} \rangle + \langle \hat{x}, L\hat{x} \rangle$$

and similarly

$$\langle y, Ly \rangle = \langle \bar{y}, L\bar{y} \rangle + \langle \hat{y}, L\hat{y} \rangle.$$

Therefore

$$\langle y, Ly \rangle \leq \langle x, Lx \rangle \text{ iff } \langle \hat{y}, L\hat{y} \rangle \leq \langle \hat{x}, L\hat{x} \rangle.$$

So that, by the G-isotonicity of Ψ on $M_G^{\perp}$, we have $\langle y, Ly \rangle \leq \langle x, Lx \rangle$, which completes proof of the characterization of G-isotone quadratic forms presented in this example.

Note by virtue of the property dual(dual D) $= D$ and Lemma 3.2, that condition (4.5) means

$$\nabla \Psi(x) \in D \text{ for any } x \in \operatorname{ri} D.$$

Such interpretation is useful when it is apriori known an analytical description of D in a concreate case of a GIC ordering.

Example 4.7. Miranda and Thompson [18] have considered the following group majorization. Let V be the linear space $\mathcal{M}_n$ of all n-by-n real matrices with the inner product $\langle x, y\rangle = \operatorname{tr} xy^T$, the trace of the matrix xy^T, where $x, y \in \mathcal{M}_n$. Put G to be the group of all linear operators

$$\mathcal{M}_n \ni x \to gxh^T \in \mathcal{M}_n,$$

g and h running over the special orthogonal group $\mathcal{SO}_n$, that is, the group of all orthogonal n-by-n real matrices with determinant 1.

These authors have proved that the group majorization induced on V by G is a GIC ordering related to the convex cone

$$D = \{x \in \mathcal{M}_n : x_{11} \geq \ldots \geq x_{n-1n-1} \geq |x_{nn}|,\ x_{ij} = 0 \text{ for } i \neq j\}.$$

Therefore in this example condition (4.5) on G-isotonicity of a G-invariant function $\Psi : \mathcal{M}_n \to R$ has the form

$$\frac{\partial\Psi(x)}{\partial x_{11}} \geq \ldots \geq \frac{\partial\Psi(x)}{\partial x_{n-1n-1}} \geq \left|\frac{\partial\Psi(x)}{\partial x_{nn}}\right| \quad \text{and} \quad \frac{\partial\Psi(x)}{\partial x_{ij}} = 0, \quad i \neq j, \quad \text{for all} \quad x \in \operatorname{ri} D, \tag{4.8}$$

where

$$\operatorname{ri} D = \{x \in \mathcal{M}_n : x_{11} > \ldots > x_{n-1n-1} > |x_{nn}|,\ x_{ij} = 0 \text{ for } i \neq j\}$$

(cf. (4.2)).

Now let us discuss some variations of Theorem 4.4. It follows from Theorem 4.3 and Lemma 3.2 that Theorem 4.4 remains valid when the two following modifications are done. First, we assume the existence of the gradient of $\Psi_{|D}$ on $\operatorname{ri} D$ together with the continuity of $\Psi_{|D}$ at least on $D \setminus \operatorname{ri} D$ instead of the existence of the gradient of Ψ on D. Secondly, in (4.5) we replace $\nabla\Psi(x)$ by $\nabla\Psi_{|D}(x)$. This is in accordance with the approach of Marshall, Walkup and Wets [16].

