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Korovkin Sets and Mean Ergodic Theorems

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Korovkin-type theorems are established, and consequently mean ergodic theorems are obtained.

1. Introduction

Let E be a normed linear space with its dual space E^* and let $B[E]$ denote the normed algebra of all bounded linear operators of E into itself with the identity operator I. Let $\mathfrak{T}$ be a subset of $B[E]$ and let $T \in \mathfrak{T}$. A subset K of E is said to be a $\mathfrak{T}$-Korovkin set for T if for any bounded sequence $\{T_n\}$ in $\mathfrak{T}$, the relation

$$\lim_{n\to\infty} \|T_n(g) - T(g)\| = 0 \qquad \text{for all } g \in K$$

implies that

$$\lim_{n\to\infty} \|T_n(f) - T(f)\| = 0 \qquad \text{for all } f \in E.$$

Let $\mathfrak{L}$ be a subset of E^* and let $\mu \in \mathfrak{L}$. A subset K of E is said to be an $\mathfrak{L}$-Korovkin set for μ if for any bounded sequence $\{\mu_n\}$ in $\mathfrak{L}$, the relation

$$\lim_{n\to\infty} \mu_n(g) = \mu(g) \qquad \text{for all } g \in K$$

implies that

$$\lim_{n\to\infty} \mu_n(f) = \mu(f) \qquad \text{for all } f \in E.$$

For the background of the Korovkin-type approximation theory, see the recent book of Altomare and Campiti [2], in which an excellent source and a vast literature of this theory can be found (cf. [3], [6], [7]).

The purpose of this paper lies in considering $\mathfrak{T}$ and $\mathfrak{L}$-Korovkin sets under certain requirements from a mean ergodic point of view. For the fundamental results about the ergogic theory, see [4; VIII] and for further extensive treatments of ergodic theorems, we refer to [8].

2. $\mathfrak{T}$ and $\mathfrak{L}$-Korovkin sets and mean ergodic theorems

If S is a subset of E, then $S^\perp$ denotes the annihilater of S. That is,

$$S^\perp = \{\mu \in E^* : \mu(f) = 0 \quad \text{for all } f \in S\}.$$

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If $\mathfrak{L}$ is a subset of E^*, then we define

$$\mathfrak{L}_\perp = \{f \in E : \mu(f) = 0 \quad \text{for all } \mu \in \mathfrak{L}\},$$

which is called the annihilater of $\mathfrak{L}$. If T is an operator in $B[E]$, then $\mathcal{R}_T$ denotes the range of $I - T$.

We shall need the following basic result.

Lemma 2.1 (see [9; Theorem 4.6.1]). *If S is a linear subspace of E, then $(S^\perp)_\perp$ coincides with the closure of S.*

Let $\mu \in E^*$ and $T \in B[E]$. Then we say that μ is T-invariant if $\mu(T(f)) = \mu(f)$ for every $f \in E$, i.e., μ belongs to $\mathcal{R}_T^\perp$. Note that μ is T-invariant if and only if it is a fixed point of the adjoint operator T^* of T, i.e., $T^*(\mu) = \mu$.

From now on, let e be any fixed non-zero element in E, and we set

$$\mathfrak{T} = \{L \in B[E] : L(e) = e\},$$

which is a closed convex subset of $B[E]$. Let φ be an element in E^* with $\varphi(e) = 1$, and we define

$$P(f) = \varphi(f)e \qquad \text{for all } f \in E. \tag{2.1}$$

Evidently, P is a projection operator on E belonging to $\mathfrak{T}$ and φ is P-invariant.

Let $T, L \in B[E]$ and $n = 1, 2, 3, \cdots$. Then we define

$$\sigma_{n,T} = \frac{1}{n} \sum_{i=0}^{n-1} T^i,$$

which is called the n-th Cesàro mean operator of T, and T is said to be norm mean stable with L if

$$\lim_{n\to\infty} \|\sigma_{n,T}(f) - L(f)\| = 0 \qquad \text{for all } f \in E. \tag{2.2}$$

The condition (2.2) implies that L is necessarily a projection operator on E and $TL = LT = L$. Furthermore, the mean ergodic theorem of Sine [16] (cf. [15]) asserts that if E is a Banach space and if $\|T\| \le 1$, then T is norm mean stable with some $L \in B[E]$ if and only if the set of all fixed points of T separates the set of all fixed points of T^*.

