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On a Variational Inequality Containing a Memory Term with an Application in Electro-Chemical Machining

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An obstacle problem with a memory term is studied in the framework of the variational inequality theory. Applying a fixed point argument and a convergence result for convex sets the existence and uniqueness of a solution are proved. Regularity results with respect to time and space are then deduced by using a penalization method. Furthermore, the time evolution of the solution is discussed. The particular evolutionary variational inequalities result from the application of a generalized Baiocchi-type transformation to degenerate free boundary problems with space- and, in particular, time-dependent coefficients. One of the applications is given by a quasi-stationary model for the electro-chemical machining problem.

Keywords: Evolutionary variational inequality, existence, uniqueness, regularity and time evolution of the solution, convex sets, penalization method, electro-chemical machining process

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1. Introduction

One of the important features of variational inequalities is their applicability to many physical problems (see e.g. the monographs [10], [2], [6] and [15]) which has strongly influenced their mathematical development closely connected with convex analysis (cf. [21], chap. 54, 55 and [3]). This contribution deals with a class of evolutionary variational inequalities with a memory term, resulting from the application of a generalized Baiocchi-type transformation to degenerate free boundary problems ('zero-specific heat' Stefan-type problems). The consideration of such problems is motivated, for instance, by the electro-chemical machining (ECM) process. In particular, the conductivity of the electrolyte is assumed to be space- and time-dependent, which causes the memory (integral) term in the evolutionary variational inequality formulation. It is worth emphasizing that such a type of variational inequality also occurs in various other engineering applications. In [18] and [17] we have considered evolutionary obstacle problems arising in non-isothermal Hele-Shaw flows. Compared to this, the inequality problems studied here are characterized by different boundary conditions and time-dependent convex constraint sets.

The present article is organized as follows. After the derivation of the variational inequality formulation in section 2, we prove the existence of a unique solution as a continuous mapping from the time interval $[0,T]$ to the Sobolev space $H^1(\Omega)$ in section 3. The proof is based on a fixed point argument in connection with a convergence result for convex sets. In order to study the regularity of the solution with respect to time and its time evolution, the variational inequality is approximated by means of a penalization method.

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This can be interpreted as a regularization of the associated multivalued operator equation (regularization of the subdifferential of the indicator function of the convex sets). Finally, the last section is concerned with the discussion of the spatial regularity, based again on an investigation of the penalized problem.

In addition to the (spatial) Sobolev spaces $W_p^2(\Omega)$ $(H^k(\Omega) = W_2^k(\Omega))$ we shall use the notation $C([0,T];X)$ for the Banach space consisting of all continuous functions $u : [0,T] \longrightarrow X$ (X as Banach space) and $L_p(0,T;X)$, $1 \le p \le \infty$ for the Lebesgue space of vector-valued functions (see e.g. [21], chap. 23). Furthermore, we shall work with

$$W_p^1(0,T;X) = \{v|\ \exists w \in L_p(0,T;X) : v(t) = v(0) + \int_0^t w(t')\,dt' \quad \forall t \in [0,T]\}$$

for $p = 2$ (in which case we abbreviate $H^1(0,T;X) = W_2^1(0,T;X)$) and $p = \infty$. We identify w with the generalized time derivative $w = \partial_t v$ for $v \in W_p^1(0,T;X)$ (see [15], sect. 9.5 and [11]).

2. Variational inequality approach for the ECM process

The electro-chemical machining process in which a metal workpiece is shaped by placing it as an anode in an electrolyte cell (ablation process of anode metal) may be modelled as a moving free boundary problem (Stefan-type problem with 'zero-specific heat'). An applied potential difference between the anode and the fixed cathode, separated by an electrolyte solution, causes a chemical reaction at the anode. As a result, anode metal is removed electrochemically, while the tool (cathode) remains unaltered. The electrolyte together with the reaction products are pumped through the gap between the electrodes. The two-dimensional annular situation corresponding to the shaping of a long cylindrical metal part by placing it inside a cylindrical tool is schematically depicted in Figure 1.

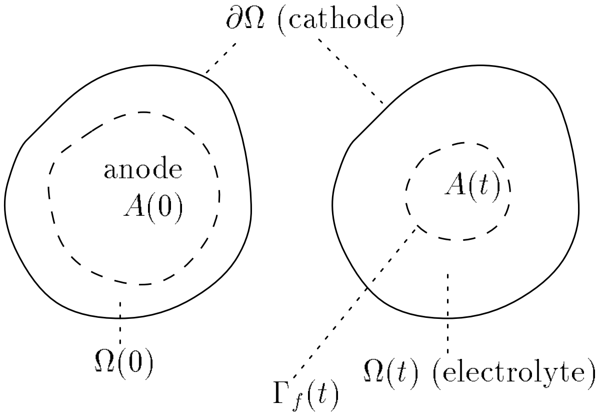

Figure 1: Cross section of an annular ECM problem. Left: initial situation. Right: situation at time $t > 0$.

We refer to [12] and [5], sect. 6.4 as well as to other textbooks concerning the physical-chemical basics. Based on an order-of-magnitude analysis performed in [13], the derivation of a quasi-stationary mathematical model can be found in [15], sect. 2.7. Moreover, a survey of mathematical methods and results for both electro-chemical machining and related problems is contained in [8] (chap.: Session on Stefan problems and applications).

For more recent studies of this application problem we refer to [20] (further developments of conformal mapping techniques) and [19]. The last paper contains investigations and a comparison of different methods (fixed domain, front tracking and level set methods) especially from the application point of view.

Based on the assumption that the electric field $\vec{E}$ between the electrodes is approximately irrotational, Ohm's law gives the relation $\vec{j} = \sigma\vec{E} = -\sigma\,\nabla\Phi$ for the electric current $\vec{j}$. Here, Φ is a potential and σ denotes the conductivity of the electrolyte. Furthermore, assuming the density of the electric charge to be constant, the conservation of charge leads to $\operatorname{div}\vec{j} = 0$, and so $\operatorname{div}(\sigma\,\nabla\Phi) = 0$ holds in $\Omega(t)$ (electrolyte region). A potential difference $\gamma_D = \gamma_D(t) > 0$ is applied across the electrodes, and we may take $\Phi = \gamma_D > 0$ at the anode and $\Phi = 0$ on $\partial\Omega$ (cathode surface).

Due to Faraday's law the dissolution rate of the workpiece (anode) is proportional to the normal magnitude of the local current density. This leads to the condition $\vec{j}\cdot\vec{n} = \lambda\,\vec{v}\cdot\vec{n}$ on $\Gamma_f(t)$ (anode surface as moving boundary), where $\vec{n}$ is the unit normal directed towards the workpiece and where $\vec{v}\cdot\vec{n} > 0$ is the normal interface velocity. The constant $\lambda < 0$ denotes the jump of the electric charge across $\Gamma_f(t)$ (the so-called electro-chemical equivalent).

Summarizing, the free boundary problem for the ECM process can be written as follows.

$$\left.\begin{array}{l} -\operatorname{div}(\sigma(x,t)\,\nabla\Phi(x,t)) = 0 \quad \text{in } \Omega(t), \quad \Phi = 0 \quad \text{on } \Gamma_D = \partial\Omega, \\ \Phi = \gamma_D(t) > 0 \quad \text{and} \quad \sigma\,\nabla\Phi\cdot\vec{n} = -\lambda\,\vec{v}\cdot\vec{n} \quad \text{on } \Gamma_f(t) = \partial A(t)\cap\partial\Omega(t) \end{array}\right| \tag{2.1}$$

for $t \in (0,T]$. The initial shapes $A(0)$, $\Omega(0) = \Omega\backslash A(0)$ are given and $0 < T < \infty$ is the machining time. We denote by Ω the (complete) region inside the cathode surface $\partial\Omega$.

By means of the maximum principle we verify $0 \le \Phi(x,t) \le \gamma_D(t)$ in $\Omega(t)$. Hence, the potential $\Phi(x,t)$ can be extended by continuity, i.e. we set $\Phi(x,t) = \gamma_D(t)$ in $A(t)$ for every $t > 0$. Furthermore, we represent the free boundary $\Gamma_f(t) = \partial A(t)\cap\partial\Omega(t)$ and the electrolyte region by an unknown function $\omega = \omega(x) \ge 0$, such that

$$\Gamma_f(t) = \{x \in \Omega : S(x,t) \equiv t - \omega(x) = 0\}, \quad \Omega(t) = \{x \in \Omega : t > \omega(x)\},$$

where $\omega(x) := 0$ is defined for $x \in \Omega(0)$.

Assuming the conductivity σ of the electrolyte to be *constant*, a change of the dependent variable is introduced by

$$u(x,t) = \int_0^t (\gamma_D(t') - \Phi(x,t'))\,dt' = \int_{\omega(x)}^t (\gamma_D(t') - \Phi(x,t'))\,dt', \quad x \in \overline{\Omega},\ t \in [0,T]$$

in [4] and [15], sect. 2.8. This integral (Baiocchi) transformation leads to a family of *elliptic* obstacle problems, where the time t appears only as parameter (see also Remark 2.1).

As a generalization of a constant conductivity, we consider a conductivity $\sigma = \sigma(x,t)$ depending both on the space variable x and, in particular, on the time t. Such a situation occurs in applications, when, for instance, the conductivity of the electrolyte depends on the temperature (cf. [5], sect. 6.4.5).