Example 4.8. Suppose a compact group $G \subset O(V)$ induces a GIC ordering with some closed convex cone D. Consider the class of the functions

$$\Psi_a(x) = \langle a, x^*\rangle, \quad x \in V,$$

where $a \in D$. We shall prove that these are the only G-isotone on V and positively linear on D real continuous functions whose restrictions to D have gradients on $\operatorname{ri} D$ (as elements of the space $\operatorname{lin} D$). Recall that a function $\Psi : V \to R$ is positively linear on D if

$$\Psi(\alpha x + \beta y) = \alpha\Psi(x) + \beta\Psi(y) \text{ for all } x, y \in D \text{ and } \alpha, \beta \geq 0.$$

In fact, by (3.4) it is understood that each Ψ_a is G-isotone on V. The continuity of Ψ_a is a consequence of the continuity of the operator $x \mapsto x^*$. The positive linearity on D is obvious. Evidently, $\Psi_{a|D}(x) = \langle a, x\rangle$ for $x \in D$, and hence $\nabla\Psi_{a|D}(x) = a$ for $x \in \operatorname{ri} D$.

On the other hand, consider a real continuous function Ψ which is G-isotone on V, positive linear on D, and has gradient on $\operatorname{ri} D$. Let $\Phi = \Psi_{|D}$. The formula

$$\Phi(x - y) = \Phi(x) - \Phi(y) \text{ for all } x, y \in D$$

extends Φ from D to the subspace $\operatorname{lin} D = D - D$. It is not hard to verify that Φ is a linear functional on $\operatorname{lin} D$. So $\Phi(x) = \langle a, x\rangle$, $x \in \operatorname{lin} D$, for some $a \in \operatorname{lin} D$, and further, $\nabla\Psi_{|D}(x) = \nabla\Phi(x) = a$, $x \in \operatorname{ri} D$. Applying Theorem 4.4 and the remark made before this example, we see that the G-isotonicity of Ψ yields

$$\langle (I-g)z, a\rangle = \langle (I-g)z, \nabla\Phi(x)\rangle \geq 0$$

for all $g \in G$ and $x, z \in \operatorname{ri} D$. Therefore, by Lemma 3.2, we have $a \in \operatorname{dual}(\operatorname{dual} D) = D$.

Since $\Phi = \Psi_{|D}$, so $\Psi(x) = \langle a, x\rangle$, $x \in D$. In addition, for any $x \in V$ we have $\Psi(x) = \Psi(x^*)$ because of the G-invariance of Ψ. Combining the last two facts we get $\Psi(x) = \langle a, x^*\rangle$ for any $x \in V$, as claimed.

This is a consequence of Theorem 4.3 that a further reduction of the gradient existence assumption in Theorem 4.4 is possible. Namely, we may only assume the existence of the gradient on a dense subset $A \subset D$ which has the form $A = \bigcup_{j=1}^{k} A_j$ with some disjoint open (in $\operatorname{lin} D$) and convex sets $A_j \subset D$, and, in addition, the continuity of the function Ψ at least on the set $D \setminus A$. Then in (4.5) we may replace the relation $x \in \operatorname{ri} D$ by $x \in A$.

Example 4.9. The group majorization considered in this example is closely related to that in Example 4.7.

Namely, take $V = R^n$ with the usual inner product and

$$G = \{g = pc : p \in \mathcal{P}_n, c \in \mathcal{C}_n, \det\ c = 1\},$$

where $\mathcal{P}_n$ is the permutation group and $\mathcal{C}_n$ is the coordinate sign changes group. Recall that members of $\mathcal{P}_n$ are n-by-n orthogonal matrices whose each entry is 0 or 1, while members of $\mathcal{C}_n$ are n-by-n diagonal matrices whose diagonal entries are ± 1. It can be proved that then G induces a GIC ordering with

$$D = \{x \in R^n \ : \ x_1 \geq \ldots \geq x_{n-1} \geq |x_n|\}.$$

Now we search conditions on a real continuous function ψ defined on R^n so that the function

$$\Psi(x) = \Psi(x_1, \ldots, x_n) = \psi(|x_1|, \ldots, |x_n|)$$

is G-isotone when ψ possesses gradient and is symmetric ($\mathcal{P}_n$-invariant).

