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A Note on Quasiconvexity and Rank-one Convexity for 2×2 Matrices

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We report on an attempt to find a counterexample to the statement that rank-one convexity does not imply quasiconvexity in the case of 2×2 matrices. The failure of such attempt is a consequence of some surprising computations.

1. Introduction

The main ingredient of the direct method of the Calculus of Variations for finding minimizers of an integral functional of the form

$$I(u) = \int_\Omega \varphi(\nabla u(x))\, dx,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, and $u : \Omega \to \mathbb{R}^m$ is a Lipschitz function, $u \in W^{1,\infty}(\Omega)$, is the weak lower semicontinuity property

$$u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(\Omega) \text{ implies } \int_\Omega \varphi(\nabla u(x))\, dx \le \liminf_{j\to\infty} \int_\Omega \varphi(\nabla u_j(x))\, dx.$$

This extremely convenient property is in turn always a consequence of some convexity condition for the integrand φ, which is a real-valued function defined on $m \times N$ matrices. In the fully vector case $N, m > 1$ the quasiconvexity condition

$$\varphi(F) \le \frac{1}{|\Omega|} \int_\Omega \varphi(F + \nabla w(x))\, dx, \tag{1.1}$$

for any matrix F and any test function $w \in W_0^{1,\infty}(\Omega)$ [10] is a necessary and sufficient condition for the above weak lower semicontinuity property. We remind the reader that an equivalent formulation for the quasiconvexity condition is

$$\varphi(F) \le \int_T \varphi(F + \nabla U(x))\, dx,$$

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for any function $U : T \to \mathbb{R}^m$ which is periodic with respect to the lattice $\mathbb{Z}^N$ and T is the unit cube in $\mathbb{R}^N$ [14]. Moreover, a slightly stronger version of the quasiconvexity condition gives the partial regularity of minimizers of I (e.g. [1, 7]). These results show that quasiconvexity is a very natural condition. However, it turns out that in general it can be very difficult to decide whether a given function is quasiconvex. This difficulty is related to the nonlocal nature of quasiconvexity [8].

Conditions which are either necessary or sufficient have been derived over the years. Some of these developments have been crucial in nonlinear elasticity [2]. One such well-known necessary condition is called rank-one convexity and it requires the usual convexity inequality

$$\varphi(tA + (1-t)B) \leq t\varphi(A) + (1-t)\varphi(B), \quad t \in [0,1]$$

provided $A - B$ is a matrix of rank-one. It is well-known and easy to see [6, 12] that each quasiconvex function is rank-one convex. This paper is motivated by the open problem of whether the opposite implication is true when $N = m = 2$. For $N \geq 2$ and $m \geq 3$ rank-one convexity does not imply quasiconvexity [9, 13]. We present here a calculation showing that, in a situation related to the counterexample constructed in [13] and when $N = m = 2$, rank-one convexity has much stronger consequences than for $m \geq 3$. More specifically, we explicitly calculate the value at 0 of both the rank-one convex and quasiconvex envelopes of a nontrivial function on 2×2 matrices. These two values turn out to be the same. This function is again motivated by the counterexample in [13]. Strictly speaking, the situation we consider here is more special since we will only work on symmetric 2×2 matrices. As a corollary of our calculation, we get an inequality which seems to be of independent interest.

In what follows, we will be using functions φ which can take on the value $+\infty$. We could avoid this by introducing (in a natural way) the notion of quasiconvexity for functions defined on selected subsets of $m \times N$ matrices. We feel that this would unnecessarily complicate the terminology and that a better solution is to allow infinite-valued functions. However, we must very strongly stress the following. From the point of view of the convexity notions we consider in this paper, there is in general an important difference between everywhere finite functions and the case where functions are allowed to take on the value $+\infty$ (see [3]). In our case, the functions under consideration will be infinite in the complement of convex, compact sets. It is not difficult to see that if a rank-one convex (respectively, quasiconvex) function φ is infinite in a complement of a convex, compact set of matrices, then φ is a pointwise limit of an increasing sequence of everywhere-finite, rank-one convex (respectively, quasiconvex) functions. In this case, any nonquasiconvex function which takes on the value $+\infty$ in the complement of a convex, compact set of matrices automatically produces an everywhere finite, nonquasiconvex function. The idea of the counterexample in [13] is to show that a suitable perturbation of the function