Let us briefly indicate the derivation of an evolutionary obstacle problem for the just mentioned situation. Calculating formally the differential equation satisfied by the new unknown $u(x,t)$, we have to distinguish the three cases

$$(\mathrm{i}) : x \in A(t), \quad (\mathrm{ii}) : x \in (A(0) \cap \Omega(t)) \quad \text{and} \quad (\mathrm{iii}) : x \in (\Omega(t) \cap \Omega(0))\,.$$

Using $\Phi(x,\omega(x)) = \gamma_D(\omega(x))$ on $\Gamma_f(t)$, we obtain

$$\nabla u(x,t) = -\int_{\omega(x)}^{t} \nabla\Phi(x,t')\; dt' = -\int_0^t \nabla\Phi(x,t')\; dt', \quad x \in \Omega,\; t \in [0,T],$$

which shows the regularizing effect of the Baiocchi transformation. Taking into account $\sigma = \sigma(x,t)$ and integrating by parts with respect to time, we verify

$$(\sigma\nabla u)(x,t) = \int_{\omega(x)}^{t} [(\partial_t\sigma\; \nabla u)(x,t') - (\sigma\; \nabla\Phi)(x,t')]\; dt'.$$

Now, let us define the differential operators

$$(Au)(x,t) = -\operatorname{div}(\sigma(x,t)\;\nabla u(x,t)) \quad \text{and} \quad (Bv)(x,t) = -\operatorname{div}(\partial_t\sigma(x,t)\;\nabla v(x,t))\,.$$

The next step is the computation of Au. We use (2.1), in particular the second (jump) condition on $\Gamma_f(t)$, which can be rewritten as $\sigma\nabla\Phi \cdot \nabla\omega(x) = -\lambda$. We obtain

$$l\;\chi_{\Omega(t)} = l\;\chi_{\Omega(0)} - (Au)(x,t) + \int_0^t (Bu)(x,t')\; dt' \quad \text{in } \Omega, \quad l = -\lambda > 0,$$

$$u = g_D(t) = \int_0^t \gamma_D(t')\; dt' \quad \text{on } \Gamma_D = \partial\Omega, \quad t \in (0,T]; \quad u(x,0) = 0, \quad \Omega(0) \text{ given.}$$

Here, $\chi_{\Omega(t)}$ (resp. $\chi_{\Omega(0)}$) denotes the characteristic function with respect to $\Omega(t)$ (resp. to $\Omega(0)$). Now, we conclude: if $\Phi \geq 0$ is a solution of the free boundary problem (2.1), then u will solve the complementarity problem

$$l \geq \int_0^t Bu\; dt' + l\;\chi_{\Omega(0)} - Au, \quad u \geq 0, \quad \left[l + Au - \int_0^t Bu\; dt' - l\;\chi_{\Omega(0)}\right] u = 0.$$

Applying Green's formula we derive the following evolutionary variational inequality.

$$\left.\begin{array}{l} \text{Find} \quad u(t) \in K(t) = \{w \in H^1(\Omega) : w \geq 0 \text{ a.e. in } \Omega,\; w = g_D(t) \text{ on } \Gamma_D = \partial\Omega\} \\ a(t; u(t), v - u(t)) \geq (F, v - u(t)) \; + \displaystyle\int_0^t b(t'; u(t'), v - u(t))\; dt' \quad \forall v \in K(t) \end{array}\right| \tag{2.2}$$

for $t \in (0,T]$ with the right-hand side $F = F(x) = (\chi_{\Omega(0)} - 1)\; l = -\chi_{A(0)}\; l$ (note that $l = -\lambda > 0$ as electro-chemical equivalent) and the initial condition $u(0) = 0$. The bilinear forms a and b are defined by

$$\left.\begin{array}{l} a(t; v(t), w) = \displaystyle\int_\Omega \sigma(x,t)\; \nabla v(x,t)\; \nabla w(x)\; dx, \\ b(t'; v(t'), w) = \displaystyle\int_\Omega \partial_t\sigma(x,t')\; \nabla v(x,t')\; \nabla w(x)\; dx. \end{array}\right| \tag{2.3}$$

We mention that, owing to $F \leq 0$ and $g_D(0) = 0$, the initial condition $u(0) = 0$ in (2.2) will automatically be satisfied if we solve the obstacle problem for $t = 0$ in (2.2).

Remark 2.1. Let us recall that such an integral transformation applied to classical one- or two-phase Stefan problems leads to parabolic variational inequalities of first or second kind (see [16] for a detailed investigation).

The elliptic inequalities with $b \equiv 0$ (i.e. $\sigma = \text{const}$) have been analysed in [11] and [15], sect. 9.5. Beside the proof of regularity results for the solution, the smoothness of the free boundary is studied in a special case (starshaped configuration). Furthermore, the convergence of the solution of the associated one-phase Stefan problem is discussed in the situation when the specific heat tends to zero.

Let us also mention that, assuming the separability of σ into $\sigma(x,t) = \sigma_1(x)\,\sigma_2(t)$, one obtains elliptic inequalities (without the memory term b) by means of the transformation $u(x,t) = \int_0^t \sigma_2(t')\,(\gamma_D(t') - \Phi(t'))\,dt'$ (see [15], sect. 2.9 and [1] concerning a similar problem arising in Hele-Shaw flows).

3. Properties of the variational solution

In the previous section we have seen that a variational inequality approach to the electrochemical machining problem with space- and time-dependent conductivity $\sigma = \sigma(x,t)$ of the electrolyte leads to evolutionary obstacle problems of the form (2.2), (2.3), which will now be investigated for more general right-hand sides $F = F(x,t)$. In the course of this, we shall pay special attention to the *time-dependent* convex sets $K(t)$ characterized by a zero obstacle and Dirichlet boundary conditions depending on the time t, but not on x.

Sometimes we will consider an equivalent inequality problem, which is obtained by a simple translation trick, i.e. we set $u(t) = \tilde{u}(t) + g_D(t)$ and get

$$\left.\begin{aligned}&\text{Find} \quad \tilde{u}(t) \in \tilde{K}(t) = \{w \in H_0^1(\Omega) : \; w \geq -g_D(t) \quad \text{a.e. in } \; \Omega\}, \; t \in [0,T],\\ &a(t;\tilde{u}(t), v - \tilde{u}(t)) \geq (F(t), v - \tilde{u}(t)) \; + \int_0^t b(t';\tilde{u}(t'), v - \tilde{u}(t))\,dt' \quad \forall v \in \tilde{K}(t).\end{aligned}\right| \tag{3.1}$$

Obviously, $a(t;u(t),v) = a(t;\tilde{u}(t),v)$ and $b(t;u(t),v) = b(t;\tilde{u}(t),v)$ for $v \in H_0^1(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary $\Gamma_D = \partial\Omega$. Furthermore, we suppose

$$\left.\begin{aligned}&\sigma \in W_\infty^1(0,T;L_\infty(\Omega)), \quad F \in C([0,T];L_2(\Omega)), \quad g_D \in C[0,T],\\ &g_D(t) \geq 0 \quad \forall t \in [0,T], \quad \sigma(x,t) \geq \sigma_0 > 0, \; x \in \Omega, \; t \in [0,T],\end{aligned}\right| \tag{3.2}$$

such that the bilinear form a is H^1-elliptic, i.e. $a(t;v,v) \geq m\,\|v\|^2_{H^1(\Omega)}$ for all $v \in H_0^1(\Omega)$ with $0 < m \neq m(t)$.

Similar to evolutionary inequality problems arising in non-isothermal Hele-Shaw flows (see [17], [18]) we prove the existence of a unique solution for (2.2)/(3.1), (2.3) by means of a fixed point argument and the theory of elliptic variational inequalities. But, due to the time dependence of K, we require in addition a convergence result for convex sets (in Mosco's sense, see [14]).

Theorem 3.1. *Under the assumptions* (3.2) *problem* (3.1), (2.3) *possesses a unique solution* $\tilde{u}(t) \in \tilde{K}(t) \;\; \forall t \in [0,T]$ *with* $\tilde{u} \in C([0,T];H_0^1(\Omega))$ *(resp.* $u \in C([0,T];H^1(\Omega))$ *for problem* (2.2), (2.3)*).*

Moreover, the estimate

$$\|u_1 - u_2\|_{C([0,T];H^1(\Omega))} \leq M \ \|F_1 - F_2\|_{C([0,T];H^{-1}(\Omega))} \quad \textit{with} \quad M = M(m,T,L) \tag{3.3}$$

is satisfied for solutions $u_i = u_i(F_i)$, $i = 1,2$, *where* $L = L(\|\sigma\|_{W^1_\infty(0,T;L_\infty(\Omega))})$.

Proof. (i). Assuming $\partial_t \sigma \equiv 0$ (i.e. without memory term b) we show at first that problem (3.1) has a unique solution $\tilde{u} \in C([0,T]; H^1_0(\Omega))$ for each $F \in C([0,T]; H^{-1}(\Omega))$.

The existence and uniqueness of $\tilde{u}(t)$ for each fixed $t \in [0,T]$ is an immediate consequence of the elliptic variational inequality theory (Lions-Stampacchia-Theorem, cf. e.g. [10], [2] and [15]).