Put $A = A_1 \cup A_2$, where

$$A_1 = \{x \in \operatorname{ri} D : x_n > 0\}$$

and

$$A_2 = \{x \in \operatorname{ri} D : x_n < 0\}.$$

Then

$$\Psi(x) = \psi(x) \ \text{ for } \ x \in A_1$$

and

$$\Psi(x) = \psi(\tilde{x}) \ \text{ for } \ x \in A_2,$$

where $\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)$. Therefore on A_1 it holds that

$$\frac{\partial \Psi}{\partial x_i}(x) = \frac{\partial \psi}{\partial x_i}(x), \ \ i = 1, \ldots, n,$$

while on A_2 we have

$$\frac{\partial \Psi}{\partial x_i}(x) = \frac{\partial \psi}{\partial x_i}(\tilde{x}), \quad i = 1, \dots, n-1,$$

and

$$\frac{\partial \Psi}{\partial x_n}(x) = -\frac{\partial \psi}{\partial x_n}(\tilde{x}).$$

By the considerations before this example, a necessary and sufficient condition for Ψ to be G-isotone is

$$\nabla \Psi(x) \in D \;\text{ for all }\; x \in A.$$

This is equivalent to

$$\frac{\partial \psi}{\partial x_1}(x_1, \dots, x_{n-1}, |x_n|) \geq \dots \geq \frac{\partial \psi}{\partial x_{n-1}}(x_1, \dots, x_{n-1}, |x_n|) \geq \left| \frac{\partial \psi}{\partial x_n}(x_1, \dots, x_{n-1}, |x_n|) \right|$$

for all $x \in R^n$ such that $x_1 > \dots > x_{n-1} > |x_n| > 0$ [cf. 4.2].

For instance, from the above criterion it is not hard to verify that the Laplace density function

$$\Phi(x) = \frac{1}{2^n} exp(-\sum_{i=1}^{n} |x_i|)$$

is antiisotone for the group majorization presented in this example.

In principle we are often interested in a minimal generator of a convex cone. Therefore it is natural to ask whether in (4.5) the set G may be replaced by a smaller one, say $G_0 \subset G$, such that the set $\{(I-g)z : g \in G_0\}$ is a generator of dual D for any $z \in \operatorname{ri} D$. Obviously, for a fixed $z \in \operatorname{ri} D$ we may exclude from G elements g satisfying $gz = z$, and put

$$G_0 = \{g \in G : gz \neq z\}.$$

For a GIC ordering it can be proved that for any $x, y \in \operatorname{ri} D$ and $g \in G$ holds

$$gx = x \;\text{ iff }\; gy = y$$

(see Lemma 3.3). Therefore we may take

$$G_0 = \{g \in G : g_{|\operatorname{lin} D} \neq I_{|\operatorname{lin} D}\}.$$

If D has nonempty interior, that is, if G is finite (see our Th. 3.4 and Th. 4.1 in [29]), then the last means

$$G_0 = \{g \in G : g \neq I\}.$$

However, in the general case of a GIC ordering none further reduction of the set G is possible. Though, in some particular cases this is possible (see Example 4.10 below).

Let us discuss the same problem in the context of condition (4.7). Now the requirements for G_0 are the following. First, the set

$$\{(I-g)z : g \in G_0\}$$

must be a generator of dual D under an arbitrary fixed $z \in \operatorname{ri} D$, and next, the property

$$gG_0g^{-1} = G_0 \text{ for all } g \in G$$

should hold. Then in a similar way as in the proof of Corollary 4.5 one can point out that in (4.7) G may be replaced by G_0.

Example 4.10. As an application of the above discussion we shall show how Corollary 4.5 implies Theorem 4.2.

Consider any finite reflection group $G \subset O(V)$. Let

$$G_0 = \{S_r : S_r \in G\},$$

where S_r is the reflection across the hyperplane orthogonal to $r \in V$, $\|r\| = 1$, i.e.

$$S_r x = x - 2\langle x, r\rangle r, \quad x \in V.$$

It is easy to check that for arbitrary $g \in G$ we obtain $gS_rg^{-1} = S_{gr}$. Therefore $gG_0g^{-1} \subset G_0$. On the other hand $G_0 \subset gG_0g^{-1}$ because for any $S_r \in G_0$ we have $S_r = gS_{g^{-1}r}g^{-1}$ with $S_{g^{-1}r} = g^{-1}S_rg \in G$. Thus $gG_0g^{-1} = G_0$ for any $g \in G$.