$$\begin{aligned} \tilde{\varphi}(X) &= xyz, \qquad \text{if } X = \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} \text{ with } \max\{|x|, |y|, |z|\} \leq 1, \\ \tilde{\varphi}(X) &= +\infty, \qquad \text{otherwise,} \end{aligned}$$

admits an everywhere-finite, rank-one, continuous extension to all of space of 3×2 matrices. This extension (still denoted $\tilde{\varphi}$) is not quasiconvex at the origin: the periodic deformation

$$U(x_1, x_2) = \frac{1}{2\pi}(\cos(2\pi x_1), \cos(2\pi x_2), \sin(2\pi(x_1 + x_2))), \quad (x_1, x_2) \in T = (0,1)^2,$$

has gradient

$$\nabla U(x_1, x_2) = \begin{pmatrix} -\sin(2\pi x_1) & 0 \\ 0 & -\sin(2\pi x_2) \\ \cos(2\pi(x_1 + x_2)) & \cos(2\pi(x_1 + x_2)) \end{pmatrix}$$

and by elementary trigonometry

$$\int_T \tilde{\varphi}(\nabla U(x_1, x_2))\, dx_1\, dx_2 = -1/4 < 0 = \tilde{\varphi}(0).$$

As remarked earlier, the goal of this note is to point out some interesting facts we came across when trying to extend this example to 2×2 matrices. There are essentially two ways of looking for such counterexamples. One can fix a rank-one convex integrand, φ, and try to find a deformation that violates the quasiconvexity inequality, or one can concentrate on a suitable, fixed deformation, ∇U, and look for a nonlinear, rank-one convex quantity for which the quasiconvexity inequality does not hold. Since nontrivial, explicit examples of rank-one convex functions are not abundant, one can reformulate the above two approaches in terms of quasiconvex and rank-one convex hulls of functions.

Let us recall that the quasiconvexification at F of a function φ is defined by

$$Q\varphi(F) = \inf_w \frac{1}{|\Omega|} \int_\Omega \varphi(F + \nabla w(x))\, dx, \tag{1.2}$$

where $w \in W_0^{1,\infty}(\Omega)$. This notion plays an important role in the analysis of variational principles [6, 12]. The quasiconvexification can equivalently be defined via the formula

$$Q\varphi(F) = \inf_U \int_T \varphi(F + \nabla U(x))\, dx,$$

where the infimum is taken over the set of periodic test deformations U. It can also be obtained by

$$Q\varphi = \sup\{\psi : \psi \text{ is quasiconvex }, \psi \leq \varphi\}. \tag{1.3}$$

In the same way, the rank-one convexification, $R\varphi$, is defined via the equality

$$R\varphi = \sup\{\psi : \psi \text{ is rank-one convex }, \psi \leq \varphi\}. \tag{1.4}$$

It can also be described [5] in a more direct way by

$$R\varphi(F) = \inf\left\{\sum_i \lambda_i \varphi(A_i) : \{(\lambda_i, A_i)\} \text{ satisfies the } (H_n) \text{ condition },\right.$$
$$\left.\sum_i \lambda_i A_i = F \text{ is a convex combination}\right\}. \tag{1.5}$$

The (H_n) condition is defined recursively. A set of pairs $\{(\lambda_i, A_i)\}$ is said to satisfy the (H_n) condition if

(i) for $n = 2$, $A_1 - A_2$ is a rank-one matrix;
(ii) for $n > 2$ and after a change of order $A_{n-1} - A_n$ is a rank-one matrix and letting

$$\begin{aligned} \mu_j &= \lambda_j, \quad j \le n-2, \\ \mu_{n-1} &= \lambda_{n-1} + \lambda_n, \\ B_j &= A_j, \quad j \le n-2, \\ B_{n-1} &= \frac{1}{\mu_{n-1}} \left(\lambda_{n-1} A_{n-1} + \lambda_n A_n \right), \end{aligned}$$

the set of pairs $\{(\mu_j, B_j)\}$ satisfies the (H_{n-1}) condition.