To show the continuity of $\tilde{u} : [0,T] \ni t \longrightarrow H^1_0(\Omega)$, we use an abstract stability result (Theorem 4.4.1 in [15]; convergence of convex sets in Mosco's sense, see also [14]). Let a sequence $\{t_n\} \subset [0,T]$ with $t_n \longrightarrow t \in [0,T]$ be given. Owing to the continuity of g_D we have $g_D(t_n) \longrightarrow g_D(t)$. We conclude that $\tilde{K}(t_n) \longrightarrow \tilde{K}(t)$ holds in Mosco's sense, i.e. the following two conditions are satisfied. For all $v \in \tilde{K}(t)$ there exists a sequence $v_n \in \tilde{K}(t_n)$ with $v_n \longrightarrow v$ in $H^1(\Omega)$ (take e.g. $v_n = \max\{v, -g_D(t_n)\}$) and for any sequence $v_n \in \tilde{K}(t_n)$ with $v_n \rightharpoonup v$ (weakly) in $H^1(\Omega)$ it follows $v \in \tilde{K}(t)$.

Furthermore, we have the convergence $a(t_n; v_n, w) \longrightarrow a(t; v(t), w)$ for any sequence $\{v_n\}$ with $\tilde{K}(t_n) \ni v_n \longrightarrow v \in \tilde{K}(t)$. Taking these properties together with the assumption $F \in C([0,T]; L_2(\Omega))$, we have shown that all the conditions of the above mentioned Theorem 4.4.1 in [15] are fulfilled. Consequently, we get $\tilde{u}(t_n) \longrightarrow \tilde{u}(t)$ in $H^1(\Omega)$ for $\tilde{u}(t_n)$ and $\tilde{u}(t)$ as solutions of (3.1) for t_n and t, respectively.

Taking $v = u_{3-i}(t)$ in the inequality for $u_i(t)$, $i = 1,2$ and adding both inequalities, we obtain the estimate (3.3) (with $M = m^{-1}$) for problem (2.2) with $\partial_t \sigma \equiv 0$.

(ii). Let us define mappings $\bar{w} = U_{\alpha_j,\alpha}\, w$ for $j = 0,1,2,\ldots$, by

$$\left.\begin{aligned} &a(t; \bar{w}(t), v - \bar{w}(t)) - \alpha_j \int_0^t b(t'; \bar{w}(t'), v - \bar{w}(t'))\, dt' \geq \ (F(t), v - \bar{w}(t)) \ + \\ &+ \ (\alpha - \alpha_j) \int_0^t b(t; w(t'), v - \bar{w}(t))\, dt' \quad \forall v \in \tilde{K}(t),\ t \in [0,T]. \end{aligned}\right| \tag{3.4}$$

First of all, we consider the case $\alpha_j = \alpha_0 = 0$, $0 < \alpha \leq 1$ for the mapping (3.4). Taking an element

$$w \in \mathcal{K} = \{w \in C([0,T]; H^1_0(\Omega)) : w(t) \in \tilde{K}(t) \ \forall t \in [0,T]\},$$

we get $F_w \in C([0,T]; H^{-1}(\Omega))$ with $\langle F_w(t), v\rangle = (F(t), v) + (\alpha - \alpha_0) \int_0^t b(t'; w(t'), v)\, dt'$. Owing to step (i) the relation $\mathcal{K} \ni \bar{w} = U_{0,\alpha}\, w$ is guaranteed. To apply the Banach Fixed-Point Theorem, we check that $U_{0,\alpha}$ is contractive for solutions $\bar{w}_i = \bar{w}_i(F_{w_i})$, $i = 1,2$ with the help of estimate (3.3). We deduce

$$\|\bar{w}_1 - \bar{w}_2\|_{C([0,T];H^1(\Omega))} \leq M \ \hat{\alpha} \ \max_{t\in[0,T]} \sup_{v \in H^1_0(\Omega)} \|v\|^{-1}_{H^1(\Omega)} \left| \int_0^t b(t'; w_1(t') - w_2(t'), v)\, dt' \right| \leq$$

$$\leq M \ \hat{\alpha} \ L \int_0^T \|w_1(t') - w_2(t')\|_{H^1(\Omega)} \ dt' \leq M \ \hat{\alpha} \ T \ L \ \|w_1 - w_2\|_{C([0,T];H^1(\Omega))}$$

with $\hat{\alpha} = \alpha - \alpha_0$. Consequently, we get a fixed point mapping $\bar{w} = U_{\alpha_0=0,\alpha} w$ for $\alpha \in (0, (LTM)^{-1})$. The evolutionary variational inequality

$$a(t; u(t), v - u(t)) \geq (F(t), v - u(t)) \ + \ \alpha \int_0^t b(t'; u(t'), v - u(t))\, dt' \quad \forall v \in K(t)$$

therefore has a unique solution $u \in C([0,T]; H^1(\Omega))$ for $\alpha \in (0, (LTM)^{-1})$.

The a priori estimate (3.3) for this inequality can be derived by means of the choice $v = u_{3-i}(t)$ in the inequality for u_i, $i = 1, 2$. The addition of both inequalities leads to

$$m \left\| u_1(t) - u_2(t) \right\|_{H^1(\Omega)} \leq \left\| F_1 - F_2 \right\|_{C([0,T]; H^{-1}(\Omega))} + L \int_0^t \left\| u_1(t') - u_2(t') \right\|_{H^1(\Omega)} \, dt'$$

for all $t \in [0,T]$, and now Gronwall's inequality implies (3.3) with $M = M(T, L, m)$.

If $(MLT)^{-1} > 1$ holds, the proof is complete. Otherwise, we repeat the above considerations successively for $\bar{w} = U_{\alpha_{j+1},\alpha} w$ with $\alpha_j < \alpha_{j+1} \leq \alpha_j + (MLT)^{-1}$ until, after a finite number of steps, the situation $\alpha = 1$ is reached. □

For the further investigation of the evolutionary variational inequality (3.1) (resp. the equivalent problem (2.2)) we will consider a penalization problem given by

$$\left.\begin{aligned} &\text{Find} \quad \tilde{z}_\varepsilon(t) \in H_0^1(\Omega): \quad a(t; \tilde{z}_\varepsilon(t), v) + \left(\xi(t)\ \beta_\varepsilon\left(\tilde{z}_\varepsilon(t) + g_D(t)\right), v\right) = \\ &= (F(t) + \xi(t), v) + \int_0^t b(t'; \tilde{z}_\varepsilon(t'), v)\, dt' \quad \forall v \in H_0^1(\Omega),\ t \in [0,T] \quad \text{with} \\ &\beta_\varepsilon(w) = \beta\left(\frac{w}{\varepsilon}\right) = \frac{(w^+/\varepsilon)^2}{(w/\varepsilon + 1/2)^2} = \frac{(w^+)^2}{(w + \varepsilon/2)^2} \quad \text{for } \varepsilon > 0 \quad \text{and } \xi = \xi(x,t) \geq 0. \end{aligned}\right| \tag{3.5}$$

In what follows we denote $z_\varepsilon(t) = \tilde{z}_\varepsilon(t) + g_D(t)$. Furthermore, throughout this paper we use the notations w^+ for the positive and w^- for the negative part of a function w, i.e.

$$w^+ = w \vee 0 = \max\{w, 0\} \geq 0 \quad \text{and} \quad w^- = w \wedge 0 = \min\{w, 0\} \leq 0,$$

such that $w = w^+ + w^-$.

Remark 3.2. It is easy to see that $\beta = \beta(r) = (r^+)^2/(r + 1/2)^2$ is a nondecreasing Lipschitz (actually a C^1) function $\beta : \mathbb{R} \longrightarrow [0,1]$ with $\beta(r) = 0$ for $r \leq 0$, $\beta(+\infty) = 1$, $(1 - \beta(r))\, r \leq 1$ for $r \geq 0$. These properties imply in a sense the convergence of $\beta_\varepsilon(r) = \beta(r/\varepsilon)$ to the Heaviside graph as $\varepsilon \longrightarrow 0$ (see also [15], sect. 5.3).

The penalization method defined in (3.5) can be interpreted as a regularization of

$$\left(F(x,t) + \int_0^t (Bu)(x,t')\, dt' - (Au)(x,t) \right) \ \in \ \partial j(u(x,t)),$$

where A and B are the elliptic differential operators associated with the bilinear forms a and b, respectively. The functional $j = j(r)$ is given by $j(r) = +\infty$ for $r < 0$ and $j(r) = 0$ for $r \geq 0$. Its subdifferential (see [3] and [21], chap. 54) is denoted by ∂j.

In the following lemma we summarize some properties of the solution $\tilde{z}_\varepsilon$ of the just defined auxiliary problem (3.5).

Lemma 3.3.