It is well-known that G, being finite reflection group, induces a GIC ordering with some convex cone D such that the cone dual D is generated by the set $\{(I - S_r)z : S_r \in G_0\}$, where z is an arbitrary point in $\operatorname{ri} D$ (cf. [10, Lemma 4.1] and (3.3)–(3.5)).

Therefore, on account of the considerations before this example and the equality

$$\langle r, x\rangle \cdot \langle r, \nabla\Psi(x)\rangle = \frac{1}{2}\langle (I - S_r)x, \nabla\Psi(x)\rangle,$$

one can see that G-isotonicity of Ψ on V is equivalent to the condition

$$\langle r, x\rangle \cdot \langle r, \nabla\Psi(x)\rangle \geq 0 \text{ for all } S_r \in G \text{ and } x \in V.$$

Thus Theorem 4.2 may be treated as a consequence of Corollary 4.5 (when the group is finite).

It is interesting in this example that a further reduction to so-called *fundamental reflections* is possible (see [10, p. 840]).

5. A characterization of GIC orderings via S-O type condition

In light of Theorem 4.4 one may ask whether or not Schur-Ostrowski type condition (4.5) is valid for a wider class of preorderings than GIC orderings. We shall show that the answer is negative. In other words, property (4.5) marks out GIC orderings among all group majorizations defined on a fixed linear space.

Remind that if conditions (A1) and (A2) hold for a convex cone D then, by Lemma 3.3, the set $F = \operatorname{ri} D$ is a relatively open fundamental region for G. Therefore in Theorem 5.1 we study (4.5) under this kind of assumptions on F.

Theorem 5.1. *Suppose G is a compact subgroup of $O(V)$. Let $F \subset V$ be a relatively open fundamental region for G such that $D = \operatorname{cl} F$ is a convex cone. Then the following statements are equivalent:*

(i) *conditions* (A1) *and* (A2) *hold for* D,

(ii) *condition* (4.5) *holds for all G-isotone real functions* Ψ *defined on* V *with gradient* $\nabla\Psi(x)$ *at each point* x *in* D,

(iii) *condition* (4.5) *holds for the function* $\Psi(x) = \|x\|^2$, $x \in V$.

Proof. (i) $\Rightarrow$ (ii). This implication follows directly from Theorem 4.4.

(ii) $\Rightarrow$ (iii). The function

$$\Psi(x) = \|x\|^2, \ x \in V,$$

is G-isotone. To see this, note that function $x \to \|x\|$ is convex and G-invariant on V. According to Theorem 2.1 this function is G-isotone, and consequently Ψ is G-isotone. Moreover, Ψ has gradient $\nabla\Psi(x) = 2x$ for any $x \in V$. Now, it is obvious that (ii) implies (iii).

(iii) $\Rightarrow$ (i). Because for $\Psi(x) = \|x\|^2$, $x \in V$, we have $\nabla\Psi(x) = 2x$, so by (iii) and (4.5) we obtain

$$0 \le \langle (I-g)z, \nabla\Psi(x)\rangle = \langle (I-g)z, 2x\rangle$$

for all $g \in G$, $x, z \in \operatorname{ri} D = F$, which together with $D = \operatorname{cl} F$ and the continuity of the inner product implies (A2).

Condition (A1) for $D = \operatorname{cl} F$ follows from the fact that F is a relatively open fundamental region for G. $\square$

Remark that in Theorem 5.1 condition (4.5) cannot be replaced by (4.4) because the latter is a necessary condition on the isotonicity for the general case of a group majorization, and obviously there exist group majorizations which are not GIC orderings.