Notice that $Q\varphi \le R\varphi$ always because the (H_n) conditions correspond to gradients or sequences of gradients.

The first approach referred to above can be restated in the following terms: find a function whose quasiconvexification at some matrix is strictly less than its rank-one convexification at the same matrix. The second is equivalent to finding a continuous function ψ such that for a given matrix F and a given deformation ∇U we have

$$R\psi(F) > \int_\Omega R\psi(\nabla U(x))\, dx.$$

The starting point of our computations is the function

$$\begin{aligned} &\varphi(X) = xyz, \qquad \text{if } X = \begin{pmatrix} x+z & z \\ z & y+z \end{pmatrix} \text{ with } \max\{|x|, |y|, |z|\} \le 1, \\ &\varphi(X) = +\infty, \qquad \text{otherwise.} \end{aligned} \tag{1.6}$$

Notice that φ is finite only for some symmetric matrices. We will always take $F = 0$. In Section 2, we will exactly compute $Q\varphi(0)$. This calculation seems to be interesting in its own right. As a consequence we will prove an interesting inequality. In Section 3, we will compute $R\varphi(0)$ and conclude that indeed $Q\varphi(0) = R\varphi(0)$ so that the counterexample cannot be found in this way. Finally, Section 4 focuses on the other approach where we concentrate on the optimal periodic deformation, ∇u, introduced in Section 2 for which

$$Q\varphi(0) = \int_T \varphi(\nabla u(x))\, dx,$$

and look for a continuous, nonlinear quantity, ψ, for which we could possibly have

$$R\psi(F) > \int_T R\psi(\nabla u(x))\, dx.$$

We again show that this is impossible.

2. The quasiconvexification

In this section we prove the following theorem.

Theorem 2.1. *Let φ be the function defined by* (1.6)*. Then*

$$Q\varphi(0) = -\frac{1}{2}.$$

To establish the lower bound for $Q\varphi(0)$ we use the new examples of quasiconvex functions found in [14]. Specifically, the function

$$\psi(X) = \begin{cases} \det X, & \text{if } X \text{ is symmetric and positive definite,} \\ 0, & \text{if } X \text{ is symmetric but not positive definite,} \\ +\infty, & \text{if } X \text{ is not symmetric,} \end{cases}$$

is quasiconvex. Through the identification

$$X = \begin{pmatrix} x+z & z \\ z & y+z \end{pmatrix} \mapsto (x,y,z), \tag{2.1}$$

and for

$$X_0 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

consider the function

$$q(x,y,z) = \frac{3}{2}\psi(X + X_0).$$

Notice that the right-hand side is a quasiconvex function of X. Let B be the cube $[-1,1]^3$.

Lemma 2.2. *For any* $(x,y,z) \in B$

$$q(x,y,z) \leq (x+1)(y+1)(z+1).$$

Proof. Notice that under the identification (2.1) the rank-one cone restricted to B is given by $xy + xz + yz = 0$. Let χ denote the right-hand side of the inequality we want to prove. The inequality follows in an elementary way after the next three remarks:

(i) q is convex along the three coordinate directions; indeed, since q is quasiconvex, it is in particular rank-one convex; observe that the coordinate directions are rank-one directions.

(ii) χ is linear along the three coordinate directions. This is trivial.

(iii) $q = \chi$ on the eight vertices of B. This is an elementary calculation.