Theorem 2.2. *Let $T \in B[E]$ and suppose that φ is T-invariant.*

(a) *If the annihilater of $\mathcal{R}_T$ is spanned by φ, then $\mathcal{R}_T$ is a $\mathfrak{T}$-Korovkin set for P.*

(b) *If $T \in \mathfrak{T}$,*

$$\lim_{n\to\infty} \frac{\|T^n(f)\|}{n} = 0 \qquad \text{for every } f \in E \tag{2.3}$$

and

$$\sup_{n\ge1} \|\sigma_{n,T}\| < \infty, \tag{2.4}$$

then the converse of (a) *is also true.*

Proof. (a) Let $\{L_n\}$ be a bounded sequence in $\mathfrak{T}$ such that for every $g \in \mathcal{R}_T$, $\lim_{n\to\infty} \|L_n(g) - P(g)\| = 0$, which is equivalent to $\lim_{n\to\infty} \|L_n(g)\| = 0$ because of $P(g) = 0$. Let $\epsilon > 0$ and $f \in E$. Then, by Lemma 2.1, there exists an element $h \in \mathcal{R}_T$ such that $\|f - P(f) - h\| < \epsilon$. Since $L_n P = P$ for all n, we have

$$\begin{aligned}\|L_n(f) - P(f)\| &\leq \|L_n(f) - P(f) - L_n(h)\| + \|L_n(h)\| \\ &\leq \|L_n\|\|f - P(f) - h\| + \|L_n(h)\| < \epsilon\|L_n\| + \|L_n(h)\|,\end{aligned}$$

and so $\lim_{n\to\infty} \|L_n(f) - P(f)\| = 0$ by virtue of $\sup_n \|L_n\| < \infty$ and $\lim_{n\to\infty} \|L_n(h)\| = 0$. Therefore, $\mathcal{R}_T$ is a $\mathfrak{T}$-Korovkin set for P.

(b) Suppose that $T \in \mathfrak{T}$, (2.3) and (2.4) hold. Then $\{\sigma_{n,T}\}$ is a bounded sequence in $\mathfrak{T}$ satisfying $\lim_{n\to\infty} \|\sigma_{n,T}(f - T(f))\| = 0$ for all $f \in E$, since

$$\sigma_{n,T}(I - T) = \frac{1}{n}(I - T^n) \qquad (n = 1, 2, 3, \cdots). \tag{2.5}$$

Assume now that $\mathcal{R}_T$ is a $\mathfrak{T}$-Korovkin set for P. Then we have that $\lim_{n\to\infty} \|\sigma_{n,T}(f) - P(f)\| = 0$ for every $f \in E$. Let μ be an arbitrary element in $\mathcal{R}_T^{\perp}$. Then for all $f \in E$, we have

$$\lim_{n\to\infty} \mu(\sigma_{n,T}(f)) = \mu(P(f)) = \varphi(f)\mu(e),$$

which implies $\mu(f) = \mu(e)\varphi(f)$, since

$$\mu(\sigma_{n,T}(f)) = \mu(f) \qquad (n = 1, 2, 3, \cdots).$$

Thus, $\mathcal{R}_T^{\perp}$ is spanned by φ. □

Remark 2.3. If T is power bounded, i.e., $\sup_{n\geq 1} \|T^n\| < \infty$, then (2.3) and (2.4) automatically hold. Also, by (2.5), (2.2) implies (2.3).

As a consequence of Theorem 2.2, we have the following.

Corollary 2.4. *Let T be an operator in $\mathfrak{T}$ satisfying* (2.3), (2.4) *and $T^*(\varphi) = \varphi$. Then the following statements are equivalent:*

(a) *$\mathcal{R}_T^{\perp}$ is spanned by φ.*
(b) *$\mathcal{R}_T$ is a $\mathfrak{T}$-Korovkin set for P.*
(c) *T is norm mean stable with P.*

Let $\mathfrak{L}$ be a subset of E^* and $\mu \in \mathfrak{L}$. Then an operator $T \in B[E]$ is said to be $\mathfrak{L}$-uniquely ergodic with μ if μ is only one T-invariant functional in $\mathfrak{L}$, or equivalently, T^* has exactly one fixed point μ in $\mathfrak{L}$, i.e.,

$$\{\lambda \in \mathfrak{L} : T^*(\lambda) = \lambda\} = \{\mu\}.$$

By [1; Corollary 1.2] and the theorem of Krein-Šmulian (see, [9; Theorem 10.2.1]), we have the following.

Remark 2.5. Suppose that E is a separable Banach space and let $\mathfrak{L}$ be a convex subset of E^* such that the set

$$\mathfrak{L} \cap \{\lambda \in E^* : \|\lambda\| \leq r\}$$

is weak*-closed for each $r > 0$. Let $T \in B[E]$, and let μ be a functional in $\mathfrak{L}$ which is T-invariant. Then T is $\mathfrak{L}$-uniquely ergodic with μ if and only if $\mathcal{R}_T$ is an $\mathfrak{L}$-Korovkin set for μ.

3. Korovkin sets and mean ergodic theorems in function spaces

In this section, let E be a function space on a non-empty set X. That is, E is a normed linear space of real or complex valued functions on X, which contains the unit function e defined by $e(x) = 1$ for all $x \in X$. Consequently, all the results obtained in the preceding section are applicable to this setting.