In Theorem 5.1 the same role as the square of the norm function plays any function of the form

$$\Psi(x) = \psi(\|x\|^2), \quad x \in V,$$

where ψ is a real function defined on the real interval $[0, \infty)$ having the derivative which is positive.

To show this, observe that the function Ψ is G-isotone, because it is the composition of the square of the norm function which is G-isotone, and the increasing function ψ. On the other hand

$$\nabla\Psi(x) = 2\psi'(\|x\|^2) \cdot x, \quad x \in V.$$

Therefore, by Theorem 4.4 we obtain

$$0 \le \langle (I-g)z, \nabla\Psi(x)\rangle = 2\psi'(\|x\|^2) \cdot \langle (I-g)z, x\rangle$$

for all $g \in G$ and $x, z \in \operatorname{ri} D = F$, which implies

$$0 \le \langle (I-g)z, x\rangle$$

for $g \in G$ and $x, z \in \operatorname{ri} D = F$, as required.

For instance, such a function on $V = R^n$ is

$$\Psi(x) = -(\sqrt{2\pi}\sigma)^{-n} \exp(-\frac{1}{2}\sigma^{-2}\sum_{i=1}^{n} x_i^2), \quad x \in R^n,$$

minus of the normal distribution density with the mean zero and the variance-covariance matrix $\sigma^2 I$.

One of the most important results in the theory of group majorizations is so-called Convolution Theorem which has a number of applications in probability and statistics (see [6, 9]). It was proved for finite reflection groups by Eaton and Perlman in [10]. Recently Suren Fernando showed that this theorem holds only for these groups (see [4, p. 123]). Therefore as a corollary from Theorem 5.1 we can get a criterion on the reflexivity for a finite group in terms of S-O type condition (4.5). Recall that for a finite group $G \subset O(V)$ there exists a fundamenatal region $F \subset V$ such that $D = \mathrm{cl}\, F$ is a convex cone.

Corollary 5.2. *Suppose $G \subset O(V)$ is a finite group with a fundamental region F such that the set $D = \mathrm{cl}\, F$ is a convex cone. Then G is a reflection group iff condition* (4.5) *holds for all G-isotone real functions Ψ defined on V with gradient $\nabla\Psi(x)$ at each point x in D.*

Proof. As we already mentioned (see Ex. 4.10), Eaton and Perlman showed that the reflexivity of a group G implies that the group majorization induced by G is a GIC ordering. It is a result of Steerneman [29, Th. 4.1] that for a finite group the converse holds. Namely, if the conditions (A1) and (A2) are met then G must be a reflection group. Now it suffices to apply Theorem 5.1. □

It is worth noting in the context of the reflexivity of a group that sometimes it is convenient to study (4.5) for some simpler function than $\Psi(x) = \|x\|^2$.

Example 5.3. Let V be the linear space $\mathcal{T}_n$ of all n-by-n upper triangular matrices with the usual inner product $\langle x, y\rangle = \mathrm{tr}\; xy^T$. Consider the group G consisting of all linear operators

$$\mathcal{T}_n \ni x \to gx \in \mathcal{T}_n,$$

where g runs over the group $\mathcal{C}_n$ of all n-by-n diagonal matrices with entries equal to 1 or -1 on the principal diagonal.

To see that G is not a reflection group when $n \geq 2$ we apply Corollary 5.2. One can easily check that

$$F = \{x \in \mathcal{T}_n : x_{ii} > 0, 1 \leq i \leq n\}$$

is a fundamental region for G and

$$D = \mathrm{cl}\, F = \{x \in \mathcal{T}_n : x_{ii} \geq 0, 1 \leq i \leq n\}$$

is a closed convex cone.