Since any point in B can be decomposed along directions parallel to the axes ending up on the vertices of B, the proof follows from (i), (ii) and (iii). □

If we now write

$$\chi = xyz + xy + xz + yz + x + y + z + 1,$$

or for $(x,y,z) \in B$

$$\chi = \varphi + L,$$

where $L = xy + xz + yz + x + y + z + 1$, then we have obtained

$$q - L \leq \varphi.$$

This inequality is trivially true outside B as well if we consider $q - L$ to be defined by $+\infty$ outside B. We claim that this extension of $q - L$ is quasiconvex. Indeed, if we take a deformation ∇u that takes on values outside B then the quasiconvex inequality is trivially true. If, on the other hand, ∇u takes on all its values on B, we know that q restricted to

B is quasiconvex, and L is in fact a null-lagrangian when restricted to B because is the sum of the determinant ($xy + xz + yz$ is the determinant in B) plus a linear functional. Since $q - L$ is quasiconvex, by (1.3) we have

$$-\frac{1}{2} = q(0,0,0) - L(0,0,0) \leq Q\varphi(0),$$

as desired.

In order to show the equality, we will consider the following periodic deformation $U : T \to \mathbb{R}^2$ defined by

$$U(x_1, x_2) = (\tilde{u}(x_1) + \tilde{w}(x_1 + x_2), \tilde{v}(x_2) + \tilde{w}(x_1 + x_2)),$$

where $\tilde{f}$ denotes the primitive of f

$$\tilde{f}(t) = \int_0^t f(s)\, ds,$$

and we take

$$u = v = \theta, \quad w = \theta(\cdot + 1/4),$$

where

$$\theta = 2\chi_{(0,1/2)} - 1$$

in the unit interval I, and extended periodically to all of $\mathbb{R}$. $\chi_{(0,1/2)}$ is the characteristic function of the interval $(0, 1/2)$. We have that

$$\nabla U(x_1, x_2) = \begin{pmatrix} u(x_1) + w(x_1 + x_2) & w(x_1 + x_2) \\ w(x_1 + x_2) & v(x_2) + w(x_1 + x_2) \end{pmatrix},$$

is symmetric, and an easy computation gives

$$\begin{aligned} \int_T \varphi(\nabla U(x))\, dx &= \frac{1}{16} [\varphi(1,1,1) + \varphi(1,-1,-1) + \varphi(-1,1,-1) + \varphi(-1,-1,1)] \\ &\quad + \frac{3}{16} [\varphi(1,1,-1) + \varphi(1,-1,1) + \varphi(-1,1,1) + \varphi(-1,-1,-1)] \\ &= -\frac{1}{2}. \end{aligned}$$

Theorem 2.1 is proved.

We would like to show how the exact computation of the quasiconvex hull of our cubic polynomial φ can be translated into an interesting inequality where we concern ourselves with the best constant for which such an inequality holds. Let us consider the following estimate

$$\left| \int_T u(x_1) v(x_2) w(x_1 + x_2)\, dx_1\, dx_2 \right| \leq C \|u\|_{L^\infty(I)} \|v\|_{L^\infty(I)} \|w\|_{L^\infty(I)}, \tag{2.2}$$

where $I = [0,1]$ is the unit interval, $T = I \times I$ is the unit cell in $\mathbb{R}^2$, u, v and w are I-periodic functions with mean value 0 over a period cell, and $C > 0$ is some constant. Obviously, taking $C = 1$, (2.2) is correct for any choice of u, v and w. The question is to determine the smallest constant C in (2.2) valid for any choice of u, v and w under the prescribed conditions. We will see that this question is closely related to determining the quasiconvexification of φ at the origin which we have already done. The best C is actually $1/2$:

$$\left|\int_T u(x_1)v(x_2)w(x_1+x_2)\,dx_1\,dx_2\right| \le \frac{1}{2}\,\|u\|_{L^\infty(I)}\,\|v\|_{L^\infty(I)}\,\|w\|_{L^\infty(I)}\,, \tag{2.3}$$

for any u, v and w I-periodic, with mean value 0.