(i) *Let the assumptions* (3.2) *and* $\xi \in C([0,T];L_2(\Omega))$ *be fulfilled. Then, there exists a unique solution* $\tilde{z}_\varepsilon \in C([0,T];H_0^1(\Omega))$ *of problem* (3.5). *Moreover, one has*

$$\|\tilde{z}_\varepsilon\|_{C([0,T];H^1(\Omega))} \le C \quad \textit{independent of} \quad \varepsilon > 0.$$

(ii) *If, additionally,* F *and* ξ *belong to* $W^1_\infty(0,T;L_2(\Omega))$ *and* $g_D \in C^{0,1}[0,T]$, *the solution* $\tilde{z}_\varepsilon$ *of* (3.5) *will be such that*

$$\tilde{z}_\varepsilon \in W^1_\infty(0,T;H_0^1(\Omega)) \quad \textit{for each fixed} \quad \varepsilon > 0 \quad \textit{and}$$
$$\|\tilde{z}_\varepsilon\|_{C^{0,1/2}([0,T];H^1(\Omega))} \le C \quad \textit{independent of} \quad \varepsilon > 0.$$

(iii) *Together with the assumptions of part* (ii) *let the following conditions be fulfilled:* $\xi(t) \in L_p(\Omega)$ *with* $p = \max\{2, n/2\}$, $\partial_t F(t) + (1-r)\ \partial_t \xi(t) \ge 0$ *a.e in* Ω, $\forall r \in [0,1]$, $\partial_t g_D(t) \ge 0$, *a.e. in* $(0,T)$. *Then,*

$$\partial_t \tilde{z}_\varepsilon \ge -\partial_t g_D \quad \textit{a.e. in } Q = \Omega \times (0,T),$$
$$\|\partial_t \tilde{z}_\varepsilon\|_{L_2(0,T;H^1(\Omega))} \le C \quad \textit{independent of } \varepsilon > 0.$$

(iv) *Under the further assumption* $\partial_t F = \partial_t \xi = 0$, *one has* $\partial_t \tilde{z}_\varepsilon \le 0$ *a.e. in* $Q = \Omega \times (0,T)$.

Proof. (i). Applying the fixed point argument used in the proof of Theorem 3.1 in connection with the theory for elliptic equations (with monotone operators), the existence of a unique solution $\tilde{z}_\varepsilon \in C([0,T];H_0^1(\Omega))$ for problem (3.5) is deduced. To show the boundedness of $\tilde{z}_\varepsilon$, we take $v = \tilde{z}_\varepsilon(t)$ in (3.5). Owing to $\xi \ge 0$ and $\beta_\varepsilon(r)\, r \ge 0$ we get

$$\left(m - \frac{\gamma_1+\gamma_2}{2}\right) \|\tilde{z}_\varepsilon(t)\|^2_{H^1(\Omega)} \le \gamma_1^{-1} \left(\|F\|^2_{C([0,T];L_2(\Omega))} + \|\xi\|^2_{C([0,T];L_2(\Omega))}\right) +$$
$$+\ C\ \|g_D\|_{C[0,T]}\ \|\xi\|_{C([0,T];L_2(\Omega))} + \frac{TL^2}{2\gamma_2}\int_0^t \|\tilde{z}_\varepsilon(t')\|^2_{H^1(\Omega)}\,dt' \quad \forall t \in [0,T]$$

by means of Young's inequality (with the parameters $\gamma_1, \gamma_2 > 0$). Finally, the application of Gronwall's inequality leads to the desired estimate.

(ii). Subtracting the equations in (3.5) for $t_i \in [0,T]$, $i = 1,2$ with $t_1 \ne t_2$, we obtain

$$a(t_1;\tilde{z}_\varepsilon(t_1) - \tilde{z}_\varepsilon(t_2), v) + \left(\xi(t_1)\ \left[\beta_\varepsilon(z_\varepsilon(t_1)) - \beta_\varepsilon(z_\varepsilon(t_2))\right], v\right) =$$
$$= \left(F(t_1) + \xi(t_1) - F(t_2) - \xi(t_2) + [\xi(t_2) - \xi(t_1)]\ \beta_\varepsilon(z_\varepsilon(t_2))\ ,\ v\right) +$$
$$+\ a(t_2;\tilde{z}_\varepsilon(t_2), v) - a(t_1;\tilde{z}_\varepsilon(t_2), v) + \int_{t_2}^{t_1} b(t';\tilde{z}_\varepsilon(t'), v)\ dt'.$$

Taking $v = \tilde{z}_\varepsilon(t_1) - \tilde{z}_\varepsilon(t_2)$ and using the monotonicity $(\beta_\varepsilon(r) - \beta_\varepsilon(s))(r-s) \ge 0$, we get

$$m\,\|\tilde{z}_\varepsilon(t_1) - \tilde{z}_\varepsilon(t_2)\|^2_{H^1(\Omega)} \le |t_1 - t_2| \left\{ \left\|\frac{F(t_1)-F(t_2)}{t_1-t_2}\right\|_{L_2(\Omega)} + 2\left\|\frac{\xi(t_1)-\xi(t_2)}{t_1-t_2}\right\|_{L_2(\Omega)} + \right.$$
$$\left. + \left\|\frac{\sigma(t_1)-\sigma(t_2)}{t_1-t_2}\right\|_{L_\infty(\Omega)} \|\tilde{z}_\varepsilon(t_2)\|_{H^1(\Omega)} + L\ \|\tilde{z}_\varepsilon\|_{C([0,T];H^1(\Omega))} \right\} \|\tilde{z}_\varepsilon(t_1) - \tilde{z}_\varepsilon(t_2)\|_{H^1(\Omega)} +$$
$$+\ |t_1 - t_2|\ \|\xi(t_1)\|_{L_2(\Omega)} \left|\frac{g_D(t_1) - g_D(t_2)}{t_1-t_2}\right|\ \|\beta_\varepsilon(z_\varepsilon(t_1)) - \beta_\varepsilon(z_\varepsilon(t_2))\|_{L_2(\Omega)}\,.$$

The crucial point is now the last term. We conclude $\tilde{z}_\varepsilon \in C^{0,1/2}([0,T];H_0^1(\Omega))$ uniformly in $\varepsilon > 0$ by means of $\|\beta_\varepsilon(z_\varepsilon(t_1)) - \beta_\varepsilon(z_\varepsilon(t_2))\|_{L_2(\Omega)} \le C$ and Young's inequality.

On the other hand, using $|\beta_\varepsilon(r) - \beta_\varepsilon(s)| \le \varepsilon^{-1}|r-s|$, we deduce $\tilde{z}_\varepsilon \in W^1_\infty(0,T;H_0^1(\Omega))$, but not uniformly in $\varepsilon > 0$.

(iii). To deduce monotonicity properties of the solution $\tilde{z}_\varepsilon$ of problem (3.5), we differentiate the penalty equation (3.5) with respect to time.

$$\left.\begin{aligned} &\text{Find } w_\varepsilon(t) \in H_0^1(\Omega) : a(t;w_\varepsilon(t),v) + \big(\xi(t)\ \beta'_\varepsilon(z_\varepsilon(t))\ (w_\varepsilon(t) + \partial_t g_D(t)), v\big) = \\ &= \big(\partial_t F(t) + \partial_t \xi(t)\,[1 - \beta_\varepsilon(z_\varepsilon(t))], v\big) \ \ \forall v \in H_0^1(\Omega), \text{ a.e. in } (0,T). \end{aligned}\right| \tag{3.6}$$

Owing to $\beta'_\varepsilon \ge 0$, $\xi \ge 0$ and the assumption $\xi(t) \in L_p(\Omega)$ (note the Sobolev embedding $H^1(\Omega) \subset L_q(\Omega)$ with q depending on the space dimension n; see e.g. [21]), problem (3.6) has the unique solution $w_\varepsilon = \partial_t \tilde{z}_\varepsilon$.

To show $\partial_t z_\varepsilon \ge 0$, we take $0 \ge v = (\partial_t z_\varepsilon(t))^- \in H_0^1(\Omega)$ in (3.6). By means of $\xi\ \beta'_\varepsilon \ge 0$, we derive

$$\|(\partial_t z_\varepsilon(t))^-\|_{H^1(\Omega)} \le 0,$$

which proves the first statement of part (iii).

To prove the boundedness of $\partial_t \tilde{z}_\varepsilon$, we put $v = \partial_t \tilde{z}_\varepsilon(t) \in H_0^1(\Omega)$ in (3.6), which leads to

$$\begin{aligned} &m\,\|\partial_t \tilde{z}_\varepsilon(t)\|^2_{H^1(\Omega)} + \big(\xi(t)\ \beta'_\varepsilon(z_\varepsilon(t))\,\partial_t\ z_\varepsilon(t),\ \partial_t z_\varepsilon(t)\big) \le \\ &\le \Big[\|\partial_t F\|_{L_2(\Omega)} + \|\partial_t \xi\|_{L_2(\Omega)}\Big]\ \|\partial_t \tilde{z}_\varepsilon(t)\|_{H^1(\Omega)} + \big(\xi(t)\ \beta'_\varepsilon(z_\varepsilon(t))\ \partial_t z_\varepsilon(t),\ \partial_t g_D(t)\big). \end{aligned}$$

Using Young's inequality and noting $0 \le \beta'_\varepsilon(z_\varepsilon(t))\ \partial_t z_\varepsilon(t) = \partial_t \beta_\varepsilon(z_\varepsilon(t))$, we arrive at

$$C_1\ \|\partial_t \tilde{z}_\varepsilon(t)\|^2_{H^1(\Omega)} \le C_2\ \Big[\|\partial_t F\|^2_{L_2(\Omega)} + \|\partial_t \xi\|^2_{L_2(\Omega)}\Big] +\ \partial_t g_D(t)\ \big(\ \partial_t \beta_\varepsilon(z_\varepsilon(t))\ ,\ \xi(t)\big).$$

By integration in time we obtain

$$\begin{aligned} &C_1\,\|\partial_t \tilde{z}_\varepsilon\|^2_{L_2(0,T;H^1(\Omega))} \le C_2\ \Big[\|\partial_t F\|^2_{L_2(0,T;L_2(\Omega))} + \|\partial_t \xi\|^2_{L_2(0,T;L_2(\Omega))}\Big]\ + \\ &+\ \|\partial_t g_D\|_{L_\infty(0,T)} \int_0^T \big(\ \partial_t \beta_\varepsilon(z_\varepsilon(t))\ ,\ \xi(t)\big)\ dt. \end{aligned}$$

Integrating the last term by parts with respect to time we get $\|\partial_t \tilde{z}_\varepsilon\|_{L_2(0,T;H^1(\Omega))} \le C$.

