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Vertical Developments of a Convex Function

François Oustry*
INRIA, Domaine de Voluceau-Rocquencourt,
B.P. 105, 78153 Le Chesnay, France.
e-mail: oustry@inria.fr

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In this paper we compare two different approaches to analyse the second-order behaviour of a convex function. The first one is classical, we call it the *horizontal approach*; the second one is more recent, it is the *vertical approach*. We prove equivalences between horizontal and vertical growth conditions. Then we derive well-known directional results. Finally we show that the vertical approach is particularly interesting to get more than a first-order (and more than directional) analysis of the maximum eigenvalue function.

Keywords: Convex analysis, second-order derivative, approximate subdifferential, semidefinite programming

1. Introduction

Let f be a finite-valued convex function from the m-dimensional Euclidean subspace $\mathbb{R}^m$ to $\mathbb{R}$. The behaviour of the ε-subdifferential of f at a point $x \in \mathbb{R}^m$, as a multifunction of the real nonnegative parameter ε, is commonly known "to give information" on the second-order behaviour of f at x. In order to examine to what extent this idea is relevant, we clarify and generalize, some results given in [12], [4], [6], [18]. We show that f satisfies a *second-order growth condition* if and only if $f'_\varepsilon(x;d)$, the approximate derivative of f at x in the direction $d \in \mathbb{R}^m$, satisfies a *half-order growth condition* uniformly with respect to d. Our main motivation is to identify some bridges between

- the classical *"horizontal" approach*, which consists in studying the limiting behaviour of the quotient

$$\frac{f(x+h) - f(x) - f'(x;h)}{\frac{1}{2}\|h\|^2}, \tag{1.1}$$

  when $\|h\|$ is small,
- the *"vertical" approach*, which consists in studying the limiting behaviour of the quotient

$$\frac{[f'_\varepsilon(x;d) - f'(x;d)]^2}{2\varepsilon}, \tag{1.2}$$

  when ε is small.

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The *horizontal/vertical* terminology is due to C. Lemaréchal and J. Zowe [12].

Our main contribution is to bound the difference quotients (1.1) and (1.2) *uniformly* with respect to the direction. We intend to keep the same point of view as in [3] where the limits of the first-order difference quotient $\frac{f(x+td)-f(x)}{t}$ are studied uniformly in $d \in \mathbb{R}^m$, $\|d\| = 1$ when $t \downarrow 0$. We shall see that this approach can be applied to the second-order difference quotient (1.1) when $\|h\| \to 0$ or equivalently for (1.2) uniformly in $d \in \mathbb{R}^m$, $\|d\| = 1$ when $\varepsilon \downarrow 0$. Geometrically, this amounts to studying the quotient

$$\frac{\Delta_H(\partial_\varepsilon f(x), \partial f(x))}{\sqrt{2\varepsilon}},$$

when $\varepsilon \downarrow 0$ and where $\Delta_H(\cdot,\cdot)$ is the Hausdorff distance.

Let us note here that this geometrical view can be compared with the geometrical interpretation of the second-order epi-derivative relatively to a sugradient $g \in \partial f(x)$ in terms of the Dupin indicatrix of the conjugate function f^* at (g, x) which are in turn connected to the difference quotient

$$\frac{\partial_\varepsilon f(x) - g}{\sqrt{2\varepsilon}},$$

when $\varepsilon \downarrow 0$ (see [6] and [13]). This approach considers vertical developments replacing in (1.2) the sublinear function $f'(x;\cdot)$ by a particular linear one $\langle g, \cdot\rangle$. In the present paper we do deal with the sublinear function $f'(x;\cdot)$.

The paper is organized as follows. After some preliminary results in § 2, we show in § 3 that f satisfies an *upper (resp. lower) horizontal growth condition* if and only if it satisfies an *upper (resp. lower) vertical growth condition.* This section considers two types of results: "uniform" ones in which the convergence in (1.2) is uniform with respect to d, and "directional" ones, in which d is fixed while $h = td$ in (1.1). In § 4, we take a directional point of view: we show that the lim sup (resp. lim inf) of (1.1) and the lim sup (resp. lim inf) of (1.2), when t and ε respectively tend to zero, are the same (possibly $+\infty$). Finally we illustrate an advantage of the vertical approach: in some situations, such as for the maximum eigenvalue function, it enables us to obtain more than a first-order (and more than a directional) sensivity analysis.

Our notation follows closely that of [4].