Now consider function

$$\Psi(x) = \sum_{j=1}^{n} x_{1j}^2.$$

This is a G-isotone function, because it is the square of a certain convex G-invariant function. To our aim it is sufficient to verify that S-O type condition (4.5) does not hold for Ψ. The gradient of the function Ψ is the matrix

$$\nabla\Psi(x) = \left(\frac{\partial\Psi}{\partial x_{ij}}(x)\right)_{1\leq i,j\leq n},$$

where

$$\frac{\partial \Psi}{\partial x_{1j}}(x) = 2x_{1j}, \quad j \geq 1$$

and

$$\frac{\partial \Psi}{\partial x_{ij}}(x) = 0, \quad i > 1, j \geq 1.$$

Therefore with $g = -I$ we have

$$\langle (I-g)z, \nabla \Psi(x) \rangle = 4 \sum_{j=1}^{n} x_{1j} z_{1j}.$$

Next, let $x, z \in F$ be such matrices that

$$x = \begin{pmatrix} 1 & n & n & \dots & n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then

$$\langle (I-g)z, \nabla \Psi(x) \rangle = 4(1-(n-1)n) < 0$$

when $n \geq 2$. This completes the proof that in this example G is not a reflection group.

6. An extension to vector-valued functions

Some results presented in Section 4 can be extented to vector-valued functions.

Namely, let V_1 and V_2 be finite-dimensional real inner product spaces. The inner products on both the spaces will be denoted by the same symbol $\langle \cdot, \cdot \rangle$. Suppose V_1 is equipped with a GIC ordering $\preceq_1$ induced by a compact subgroup G of the orthogonal group $O(V_1)$. Let $D \subset V_1$ be a closed convex cone $D \subset V_1$ related to $\preceq_1$ as in Definition 3.1. Assume that V_2 is provided with a cone preordering $\preceq_2$ induced by a closed convex cone $C \subset V_2$.

Our goal in this section is to characterize *G-C-isotone* vector-valued functions $\Psi : V_1 \to V_2$, i.e. functions such that

$$y \preceq_1 x \text{ implies } \Psi(y) \preceq_2 \Psi(x) \text{ for all } x, y \in V_1,$$

and to characterize *C-G-antiisotone* (on convex sets $A \subset V_2$) vector-valued functions $\Phi : V_2 \to V_1$, i.e. functions satisfying

$$y \preceq_2 x \text{ implies } \Phi(x) \preceq_1 \Phi(y) \text{ for all } x, y \in A.$$

Theorem 4.3 being a main source of results in Section 4 must be now replaced by its vector version.

Theorem 6.1 ([22]). *Let Ψ be a continuous function defined on a convex set $A_1 \subset V_1$ with values in a set $A_2 \subset V_2$. Suppose $\preceq_1$ is a preordering on A_1 induced by a convex cone $C_1 \subset V_1$, and $\preceq_2$ is a preordering on A_2 induced by a closed convex cone $C_2 \subset V_2$. If the directional derivative $\nabla_t \Psi(x)$ at the point x and in the direction t exists for all $x \in \operatorname{ri} A_1$ and $t \in C_1 \cap \operatorname{lin} A_1$ then the following conditions are equivalent:*

(i) $y \preceq_1 x$ *implies* $\Psi(y) \preceq_2 \Psi(x)$ *for all* $x, y \in A_1$,
(ii) $0 \preceq_1 t$ *implies* $0 \preceq_2 \nabla_t \Psi(x)$ *for all* $x \in \operatorname{ri} A_1$ *and* $t \in \operatorname{lin} A_1$.

Employing the above theorem one can prove the next two corollaries. (We will not repeat the assumptions stated at the beginning of this section).

Corollary 6.2. *Let* $\Psi : V_1 \to V_2$ *be a continuous G-invariant function such that the Gâteaux derivative* $\Psi'(x)$ *exists for all* $x \in \operatorname{ri} D$. *Then* Ψ *is G-C-isotone on* V_1 *iff* $0 \preceq_2 \Psi'(x)t$ *for all* $x \in \operatorname{ri} D$ *and* $t \in T$, *where T is a generator of the cone* dual $D \cap \operatorname{lin} D$.