(iv). The statement follows with $0 \le v = (\partial_t \tilde{z}_\varepsilon(t))^+ \in H_0^1(\Omega)$ in (3.6). Recalling the special assumption of part (iv), i.e. $\partial_t F = \partial_t \xi = 0$, we have

$$\begin{aligned} &m\,\big\|(\partial_t \tilde{z}_\varepsilon(t))^+\big\|^2_{H^1(\Omega)} + \big(\xi(t)\ \beta'_\varepsilon(z_\varepsilon(t))\ \partial_t \tilde{z}_\varepsilon(t)\ ,\ (\partial_t \tilde{z}_\varepsilon(t))^+\big) \le \\ &\le\ -\ \big(\xi(t)\ \beta'_\varepsilon(z_\varepsilon(t))\ \partial_t g_D(t)\ ,\ (\partial_t \tilde{z}_\varepsilon(t))^+\big) \le 0 \end{aligned}$$

and, hence, $(\partial_t \tilde{z}_\varepsilon(t))^+ = 0$ in $H^1(\Omega)$. $\square$

Let us now perform the passage to the limit as $\varepsilon \longrightarrow 0$ in the penalization problem (3.5). As a result we will derive monotonicity and regularity properties of the solution u of the variational inequality (2.2), (2.3).

Theorem 3.4. *In addition to the preceding assumptions of* Lemma 3.3(iii) *we demand the condition* $F(x,0)+\xi(x,0)\geq 0$ *a.e. in* Ω.

(i) *The unique solution* $\tilde{z}_\varepsilon = \tilde{z}_\varepsilon(t)$ *of the penalization problem* (3.5) *belongs to* $\tilde{K}(t)$ *(resp.* $z_\varepsilon(t) = (\tilde{z}_\varepsilon(t)+g_D(t)) \in K(t)$*) for each* $\varepsilon > 0$ *and for all* $t \in [0,T]$. *Moreover, the estimate*

$$\|u - z_\varepsilon\|_{C([0,T];H^1(\Omega))} \leq C\ \sqrt{\varepsilon}\ \|\xi\|^{1/2}_{C([0,T];L_1(\Omega))}$$

is satisfied, where u *is the unique solution of the obstacle problem* (2.2), (2.3).

(ii) *The unique solution* u *of* (2.2), (2.3) *is such that*

$$u \in H^1(0,T;H^1(\Omega)) \quad \text{and} \quad \partial_t u \geq 0 \quad \text{a.e. in } Q = \Omega\times(0,T).$$

Supposing additionally $\partial_t F = \partial_t \xi = 0$, *one has* $0 \leq \partial_t u \leq \partial_t g_D$ *a.e. in* Q.

Proof. (i). We consider the penalization problem (3.5) for $t=0$, which coincides with the elliptic equation

$$a(0;z_\varepsilon(0),v) + (\xi(0)\ \beta_\varepsilon\left(z_\varepsilon(0)\right),v) = (F(0)+\xi(0),v) \quad \forall v \in H^1_0(\Omega).$$

Due to $g_D(0)\geq 0$, we can take $0 \geq v = (z_\varepsilon(0))^- \in H^1_0(\Omega)$ in this equation. Therefore, $z_\varepsilon(0) \geq 0$ holds and, hence, $z_\varepsilon(0) \in K(0)$ is guaranteed. Owing to Lemma 3.3(iii) we have $z_\varepsilon(t) \geq z_\varepsilon(0) \geq 0$ in Ω for $t \geq 0$. Hence, it follows $z_\varepsilon(t) \in K(t)$ for all $t \in [0,T]$.

Consequently, it is allowed to take $v = z_\varepsilon(t)$ in (2.2) and $v = (u(t)-z_\varepsilon(t)) \in H^1_0(\Omega)$ in (3.5). Recalling the notation $\tilde{u} = u - g_D$, we obtain

$$\begin{aligned} a(t;\tilde{z}_\varepsilon(t)-\tilde{u}(t),\tilde{z}_\varepsilon(t)-\tilde{u}(t)) &\leq (\xi(t)\ \left[1-\beta_\varepsilon\left(z_\varepsilon(t)\right)\right], z_\varepsilon(t)-u(t))\ + \\ &+ \int_0^t b(t';\tilde{z}_\varepsilon(t')-\tilde{u}(t'),\tilde{z}_\varepsilon(t)-\tilde{u}(t))\ dt' \end{aligned}$$

by subtraction. Due to $(1-\beta(r))\ r \leq 1$ for $r \geq 0$ (see Remark 3.2) and the relations $\xi \geq 0$, $\beta_\varepsilon\left(z_\varepsilon(t)\right) \leq 1$, $u \geq 0$, the first term on the right-hand side can be estimated by

$$\left(\xi(t)\ [1-\beta_\varepsilon\left(z_\varepsilon(t)\right)]\ ,\ z_\varepsilon(t)-u(t)\right) \leq \left(\xi(t)\ [1-\beta_\varepsilon\left(z_\varepsilon(t)\right)]\ ,\ z_\varepsilon(t)\right) \leq \varepsilon\ \ (\xi(t),1)\,.$$

Now, using Young's inequality, we arrive at

$$\begin{aligned} m\,\|\tilde{z}_\varepsilon(t)-\tilde{u}(t)\|^2_{H^1(\Omega)} &\leq \varepsilon\ \ \|\xi\|_{C([0,T];L_1(\Omega))} + \frac{L\,\gamma}{2}\ \|\tilde{z}_\varepsilon(t)-\tilde{u}(t)\|^2_{H^1(\Omega)} + \\ &+ \frac{L\,T}{2\gamma}\int_0^t \|\tilde{z}_\varepsilon(t)-\tilde{u}(t')\|^2_{H^1(\Omega)}\ dt' \quad \forall t\in[0,T],\ \gamma>0, \end{aligned}$$

such that after the application of Gronwall's inequality the statement is concluded.

(ii). Owing to the statements of Lemma 3.3(i), (iii) we can extract a subsequence $\tilde{z}_{\varepsilon'}$ of $\tilde{z}_\varepsilon$, such that $\tilde{z}_{\varepsilon'} \rightharpoonup z$ and $\partial_t \tilde{z}_{\varepsilon'} \rightharpoonup \partial_t z$ (weakly) in $L_2(0,T;H^1(\Omega))$. The function z belongs to $H^1(0,T;H^1_0(\Omega))$.

But in the previous part (i) we have proved the strong convergence $\tilde{z}_\varepsilon \longrightarrow \tilde{u} = u - g_D$ in $C([0,T]; H^1(\Omega))$, such that z and $u - g_D$ coincide as mappings from $[0,T] \longrightarrow H_0^1(\Omega)$. Hence, we have $u \in H^1(0,T; H^1(\Omega))$.

Finally, the monotonicity results for the variational inequality solution u are obtained by means of the statements of Lemma 3.3(iii), (iv). □

Remark 3.5. In the situation $\partial_t F \geq 0$ the monotony assumptions of Lemma 3.3(iii) and Theorem 3.4 are satisfied, for instance, by the choice $\xi = \xi(t) = \xi(0) = (-F(0))^+$ independent of t. Moreover, in the electro-chemical machining problem one even has $\partial_t F = 0$, such that the last statement of Theorem 3.4(ii) is also fulfilled.

Finally, let us prove that the electro-chemical machining inequality (2.2), (2.3), which was given in section 2 with $F = F(x) = -\chi_{A(0)}\, l \leq 0$ and $g_D(t) = \int_0^t \gamma_D(t')\, dt'$ as a special case of the inequalities investigated in this section, has a unique solution u belonging to $W^1_\infty(0,T; H^1(\Omega))$. In order to demonstrate this property, we first propose a general criteria and then this criteria will be checked for the special ECM inequality.

Lemma 3.6. *Consider* (3.2) *together with the conditions* $F \in W^1_\infty(0,T,L_2(\Omega))$ *and* $g_D \in C^{0,1}[0,T]$. *Suppose that a function*

$$\delta \in W^1_\infty(0,T; H^1(\Omega)) \ \ \textit{with} \ \ \delta(x,t) = -g_D(t) \ \ \textit{for} \ x \in \partial\Omega \quad \textit{and} \quad \delta(x,t) \leq 0 \ \ \textit{for} \ x \in \Omega$$

is given for all $t \in [0,T]$, *such that the function* $\eta = \eta(x,t)$ *defined by* $\eta := u + \delta$ *satisfies* $\eta(x,t) \geq 0$.

Then, the solution u *of the problem* (2.2), (2.3) *belongs to* $W^1_\infty(0,T; H^1(\Omega))$.