Let us explain how the connection of (2.2) with the quasiconvexification of φ at the origin is established. By dividing (2.2) by the L^∞ norms of u, v and w, we may assume in addition that the three functions are indeed bounded by 1. On the other hand, since the restrictions on u, v and w are preserved by changes in sign, we are in search of the minimum value, m, of the integrals

$$\int_T u(x_1)v(x_2)w(x_1+x_2)\,dx_1\,dx_2. \tag{2.4}$$

Consider the periodic deformation $U: T \to \mathbb{R}^2$ defined by

$$U(x_1,x_2) = (\tilde{u}(x_1)+\tilde{w}(x_1+x_2), \tilde{v}(x_2)+\tilde{w}(x_1+x_2)), \tag{2.5}$$

where again $\tilde{f}$ represents the primitive of f. As before, we have that

$$\nabla U(x_1,x_2) = \begin{pmatrix} u(x_1)+w(x_1+x_2) & w(x_1+x_2) \\ w(x_1+x_2) & v(x_2)+w(x_1+x_2) \end{pmatrix},$$

is symmetric, and clearly (2.4) is written

$$\int_T \varphi(\nabla U(x))\,dx.$$

In this fashion we see that $Q\varphi(0) \le m$. Since we know that $Q\varphi(0) = -1/2$, then (2.3) holds. The fact that (2.3) is sharp is shown by considering again the optimal deformation

$$u = v = \theta, \quad w = \theta(\cdot + 1/4),$$

where

$$\theta = 2\chi_{(0,1/2)} - 1.$$

3. The rank-one convexification

In this section, we prove that $R\varphi(0) = -1/2$ as well.

Theorem 3.1. *Let φ be the function defined by* (1.6). *Then*

$$R\varphi(0) = -\frac{1}{2}.$$

On the one hand, the quasiconvexification at a particular matrix is always a lower bound for the rank-one convexification, so that $-1/2 = Q\varphi(0) \le R\varphi(0)$. On the other hand, since $R\varphi(0)$ is obtained in (1.5) as an infimum, it suffices to find some set of pairs $\{(\lambda_i, A_i)\}$ satisfying the (H_n) condition for which

$$\sum_i \lambda_i \varphi(A_i) = -\frac{1}{2}.$$

These computations are inspired by [11].

For the sake of simplicity, we are going to stick to the identification (2.1), so that rank-one directions are given by the vectors (x, y, z) such that $xy + xz + yz = 0$. We claim that the set of pairs

$$\left\{\left(\frac{3}{16}, (-1,-1,-1)\right), \left(\frac{3}{16}, (-1,-1,1)\right), \left(\frac{2}{16}, (1,-1,1)\right), \right.$$
$$\left. \left(\frac{2}{16}, (-1,1,1)\right), \left(\frac{1}{16}, (1,1,1)\right), \left(\frac{5}{16}, (1,1,-1)\right)\right\} \tag{3.1}$$

satisfies some (H_n) condition. To convince the reader of this assertion, let us consider the following set of points in the cube B

$$P_0 = (0,0,0), \quad P_1 = (-1/2, 1, 1),$$
$$P_2 = (1/10, -1/5, -1/5), \quad P_3 = (1, -5/7, 1),$$
$$P_4 = (-1/11, -1/11, -5/11), \quad P_5 = (1, 1, -1),$$
$$P_6 = (-1,-1,0), \quad P_7 = (-1,-1,1),$$
$$P_8 = (-1,-1,-1).$$

The following facts can be verified after some careful computations:

(i) $P_j - P_{j-1}$ are rank-one directions for $j = 2, 4, 6, 8$;

(ii)

$$P_0 = \frac{1}{6}P_1 + \frac{5}{6}P_2, \quad P_2 = \frac{21}{120}P_3 + \frac{99}{120}P_4,$$
$$P_4 = \frac{5}{11}P_5 + \frac{6}{11}P_6, \quad P_6 = \frac{1}{2}P_7 + \frac{1}{2}P_8,$$
$$P_3 = \frac{6}{7}(1,-1,1) + \frac{1}{7}(1,1,1).$$

If we begin with P_0 and decompose

$$P_0 = \frac{1}{6}P_1 + \frac{5}{6}P_2, \quad P_2 - P_1 \text{ rank-one},$$

and then substitute P_2 by

$$P_2 = \frac{21}{120}P_3 + \frac{99}{120}P_4, \quad P_4 - P_3 \text{ rank-one},$$

and so on according to the decompositions in (ii) above until all points involved are vertices of B, then we prove that the set of pairs (3.1) truly satisfies some (H_n) condition. Then,

$$\begin{aligned} R\varphi(0) &\leq \frac{3}{16}\varphi(-1,-1,-1)+\frac{3}{16}\varphi(-1,-1,1)+\frac{2}{16}\varphi(1,-1,1) \\ &+\frac{2}{16}\varphi(-1,1,1)+\frac{1}{16}\varphi(1,1,1)+\frac{5}{16}\varphi(1,1,-1) \\ &= -\frac{1}{2}. \end{aligned}$$

This finishes the proof of Theorem 3.1.