Example 6.3. Put V_1 to be $\mathcal{S}_n$, the space of all n-by-n real symmetric matrices with the trace inner product. Consider the group G of all linear operators of the form

$$\mathcal{S}_n \ni x \to gxg^T \in \mathcal{S}_n,$$

where g runs over the orthogonal group $\mathcal{O}_n$ acting on R^n.

Take V_2 to be also $\mathcal{S}_n$ and C to be the Loewner cone $\mathcal{L}_n$ of all non-negative definite matrices in $\mathcal{S}_n$.

It is well-known that G induces on V_1 a GIC ordering with the convex cone

$$D = \{x \in \mathcal{D}_n : x_{11} \geq x_{22} \geq \ldots \geq x_{nn}\},$$

where by $\mathcal{D}_n$ is denoted the space of all n-by-n real diagonal matrices (see [7, p. 17]). Then $\operatorname{lin} D$ is the space $\mathcal{D}_n$, while

$$\operatorname{dual} D \cap \operatorname{lin} D = \{x \in \mathcal{D}_n \ : \ \sum_{j=1}^{n} x_{jj} = 0, \ \sum_{j=1}^{i} x_{jj} \geq 0, \ i = 1, 2, \ldots, n-1\}.$$

A generator of dual $D \cap \operatorname{lin} D$ is the set

$$T = \{t^{(i)} = e^{(i)} - e^{(i+1)} : i = 1, \ldots, n-1\},$$

where $e^{(k)}$, $k = 1, \ldots, n$, is n-by-n diagonal matrix whose kth diagonal entry is 1 and the remaining ones are 0.

Therefore a necessary and sufficient condition for a continuous G-invariant function

$$\Psi = \begin{pmatrix} \Psi_{11} & \ldots & \Psi_{1n} \\ . & \ldots & . \\ . & \ldots & . \\ \Psi_{n1} & \ldots & \Psi_{nn} \end{pmatrix}$$

defined on $V_1 = \mathcal{S}_n$ with values in $V_2 = \mathcal{S}_n$ to be G-C-isotone on V_1 is

$$\frac{\partial \Psi}{\partial x_{11}}(x) \succeq_2 \frac{\partial \Psi}{\partial x_{22}}(x) \succeq_2 \ldots \succeq_2 \frac{\partial \Psi}{\partial x_{nn}}(x) \quad \text{for all} \quad x \in \operatorname{ri} D, \tag{6.1}$$

where

$$\frac{\partial \Psi}{\partial x_{kk}}(x) = \Psi'(x) e^{(k)} = \begin{pmatrix} \frac{\partial \Psi_{11}(x)}{\partial x_{kk}} & \ldots & \frac{\partial \Psi_{1n}(x)}{\partial x_{kk}} \\ . & \ldots & . \\ . & \ldots & . \\ \frac{\partial \Psi_{n1}(x)}{\partial x_{kk}} & \ldots & \frac{\partial \Psi_{nn}(x)}{\partial x_{kk}} \end{pmatrix}, \quad 1 \leq k \leq n.$$

Thus we obtain an analogon of Schur-Ostrowski condition (4.2) for matrix-valued functions. Condition (6.1) means that the matrix $(\partial\Psi/\partial x_{i+1i+1})(x)$ is not greater than $(\partial\Psi/\partial x_{ii})(x)$ in the Loewner ordering sense i.e. that the matrix

$$\frac{\partial\Psi}{\partial x_{ii}}(x)-\frac{\partial\Psi}{\partial x_{i+1i+1}}(x),\quad 1\le i\le n-1,$$

is non-negative definite for each n-by-n diagonal matrix x such that $x_{11}>x_{22}>\ldots>x_{nn}$.