Proof. To verify the assertion, we substitute $u = \eta - \delta$ into (2.2), which leads to

$$\begin{aligned} a(t;\eta(t), w(t)-\eta(t)) &\geq (F(t), w(t)-\eta(t)) \ + \\ &+ \ a(t;\delta(t), w(t)-\eta(t)) + \int_0^t b(t'; u(t'), w(t)-\eta(t))\, dt' \end{aligned}$$

for all functions $w(t) = v + \delta(t)$ with $v \in K(t)$, i.e. $w(t) \in H_0^1(\Omega)$, $w(t) \geq \delta(t)$, $t \in [0,T]$. Owing to $\delta(x,t) \leq 0$ and $H_0^1(\Omega) \ni \eta(t) \geq 0$ it is now allowed to take $w(t_i) = \eta(t_{3-i})$ for $t_i \in [0,T]$, $i = 1,2$. Adding the inequalities for t_1 and t_2, we arrive at

$$\begin{aligned} & a(t_1; \eta(t_1)-\eta(t_2), \eta(t_1)-\eta(t_2)) \leq (F(t_1)-F(t_2), \eta(t_1)-\eta(t_2)) \ + \\ & + \ a(t_2; \eta(t_2)-\delta(t_2), \eta(t_1)-\eta(t_2)) - \ a(t_1; \eta(t_2)-\delta(t_2), \eta(t_1)-\eta(t_2)) \ + \\ & + \ a(t_1; \delta(t_1)-\delta(t_2), \eta(t_1)-\eta(t_2)) + \int_{t_2}^{t_1} b(t'; u(t'), \eta(t_1)-\eta(t_2))\, dt'. \end{aligned}$$

By means of the estimate

$$\begin{aligned} & m\, \|\eta(t_1)-\eta(t_2)\|_{H^1(\Omega)} \leq \|F(t_1)-F(t_2)\|_{L_2(\Omega)} + \|\sigma(t_1)\|_{L_\infty(\Omega)}\, \|\delta(t_1)-\delta(t_2)\|_{H^1(\Omega)} \ + \\ & + \ \|\sigma(t_2)-\sigma(t_1)\|_{L_\infty(\Omega)} \left[\|\eta(t_2)\|_{H^1(\Omega)} + \|\delta(t_2)\|_{H^1(\Omega)} \right] + L\ |t_1-t_2|\ \ \|u\|_{C([0,T];H^1(\Omega))}, \end{aligned}$$

we verify $\eta \in W^1_\infty(0,T; H^1(\Omega))$ and, thus, $u \in W^1_\infty(0,T; H^1(\Omega))$ is proved. □

Let us now establish corresponding functions δ and η for the electro-chemical machining problem introduced in section 2. We emphasize that the underlying idea is a generalization of an argument which has already been used in [11] and [15] for the ECM problem with $\sigma = \text{const}$ (cf. Remark 2.1).

Corollary 3.7. *Under the assumptions* (3.2) *the solution* u *of the ECM problem* (2.2), (2.3) *with* $F(x) = -\chi_{A(0)}\, l$ *and* $g_D(t) = \int_0^t \gamma_D(t')\, dt'$, $\gamma_D \in C[0,T]$, $\gamma_D(t) \geq \gamma_0 > 0$ *for all* $t \in [0,T]$ *is such that* $u \in W^1_\infty(0,T; H^1(\Omega))$.

Proof. Based on the penalization problem (3.5) and its time derivative (3.6) we construct a function

$$H^1_0(\Omega) \ni \eta_\varepsilon(t) = (z_\varepsilon(t) + \delta(t)) \geq 0, \ t \in [0,T].$$

Then the desired function $0 \leq \eta = u + \delta$ will be obtained by passing to the limit as $\varepsilon \longrightarrow 0$.

We mention that, owing to $\partial_t F = 0$, we take $0 \leq \xi = \xi(x) = (-F(x))^+$ in the penalization problem (3.5) (cf. Remark 3.5).

Let G be a smooth closed subset of Ω, such that $A(0) \subset G \subset \Omega$ (cf. Figure 1). We define the function δ by

$$\delta(x,t) := \int_0^t \rho(x,t')\, \gamma_D(t')\, dt',$$

where $\rho = \rho(x,t)$ is given as solution of the auxiliary problem

$$\left.\begin{aligned} &\int_{\Omega\setminus G} \sigma(x,t)\, \nabla\rho(x,t)\, \nabla v(x)\, dx = 0 \quad \forall v \in H^1(\Omega\setminus G), \quad v = 0 \quad \text{on } \partial\Omega,\ \partial G; \\ &\rho(x,t) = -1,\ \ x \in \partial\Omega; \quad \rho(x,t) = 0,\ \ x \in \partial G; \quad t \in [0,T]. \end{aligned}\right| \tag{3.7}$$

The function ρ is extended by 0 into the subset G, such that $\rho \in C([0,T]; H^1(\Omega))$. By means of the maximum principle we get $0 \geq \rho(x,t) \geq -1$. Therefore, we have

$$\delta \in W^1_\infty(0,T; H^1(\Omega)), \quad \delta(x,0) = 0, \quad 0 \geq \delta(x,t) \geq -\int_0^t \gamma_D(t')\, dt' = -g_D(t).$$

Thus, we obtain $0 \leq \eta_\varepsilon(0) = z_\varepsilon(0) \in K(0)$ (cf. Theorem 3.4(i)). To have $0 \leq \eta_\varepsilon(t)$ for $t \geq 0$, it remains to show $\partial_t \eta_\varepsilon(t) \geq 0$ in Ω.

By virtue of $\partial_t \eta_\varepsilon = \partial_t z_\varepsilon \geq 0$ in G and $\partial_t \eta_\varepsilon = 0$ on $\partial\Omega$, we have $(\partial_t \eta_\varepsilon(t))^- = 0$ in G and on $\partial\Omega$. By means of problem (3.6) (note $\xi = \chi_{A(0)}\, l$, $A(0) \subset G$) and the auxiliary problem (3.7) we deduce

$$\int_\Omega \sigma\, \nabla\partial_t \tilde{z}_\varepsilon\, \nabla\left((\partial_t\eta_\varepsilon)^-\right)\, dx = -\int_\Omega \beta'_\varepsilon(z_\varepsilon)\, \partial_t z_\varepsilon\, (\partial_t\eta_\varepsilon)^-\, \chi_{A(0)}\, l\, dx = 0 \quad \text{and}$$

$$(\gamma_D(t))^{-1} \int_{\Omega\setminus G} \sigma\, \nabla(\partial_t\delta)\, \nabla\left((\partial_t\eta_\varepsilon)^-\right)\, dx = \int_{\Omega\setminus G} \sigma\, \nabla\rho\, \nabla\left((\partial_t\eta_\varepsilon)^-\right)\, dx = 0.$$

Consequently, we obtain

$$m \left\|(\partial_t\eta_\varepsilon(t))^-\right\|^2_{H^1(\Omega)} \leq a(t;(\partial_t\eta_\varepsilon(t))^-,(\partial_t\eta_\varepsilon(t))^-) = a(t;\partial_t(z_\varepsilon(t)+\delta(t)),(\partial_t\eta_\varepsilon(t))^-) \leq 0,$$

such that $(\partial_t\eta_\varepsilon(t))^- = 0$ in $H^1(\Omega)$ holds and therefore $\partial_t\eta_\varepsilon(t) \geq 0$. Hence, we end up with $\eta_\varepsilon(t) = \eta_\varepsilon(0) + \int_0^t \partial_t\eta_\varepsilon(t')\,dt' \geq 0$. □

4. Spatial regularity

In this section we concentrate our study on the spatial smoothness of the solution of the evolutionary obstacle problems (2.2)/(3.1), (2.3). We shall employ a semi-discretization in time procedure (Rothe's method, see [9]) in connection with the regularity theory and a priori estimates for second order elliptic equations. These auxiliary tools are applied to the penalization problem and after that a passage to the limit as $\varepsilon \longrightarrow 0$ yields the assertions for the solution of the variational inequalities. This framework is similar to the investigations of evolutionary variational inequalities with Robin boundary conditions arising in non-isothermal Hele-Shaw flows (see [17]).

Let us recall the regularity statement that the solution w of the Dirichlet boundary value problem

$$-\operatorname{div}(k(x)\,\nabla w) = f(x) \quad \text{in } \Omega \qquad \text{with } w = 0 \quad \text{on} \quad \Gamma_D = \partial\Omega$$

belongs to the Sobolev space $W^2_p(\Omega)$ $(2 \leq p < \infty)$, provided $k \in C^1(\overline{\Omega})$, $k(x) \geq k_0 > 0$, $f \in L_p(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a domain with a $C^{1,1}$ boundary (see e.g. [7], chap. 2 and [15], sect. 3.7). As a second major tool we will exploit a general a priori estimate of the form

$$\|w\|_{W^2_p(\Omega)} \leq C\ \left(\|\operatorname{div}(k\nabla w)\|_{L_p(\Omega)} + \|w\|_{W^1_p(\Omega)}\right) \quad \forall w \in W^2_p(\Omega) \tag{4.1}$$

with a constant $C > 0$ (see [7], sect. 2.3.3). In addition to the conditions (3.2) we shall assume

$$\sigma \in W^1_\infty(0,T;C^1(\overline{\Omega})), \quad \text{boundary } \Gamma_D = \partial\Omega \text{ of } \Omega \text{ is of the class } C^{1,1}. \tag{4.2}$$

In the following lemma we discuss the spatial regularity of the solution of an auxiliary problem given by: Find $z(t) \in H^1_0(\Omega)$ such that

$$a(t;z(t),v) = (F(t),v)\ + \int_0^t b(t';z(t'),v)\,dt' \quad \forall v \in H^1_0(\Omega),\ t \in [0,T] \tag{4.3}$$

with the bilinear forms a, b defined in (2.3). Moreover, the regularity of z as the solution of problem (4.3) will be extended to the solution $\tilde{z}_\varepsilon$ of the penalization problem (3.5).