From now on, let X be a compact metric space and let $C(X)$ denote the Banach space of all real valued continuous functions on X with the usual supremum norm. Note that $C(X)$ is separable. Let E be a linear subspace containing the unit function e. For a point $x \in X$, we define the point evaluation functional δ_x at x by $\delta_x(f) = f(x)$ for all $f \in E$.

If $\mathfrak{L}$ is a subset of E^*, then $\mathfrak{T}(\mathfrak{L})$ denotes the set of all operators $L \in B[E]$ such that $\delta_x \circ L$ belongs to $\mathfrak{L}$ for every $x \in X$. Set

$$\mathfrak{L}^1 = \{\mu \in E^* : \mu(e) = 1\}$$

and

$$\mathfrak{T}^1 = \{L \in B[E] : L(e) = e\}.$$

Then we have $\mathfrak{T}(\mathfrak{L}^1) = \mathfrak{T}^1$. Let $\mathfrak{L}_+$ denote the set of all positive linear functionals on E, and we put $\mathfrak{T}_+ = \mathfrak{T}(\mathfrak{L}_+)$, which consists of all positive linear operators of E into itself. Furthermore, we set $\mathfrak{L}^1_+ = \mathfrak{L}_+ \cap \mathfrak{L}^1$ and $\mathfrak{T}^1_+ = \mathfrak{T}(\mathfrak{L}^1_+)$, which coincides with $\mathfrak{T}_+ \cap \mathfrak{T}^1$.

Recall that $\varphi \in \mathfrak{L}^1$ and P is the projection operator in $\mathfrak{T}^1$ defined by (2.1).

Theorem 3.1. *Let $T \in \mathfrak{T}^1_+$. Suppose that $\varphi \in \mathfrak{L}^1_+$ and $T^*(\varphi) = \varphi$. Then $\mathcal{R}_T$ is a $\mathfrak{T}^1_+$-Korovkin set for P if and only if T is norm mean stable with P.*

Proof. Note that $\{\sigma_{n,T}\}$ is a bounded sequence in $\mathfrak{T}^1_+$ with $\|\sigma_{n,T}\| = 1$ for all $n = 1, 2, 3, \cdots$. Since $\|T\| = 1$ and P vanishes on $\mathcal{R}_T$, (2.5) yields that $\lim_{n\to\infty} \|\sigma_{n,T}(g) - P(g)\| = 0$ for all $g \in \mathcal{R}_T$. Therefore, if $\mathcal{R}_T$ is a $\mathfrak{T}^1_+$-Korovkin set for P, then T is norm mean stable with P.

Conversely, suppose that T is norm mean stable with P. Let λ be any functional in $\mathfrak{L}^1_+$ with $T^*(\lambda) = \lambda$. Then we are able to extend λ to a positive linear functional ν on the whole space $C(X)$. By the Riesz representation theorem, there exists a probability measure ρ on X such that

$$\nu(f) = \int_X f(x)\, d\rho(x) \qquad \text{for all } f \in C(X).$$

Let g be an arbitrary function in E. Then we have

$$|\sigma_{n,T}(g)(x)| \le \|\sigma_{n,T}\|\|g\| = \|g\|$$

for all $x \in X$ and for each $n = 1, 2, 3, \cdots$. Therefore, it follows that

$$\begin{aligned}\varphi(g) &= \int_X P(g)(x)\, d\rho(x) = \lim_{n\to\infty} \int_X \sigma_{n,T}(g)(x)\, d\rho(x) \\ &= \lim_{n\to\infty} \nu(\sigma_{n,T}(g)) = \lim_{n\to\infty} \lambda(\sigma_{n,T}(g)) = \lambda(g).\end{aligned}$$

Thus we have $\lambda = \varphi$, and so it follows from [5; Theorems 1.1 and 1.2] that $\mathcal{R}_T$ is a $\mathfrak{T}^1_+$-Korovkin set for P. □

Remark 3.2. Let $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be a finite set of continuous mappings from X into itself and $F = \{f_1, f_2, \cdots, f_m\}$ a finite subset of E. We define

$$T_{\alpha,F}(f) = \sum_{i=1}^{m} (f \circ \alpha_i) f_i$$

for all $f \in E$. Then $T_{\alpha,F}$ is a bounded linear operator of E into $C(X)$. Assume that $T_{\alpha,F}$ maps E into itself. Then all the results presented in this section are applicable to $T = T_{\alpha,F}$.

Finally, in view of the study of the rate of convergence for approximation processes of positive linear operators, we notice that our forthcoming topic is to give a quantitative version of Theorem 3.1, with an optimal order of approximation (cf. [10], [11], [12], [13], [14]).

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