Corollary 6.4. *Let Φ be a continuous function defined on a convex set $A\subset V_2$ with values in $D\subset V_1$. Suppose that the Gâteaux derivative $\Phi'(x)$ there exists for all $x\in\operatorname{ri}A$. Then Φ is C-G-antiisotone on A iff $\langle u,\Phi'(x)t\rangle\le 0$ for all $x\in\operatorname{ri}A$, $t\in T$ and $u\in U$, where T is a generator of the cone $C\cap\operatorname{lin}A$ and U is a generator of the cone D.*

Proof. It is sufficient to employ Th. 6.1 for the function $-\Phi$ which has values in the set $-D$ with the cone preordering induced by dual D. □

Example 6.5. In [1, Th. 1-16] we can found the following theorem.

If $x,y\in R^n$ and

$$x_i\ge x_{i+1},\quad y_i\ge y_{i+1},\quad\text{and}\quad x_i-x_{i+1}\ge y_i-y_{i+1}\quad\text{for all}\quad i=1,\ldots,n-1$$

then

$$\varrho(x)\preceq\varrho(y),$$

where $\preceq$ is the classical majorization on R^n and

$$\varrho(a)=\left(\frac{\exp a_1}{\sum_{k=1}^n\exp a_k},\ldots,\frac{\exp a_n}{\sum_{k=1}^n\exp a_k}\right)^T,\quad a\in R^n,$$

is the Gibbsian states function.

This theorem gives reasons to establish a necessary and sufficient condition for a continuous function $\Phi:D\to D$ to be D-$\mathcal{P}_n$-antiisotone, where

$$D=\{x\in R^n:x_1\ge x_2\ge\ldots\ge x_n\}$$

induces the cone preordering $\preceq_2$ on the domain of Φ, and the permutation group $\mathcal{P}_n$ induces the classical majorization preordering $\preceq_1$ on the range of the function.

We shall apply Corollary 6.4. Take $V_1=V_2=R^n$ with the usual inner product. Next put $G=\mathcal{P}_n$, $A=D$, and $C=D$. Then a continuous function

$$\Phi=(\Phi_1,\ldots,\Phi_n)^T$$

is D-$\mathcal{P}_n$-antiisotone on D iff

$$\langle t^{(i)},\Phi'(x)t^{(j)}\rangle\le 0,\ \ 1\le i,j\le n+1,\ \ x\in\operatorname{ri}D,$$

where $t^{(k)}\in R^n$, $k=1,2,\ldots,n$, is the vector whose the first k entries are 1 and the remaining ones are 0, and $t^{(n+1)}=-t^{(n)}$. Because

$$\Phi'(x)=\begin{pmatrix}\frac{\partial\Phi_1(x)}{\partial x_1}&\ldots&\frac{\partial\Phi_1(x)}{\partial x_n}\\ \cdot&\ldots&\cdot\\ \cdot&\ldots&\cdot\\ \frac{\partial\Phi_n(x)}{\partial x_1}&\ldots&\frac{\partial\Phi_n(x)}{\partial x_n}\end{pmatrix}$$

so the condition on D-$\mathcal{P}_n$-antiisotonicity of Φ takes the form

$$\sum_{p=1}^{i}\sum_{q=1}^{j}\frac{\partial\Phi_p(x)}{\partial x_q}\le 0,\quad i,j=1,2,\dots,n,\quad x\in \operatorname{ri} D,$$

with the equality when $i=n$ or $j=n$, where $\operatorname{ri} D=\{x\in R^n : x_1>x_2>\ldots>x_n\}$. The equality is equivalent to the condition that

$$\sum_{p=1}^{n}\frac{\partial\Phi_p(x)}{\partial x_k}=0 \text{ and } \sum_{q=1}^{n}\frac{\partial\Phi_k(x)}{\partial x_q}=0,\quad k=1,2,\dots,n,\quad x\in \operatorname{ri} D.$$

In other words, the condition means that if we take any left upper submatrix of $\Phi'(x)$ then the sum of all its entries is nonpositive and the sum of all entries in each column and row of $\Phi'(x)$ is 0.

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