Lemma 4.1.

(i) *Suppose that the conditions* (3.2) *and* (4.2) *hold. Then, for every right-hand side* $F \in L_2(0,T;L_p(\Omega))$, *there exists a function*

$$z \in \left(C([0,T];H^1_0(\Omega))\ \cap\ L_2(0,T;W^2_p(\Omega))\right),$$

which is the unique solution of problem (4.3) *and satisfies*

$$\|z\|_{L_2(0,T;W_p^2(\Omega))} \leq C_1 \ \|F\|_{L_2(0,T;L_p(\Omega))} + C_2 \ \|F\|_{C([0,T];L_2(\Omega))}, \quad 2 \leq p < \infty.$$

Assume additionally that $\xi \in L_2(0,T;L_p(\Omega)) \cap C([0,T];L_2(\Omega))$ *holds. Then, the unique solution* $\tilde{z}_\varepsilon$ *of the penalization problem* (3.5) *satisfies*

$$\tilde{z}_\varepsilon \in \left(C([0,T];H_0^1(\Omega)) \ \cap \ L_2(0,T;W_p^2(\Omega))\right),$$
$$\|\tilde{z}_\varepsilon\|_{L_2(0,T;W_p^2(\Omega))} \leq C \quad \textit{uniformly in} \quad \varepsilon > 0.$$

(ii) *In addition to part* (i) *suppose that* F, ξ *belong to* $L_\infty(0,T;L_p(\Omega))$. *Then, the solution* z *of* (4.3) *belongs to* $L_\infty(0,T;W_p^2(\Omega))$ *and there exist constants* $C_1, C_2 > 0$, *such that*

$$\|z\|_{L_\infty(0,T;W_p^2(\Omega))} \leq C_1 \ \|F\|_{L_\infty(0,T;L_p(\Omega))} + C_2 \ \|F\|_{C([0,T];L_2(\Omega))}, \quad 2 \leq p < \infty.$$

Furthermore, the unique solution $\tilde{z}_\varepsilon$ *of* (3.5) *fulfills*

$$\tilde{z}_\varepsilon \in \left(C([0,T];H_0^1(\Omega)) \ \cap \ L_\infty(0,T;W_p^2(\Omega))\right), \quad \|\tilde{z}_\varepsilon\|_{L_\infty(0,T;W_p^2(\Omega))} \leq C$$

uniformly in $\varepsilon > 0$.

Proof. Recalling the existence results derived in section 3, it remains to show the spatial regularity. Using a time discretization procedure we prove at first $z \in L_2(0,T;W_p^2(\Omega))$. Afterwards we deduce $z \in L_\infty(0,T;W_p^2(\Omega))$ in part (ii) and, finally, the spatial regularity is extended to the solution $\tilde{z}_\varepsilon$ of the penalization problem (3.5) in part (iii).

(i). Let $[0,T]$ be discretized into $\{t_j = j\tau, \ j = 0,\ldots,N\}$, $N\tau = T$, where N will later tend towards $+\infty$. We define a sequence $\{w_j\}_{j=0}^{N-1}$ by the solutions of the elliptic problems

$$a(t_j;w_j,v) = (F_j,v) + \sum_{i=0}^{j-1} \int_{i\tau}^{(i+1)\tau} b(t';w_i,v) \ dt' \quad \forall v \in H_0^1(\Omega) \tag{4.4}$$

with $F_j = \frac{1}{\tau}\int_{j\tau}^{(j+1)\tau} F(t') \ dt'$. On the one hand, taking $v = w_j$ in (4.4) and applying Gronwall's inequality in its discrete version (cf. Remark 4.2(ii)), we get

$$\|w_j\|_{H^1(\Omega)} \leq c_1 \ \|F_j\|_{L_2(\Omega)} + c_2 \ \|F\|_{L_2(0,T;L_2(\Omega))}, \tag{4.5}$$

where c_1, c_2 do not depend on j and τ. On the other hand, due to the elliptic regularity (cf. [7], Theorem 2.4.2.5 and [15], Theorem 3.7.4) we successively conclude $w_j \in W_p^2(\Omega)$ for $j = 0,\ldots,N-1$. Furthermore, there is an a priori estimate of the form

$$\|w_j\|_{W_p^2(\Omega)} \leq C \ \left(\|\hat{F}_j\|_{L_p(\Omega)} + \|w_j\|_{H^1(\Omega)}\right), \quad \hat{F}_j = F_j + \sum_{i=0}^{j-1} \int_{i\tau}^{(i+1)\tau} B(t') \ w_i \ dt' \tag{4.6}$$

with the differential operator $B(t') \ w = -\operatorname{div}(\partial_t\sigma(t') \ \nabla w)$. In particular, the constant C can be chosen independent of j and τ (see (4.1) and cf. Remark 4.2(i) below). Due to the regularity $w_i \in W_p^2(\Omega)$ we have $\|\hat{F}_j\|_{L_p(\Omega)} \leq \|F_j\|_{L_p(\Omega)} + \bar{C} \ \tau \ \sum_{i=0}^{j-1} \|w_i\|_{W_p^2(\Omega)}$.

Therefore, by means of (4.5) and the discrete Gronwall inequality, the estimate (4.6) is rewritten as

$$\|w_j\|_{W_p^2(\Omega)} \leq \tilde{C}_1 \ \|F\|_{L_2(0,T;L_2(\Omega))} + \tilde{C}_2 \ \|F_j\|_{L_p(\Omega)} + \tilde{C}_3 \ \tau \sum_{i=0}^{j-1} \|F_i\|_{L_p(\Omega)} \, , \tag{4.7}$$

where the constants $\tilde{C}_i$, $i = 1, 2, 3$ are independent of j and τ. Now, we define step functions F_τ and w_τ by

$$F_\tau(t) = F_j \quad \text{for} \quad j\tau \leq t < (j+1)\tau, \quad j = 0, \ldots, N-1 \quad \text{and} \quad F_\tau(T) = F_{N-1}$$

(analogously w_τ based on w_j). By means of Hölder's inequality we get

$$\|F_\tau\|_X = \left(\tau \sum_{j=0}^{N-1} \|F_j\|^2_{L_p(\Omega)} \right)^{1/2} \leq \|F\|_X \, , \quad \tau \sum_{i=0}^{j-1} \|F_i\|_{L_p(\Omega)} \leq \ \sqrt{t_j} \ \|F\|_{L_2(0,t_j;L_p(\Omega))}$$

with $X = L_2(0, T; L_p(\Omega))$. Combining these estimates with (4.7) and performing some calculations, we find that w_τ can be bounded by $\|w_\tau\|_{L_2(0,T;W_p^2(\Omega))} \leq \bar{C}_1 \ \|F\|_{L_2(0,T;L_p(\Omega))}$. Therefore, w_τ remains in a bounded subset of $L_2(0, T; W_p^2(\Omega))$. Hence, extracting a subsequence, again denoted by w_τ, we get $w_\tau \rightharpoonup w$ in $L_2(0, T; W_p^2(\Omega))$ (weakly). Integrating (4.4) in time and multiplying it by $v \in L_2(0, T; H_0^1(\Omega))$, we derive

$$\int_0^T a(t; w_\tau(t), v(t)) \, dt - \int_0^T \big(a(t; w_\tau(t), v(t)) - a(\tilde{t}; w_\tau(t), v(t)) \big) \ dt =$$
$$= \int_0^T (F_\tau(t), v(t)) \, dt \ + \int_0^T \left(\int_0^t b(t'; w_\tau(t'), v(t)) \, dt' - \int_{\tilde{t}}^t b(t'; w_\tau(t'), v(t)) \, dt' \right) dt,$$

where $\tilde{t} = [\frac{t}{\tau}]\tau$ for $t \in [0, T)$ ($[\frac{t}{\tau}]$ as integer part of $\frac{t}{\tau}$) and $\tilde{t} = T - \tau$ for $t = T$. Now, in passing to the limit as $\tau \longrightarrow 0$, we obtain

$$\int_0^T a(t; w(t), v(t)) \, dt = \int_0^T (F(t), v(t)) \, dt \ + \int_0^T \left(\int_0^t b(t'; w(t'), v(t)) \, dt' \right) \ dt$$

for all $v \in L_2(0, T; H_0^1(\Omega))$ and hence

$$a(t; w(t), v) = (F(t), v) + \int_0^t b(t', w(t'), v) \ dt' \quad \forall v \in H_0^1(\Omega) \quad \text{a.e. in} \quad (0, T).$$

This equation implies, together with (4.3), that $w(t) = z(t)$ holds a.e. in $(0, T)$. Thus, $z \in L_2(0, T; W_p^2(\Omega))$ is proved.