- $\mathbb{R}^m$ m-dimensional Euclidean space
- $\langle x, y\rangle$ scalar product of $x, y \in \mathbb{R}^m$
- $\|x\| := \sqrt{\langle x, x\rangle}$ Euclidean norm of $x \in \mathbb{R}^m$
- S is the unit sphere of $\mathbb{R}^m$
- $B(x, \varepsilon)$ is the ball centered at $x \in \mathbb{R}^m$ with radius $\varepsilon > 0$
- $\mathbb{R}^m \ni d \mapsto \sigma_G(d) := \sup_{g\in G}\langle g, d\rangle$ is the support function of the nonempty set $G \subset \mathbb{R}^m$
- $\Delta_H(G_1, G_2)$ is the Hausdorff distance between two nonempty compact sets $G_1, G_2 \subset \mathbb{R}^m$; if in addition G_1 and G_2 are convex, it can be characterized by [8, Theorem V.3.3.8]:

$$\Delta_H(G_1, G_2) = \max_{\|d\|\le 1} |\sigma_{G_1}(d) - \sigma_{G_2}(d)|,$$

which can also be written, using sublinearity of $\mathbb{R}^m \ni d \mapsto |\sigma_{G_1}(d) - \sigma_{G_2}(d)|$,

$$\Delta_H(G_1, G_2) = \max_{d \in S} |\sigma_{G_1}(d) - \sigma_{G_2}(d)|$$

- $\partial f(x)$ is the subdifferential of f at x:

$$\partial f(x) := \{s \in \mathbb{R}^m : f(y) - f(x) \geq \langle s, y - x\rangle \text{ for all } y \in \mathbb{R}^m\}$$

- $f'(x; d)$ is the directional derivative of f at $x \in \mathbb{R}^m$ in the direction $d \in \mathbb{R}^m$

$$f'(x; d) := \inf_{t>0} \frac{f(x + td) - f(x)}{t}$$

  or equivalently (see [8, § VI.1]), $f'(x; \cdot)$ is the support function of $\partial f(x)$

$$f'(x; \cdot) := \sigma_{\partial f(x)}(\cdot)$$

- $\partial_\varepsilon f(x)$ is the ε-subdifferential of f at $x \in \mathbb{R}^m$:

$$\partial_\varepsilon f(x) := \{s \in \mathbb{R}^m : f(y) - f(x) \geq \langle s, y - x\rangle - \varepsilon \text{ for all } y \in \mathbb{R}^m\}$$

- $f'_\varepsilon(x; d)$, the ε-directional derivative of f at $x \in \mathbb{R}$ in the direction $d \in \mathbb{R}$, is the support function of $\partial_\varepsilon f(x)$:

$$f'_\varepsilon(x; \cdot) := \sigma_{\partial_\varepsilon f(x)}(\cdot)$$

- $I_G(x) = \begin{cases} 0 & \text{if } x \in G \\ +\infty & \text{otherwise} \end{cases}$ is the indicator function of the nonempty set $G \subset \mathbb{R}^m$; it is convex if and only if G is convex.
- $N_{G,\varepsilon}(x) := \{s \in \mathbb{R}^m : \langle s, y - x\rangle \leq \varepsilon, \text{ for all } y \in G\}$ ε-normal set to the closed convex set G at $x \in G$ and $N_G(x) = N_{G,0}(x)$ is the usual normal cone.

2. Preliminary results

First we recall the following fundamental relation.

Theorem 2.1 ([8, Th. XI.2.1.1]). *For $x \in \mathbb{R}^m$ and $\varepsilon > 0$, the ε-directional derivative is the infimum over $t > 0$ of the ε-*approximate difference quotient*:*

$$f'_\varepsilon(x; h) = \inf_{t>0} \frac{f(x + th) - f(x) + \varepsilon}{t} \quad \textit{for all } h \in \mathbb{R}^m. \tag{2.1}$$

Another significant property of finite valued convex functions : they are *locally Lipschitz* (see e.g., [8, Theorem IV.3.1.2]), i.e., for every $x \in \mathbb{R}^m$,

$$\exists l, L > 0 : \; |f(y) - f(z)| \leq L\|y - z\| \text{ for all } y, z \in B(x, l). \tag{2.2}$$

From this Lipschitz property C. Lemaréchal and J. Zowe derive in [12] the following Lemma.

Lemma 2.2 ([12, Lemma 1.1]). *For each x and h in $\mathbb{R}^m$, one has*

$$f(x+h) - f(x) = \max\{f'_\varepsilon(x;h) - \varepsilon : \varepsilon \geq 0\}\,.$$

Set $\mathcal{E}(h) := \{\varepsilon \geq 0 : f(x+h) - f(x) = f'_\varepsilon(x;h) - \varepsilon\}$. Then

$$\mathcal{E}(h) = \{f(x) - f(x+h) + \langle s, h\rangle : s \in \partial f(x+h)\}\,.$$

Suppose $\|h\| \leq \frac{l}{2}$ where l is defined in (2.2). Then

$$\varepsilon \leq 2L\|h\| \text{ for all } \varepsilon \in \mathcal{E}(h)\,.$$

Finally we observe that, when $\varepsilon \downarrow 0$, $\partial_\varepsilon f(x)$ decreases to $\partial f(x)$; equivalently the function $f'_{[\cdot]}(x;d)$ is continuous at $\varepsilon = 0$ uniformly for $d \in S$:

Theorem 2.3. *When $\varepsilon \downarrow 0$,*

$$\lim_{\varepsilon \downarrow 0} f'_\varepsilon(x;d) = f'(x;d)\,,$$

uniformly with respect to $d \in S$.