4. The optimal deformation

In this final section, we investigate the possibility of fixing an appropriate periodic deformation, ∇U, and then looking for a suitable continuous function, ψ such that

$$R\psi(0) > \int_T R\psi(\nabla U(x))\,dx. \tag{4.1}$$

The first step is to choose a good candidate for having such an inequality. The choice we tried is the optimal deformation in the computation of the quasiconvexification of the cubic polynomial φ, that is to say the deformation U such that

$$\begin{aligned} \int_T \psi(\nabla U(x))\,dx &= \frac{1}{16}\left[\psi(1,1,1)+\psi(1,-1,-1)+\psi(-1,1,-1)+\psi(-1,-1,1)\right] \\ &+\frac{3}{16}\left[\psi(1,1,-1)+\psi(1,-1,1)+\psi(-1,1,1)+\psi(-1,-1,-1)\right] \\ &= -\frac{1}{2}. \end{aligned}$$

Is it possible to find a continuous function on the cube B such that (2.2) holds? The answer is once again negative. This has been shown in [11]. As a matter of fact, if we consider the probability measure ν supported on the set of vertices of the cube

$$\begin{aligned} \nu &= \frac{1}{16}\left[\delta_{(1,1,1)}+\delta_{(1,-1,-1)}+\delta_{(-1,1,-1)}+\delta_{(-1,-1,1)}\right] \\ &+\frac{3}{16}\left[\delta_{(1,1,-1)}+\delta_{(1,-1,1)}+\delta_{(-1,1,1)}+\delta_{(-1,-1,-1)}\right], \end{aligned}$$

then

$$\nu = \frac{1}{3}\nu_1+\frac{1}{3}\nu_2+\frac{1}{3}\nu_3,$$

where ν_1 is the probability measure verifying some (H_n) condition introduced in Section 3

$$\begin{aligned} \nu_1 &= \frac{3}{16}\delta_{(-1,-1,-1)}+\frac{3}{16}\delta_{(-1,-1,1)}+\frac{2}{16}\delta_{(1,-1,1)} \\ &+\frac{2}{16}\delta_{(-1,1,1)}+\frac{1}{16}\delta_{(1,1,1)}+\frac{5}{16}\delta_{(1,1,-1)}, \end{aligned}$$

and ν_2 and ν_3 are obtained from ν_1 by symmetry:

$$\begin{aligned}
\nu_2 &= \frac{3}{16}\left(\delta_{(-1,-1,-1)} + \delta_{(-1,1,-1)}\right) \\
&\quad + \frac{2}{16}\left(\delta_{(-1,1,1)} + \delta_{(1,1,-1)}\right) \\
&\quad + \frac{1}{16}\left(\delta_{(1,1,1)} + 5\delta_{(1,-1,1)}\right), \\
\nu_3 &= \frac{3}{16}\left(\delta_{(-1,-1,-1)} + \delta_{(1,-1,-1)}\right) \\
&\quad + \frac{2}{16}\left(\delta_{(1,1,-1)} + \delta_{(1,-1,1)}\right) \\
&\quad + \frac{1}{16}\left(\delta_{(1,1,1)} + 5\delta_{(-1,1,1)}\right).
\end{aligned}$$

Thus, ν_2 and ν_3, and consequently ν itself, satisfy some (H_n) condition. This implies that for a rank-one convex function, $R\psi$, we always have

$$R\psi(0) \le \langle R\psi, \nu\rangle = \int_T R\psi(\nabla U(x))\,dx,$$

against (4.1).

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