In order to show the estimates for z we first take $v = z(t) \in H_0^1(\Omega)$ in (4.3) and apply Gronwall's inequality. As a result we obtain $\|z\|_{C([0,T];H^1(\Omega))} \leq c \ \|F\|_{C([0,T];L_2(\Omega))}$. Owing to the previously proved regularity $z \in L_2(0, T; W_p^2(\Omega))$ for the solution of (4.3), we have the a priori estimate

$$\|z(t)\|_{W_p^2(\Omega)} \leq C \ \left(\|z(t)\|_{H^1(\Omega)} + \|F(t)\|_{L_p(\Omega)} + \int_0^t \|(Bz)(t')\|_{L_p(\Omega)} \ dt' \right) \tag{4.8}$$

a.e. in $(0,T)$, where the constant C does not depend on t (see (4.1) and cf. Remark 4.2(i)) and $(Bz)(t') = -\operatorname{div}(\partial_t\sigma(t')\ \nabla z(t'))$. Combining the last two inequalities, we obtain

$$\|z(t)\|^2_{W^2_p(\Omega)} \leq C_1\,\|F\|^2_{C([0,T];L_2(\Omega))} + C_2\,\|F(t)\|^2_{L_p(\Omega)} + C_3 \int_0^t \|z(t')\|^2_{W^2_p(\Omega)}\ dt' \tag{4.9}$$

a.e. in $(0,T)$. Integrating (4.9) in time and using again Gronwall's inequality, we get

$$\int_0^t \|z(t')\|^2_{W^2_p(\Omega)}\ dt' \leq \bar{C}_1\,\|F\|^2_{C([0,T];L_2(\Omega))} + \bar{C}_2 \int_0^t \|F(t')\|^2_{L_p(\Omega)}\ dt' \quad \forall t \in [0,T]. \tag{4.10}$$

The estimate for $\|z\|_{L_2(0,T;W^2_p(\Omega))}$ follows by means of (4.10) for $t = T$.

(ii). Substituting (4.10) into (4.9) we obtain $z \in L_\infty(0,T;W^2_p(\Omega))$ and the desired estimate for $\|z\|_{L_\infty(0,T;W^2_p(\Omega))}$.

(iii). The statements for the solution $\tilde{z}_\varepsilon$ of problem (3.5) follow from the parts (i), (ii), which are applied with a modified right-hand side $\breve{F}(t) = F(t) + [1-\beta_\varepsilon(z_\varepsilon(t))]\ \xi(t)$. The assumptions guarantee that $\breve{F}$ belongs to $L_2(0,T;L_p(\Omega))$ (resp. to $L_\infty(0,T;L_p(\Omega))$). $\square$

Remark 4.2. (i) Let us briefly point out that the constants in the a priori estimates (4.6), (4.8) can be chosen independent of t and τ. Considering the elliptic problem

$$-\operatorname{div}(\sigma(x,t)\ \nabla v) = F(x,t) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

for fixed $t \in [0,T]$, one has $v \in W^2_p(\Omega)$ resulting from the elliptic regularity theory (cf. [7], Theorem 2.4.2.5 and [15], Theorem 3.7.4). Thus, the differential equation can be multiplied by $|\triangle v|^{p-1} \in L_{\frac{p}{p-1}}(\Omega)$. Integrating over Ω as well as using Young's and Hölder's inequality, we obtain $\|\triangle v\|_{L_p(\Omega)} \leq C\ \left(\|v\|_{W^1_p(\Omega)} + \|F(t)\|_{L_p(\Omega)}\right)$ with a constant $C = C(p,\sigma_0,\|\sigma(t)\|_{C^1(\overline{\Omega})})$. Owing to the assumptions (4.2), this constant C can be taken independent of t.

Applying now the general a priori estimate (4.1) to $\triangle v$ with $v \in W^2_p(\Omega)$ and $v = 0$ on $\partial\Omega$, we deduce

$$\begin{aligned} \|v\|_{W^2_p(\Omega)} &\leq \bar{C}\left(\|v\|_{W^1_p(\Omega)} + \|F(t)\|_{L_p(\Omega)}\right), \\ \|v\|_{W^2_p(\Omega)} &\leq \tilde{C}\left(\|v\|_{L_p(\Omega)} + \|F(t)\|_{L_p(\Omega)}\right). \end{aligned} \tag{4.11}$$

The second estimate is obtained from the previous one using a Gagliardo-Nirenberg inequality (cf. [21], sect. 21.17, 21.19). Combining (4.11) and the Sobolev embeddings $W^j_r(\Omega) \subset L_q(\Omega)$, $j = 1,2$ for corresponding values of r and q (depending on $2 \leq p < \infty$ and the space dimension n a multiple application can be necessary), we arrive at

$$\|v\|_{W^2_p(\Omega)} \leq C\ \left(\|v\|_{H^1(\Omega)} + \|F(t)\|_{L_p(\Omega)}\right), \quad 2 \leq p < \infty,$$

where C does not depend on t.

(ii) In the proof of Lemma 4.1 we have used several times a discrete version of the Gronwall inequality. Suppose, that the discrete functions $d_j = d(t_j) \geq 0$ and $g_j = g(t_j) \geq 0$ satisfy $d_0 \leq g_0$ and $d_j \leq c\ \sum_{i=0}^{j-1} \tau d_i + g_j$ for all $j = 1,\dots,N$, $0 \leq c =$ const. Then, the estimate $d_j \leq g_j + c\ e^{c\ t_{j-1}} \sum_{i=0}^{j-1} \tau g_i$ holds for all $j = 1,\dots,N$, where $\{t_j = j\tau,\ j = 0,\dots,N\}$, $N\tau = T$.

We come now to the global spatial regularity of the solution of the obstacle problems (2.2), (3.1). These main statements are obtained connecting the results of Theorem 3.4(i) (strong convergence of $\tilde{z}_\varepsilon$ to $\tilde{u} = u - g_D$) and Lemma 4.1 (boundedness of $\tilde{z}_\varepsilon$).

Theorem 4.3. *Under the assumptions of* Theorem 3.4 *and* Lemma 4.1(i) *the solution* u *of problem* (2.2), (2.3) *is such that*

$$u \in \left(H^1(0,T;H^1(\Omega)) \ \cap \ L_2(0,T;W_p^2(\Omega))\right), \quad 2 \le p < \infty.$$

Furthermore, under the additional assumptions of Lemma 4.1(ii)*, one has*

$$u \in \left(H^1(0,T;H^1(\Omega)) \ \cap \ L_\infty(0,T;W_p^2(\Omega))\right), \quad 2 \le p < \infty$$

for the solution u *of* (2.2)*,* (2.3)*.*

Proof. Owing to Lemma 4.1(i) the family $\{\tilde{z}_\varepsilon\}$ of penalty solutions is bounded in the space $L_2(0,T;W_p^2(\Omega))$ (independent of ε). Hence, there exists a subsequence $\tilde{z}_{\varepsilon'}$, which converges in $L_2(0,T;W_p^2(\Omega))$ weakly to an element $w \in L_2(0,T;W_p^2(\Omega))$.

Thus, we have, on the one hand, $\tilde{z}_{\varepsilon'} \rightharpoonup w$ in $L_2(0,T;H^1(\Omega))$ (weakly). On the other hand, owing to Theorem 3.4(i), we obtain $\tilde{z}_\varepsilon \longrightarrow \tilde{u}$ in $L_2(0,T;H^1(\Omega))$, such that $u - g_D = w$ holds.

The second statement $u \in L_\infty(0,T;W_p^2(\Omega))$ is proved by means of the weak* convergence. We observe that $L_1(0,T;(W_p^2(\Omega))^*)$ is a separable Banach space (cf. [21], chap. 23, in particular, problem 23.12).

Hence, due to the boundedness of $\tilde{z}_\varepsilon$ in the space $L_\infty(0,T;W_p^2(\Omega))$ (see Lemma 4.1(ii)), there exists a subsequence $\tilde{z}_{\varepsilon'}$ of $\tilde{z}_\varepsilon$ with $\tilde{z}_{\varepsilon'} \overset{*}{\rightharpoonup} w$ in $L_\infty(0,T;W_p^2(\Omega))$. Again, the strong convergence $\tilde{z}_\varepsilon \longrightarrow \tilde{u}$ in $C([0,T];H^1(\Omega))$ derived in Theorem 3.4(i) implies that $w = u - g_D$ solves the variational inequality (2.2). □

In the end let us briefly return to the special ECM problem from section 2, the regularity of which with respect to time is discussed in Corollary 3.7. The solution u of this problem has been shown to satisfy

$$u \in \left(W_\infty^1(0,T;H^1(\Omega)) \ \cap \ L_\infty(0,T;W_p^2(\Omega))\right).$$

Moreover, owing to the Sobolev embedding $W_p^2(\Omega) \subset C^{1,\alpha}(\overline{\Omega})$ for $0 < \alpha = 1 - n/p$, the solution u belongs to $L_\infty(0,T;C^{1,\alpha}(\overline{\Omega}))$. Recalling Theorem 3.4(ii) (observe $\partial_t F = 0$) we conclude that all derivatives of u are bounded. Therefore, we end up with

$$u \in C^{0,1}(\overline{\Omega} \times [0,T]) = W_\infty^1(\Omega \times (0,T)).$$

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