Proof. The pointwise convergence is [16, Theorem 23.6]. Because $f'_\varepsilon(x;\cdot)$ is finite-valued for all $\varepsilon \geq 0$, the convergence is uniform with respect to $d \in S$ ([8, Theorem IV.3.1.5]). ☐

3. Vertical and horizontal growth conditions

In this section we consider the *second-order horizontal difference quotient* at $x \in \mathbb{R}^m$ in the direction $h \in \mathbb{R}^m$

$$H(x,h) := \frac{f(x+h) - f(x) - f'(x;h)}{\frac{1}{2}\|h\|^2}\,, \tag{3.1}$$

and for $\varepsilon > 0$, the *half-order vertical difference quotient* at $x \in \mathbb{R}^m$ in the direction $d \in S$:

$$V(x,d,\varepsilon) := \frac{[f'_\varepsilon(x;d) - f'(x;d)]^2}{2\varepsilon}\,. \tag{3.2}$$

According to the terminology of [12], h will be called the *horizontal decrement* and ε *the vertical decrement.* The analysis of $V(x,d,\varepsilon)$ when ε is small will lead us to developments of $f'_\varepsilon(x;d)$ in $\varepsilon^{1/2}$. This motivates the *half-order* terminology. In this section we take first a uniform point of view with respect to $h \in \mathbb{R}^m$: we study uniform upper and lower growth conditions. Then, concerning lower growth conditions we take a directional point of view to obtain sharper results.

3.1. Uniform upper growth conditions

We prove here that it is equivalent to bound, from above and uniformly with respect to the direction, the second-order horizontal or the half-order vertical difference quotients.

Theorem 3.1. *Let $x \in \mathbb{R}^m$. The following statements are equivalent:*

$$\exists \rho, C > 0 : \|h\| \leq \rho \Rightarrow H(x,h) \leq C\,, \tag{3.3}$$

$$\exists \bar{\varepsilon}, C > 0 : \varepsilon \leq \bar{\varepsilon} \Rightarrow V(x,d,\varepsilon) \leq C \;\; \textit{for all} \;\; d \in S\,, \tag{3.4}$$

The constant C is the same in (3.3) and (3.4).

Proof. [(3.3) $\Rightarrow$ (3.4)]. For a given $d \in S$, we have from (3.3)

$$f(x+td) \leq f(x) + tf'(x;d) + \frac{C}{2}t^2 \;\text{ for all }\; t \in (0,\rho]\,.$$

Adding ε and dividing by t, we obtain

$$\frac{f(x+td) - f(x) + \varepsilon}{t} - f'(x;d) \leq t\frac{C}{2} + \frac{\varepsilon}{t} \;\text{ for all }\; t \in (0,\rho]\,,$$

so Theorem 2.1 gives

$$f'_\varepsilon(x;d) - f'(x;d) \leq t\frac{C}{2} + \frac{\varepsilon}{t} \;\text{ for all }\; t \in (0,\rho]\,. \tag{3.5}$$

Now, the function $\mathbb{R}_+ \setminus \{0\} \ni t \mapsto t\frac{C}{2} + \frac{\varepsilon}{t}$ is strictly convex; its unique minimum is attained for $\bar{t} = \sqrt{\frac{2\varepsilon}{C}}$ which is smaller than ρ when

$$\varepsilon \leq \bar{\varepsilon} := \frac{1}{2}C\rho^2\,.$$

Writing (3.5) for $t = \bar{t}$, we get

$$f'_\varepsilon(x;d) - f'(x;d) \leq \sqrt{2\varepsilon C} \;\text{ for all }\; \varepsilon \in [0,\bar{\varepsilon}]\,.$$

which leads directly to (3.4).

[(3.4)$\Rightarrow$(3.3)]. Conversely suppose (3.4). Then consider $h \in \mathbb{R}^m \setminus \{0\}$ and set $d = \frac{h}{\|h\|}$ and $t = \|h\|$. Taking the square root in (3.4), multipliying by $t\sqrt{2\varepsilon}$ and subtracting ε we get

$$\begin{aligned} f'_\varepsilon(x;h) - f'(x;h) - \varepsilon &\leq t\sqrt{2\varepsilon C} - \varepsilon \;\text{ for all }\; \varepsilon \in [0,\bar{\varepsilon}] \\ &\leq \max_{\varepsilon \geq 0}\{\|h\|\sqrt{2\varepsilon C} - \varepsilon\} = \tfrac{1}{2}C\|h\|^2\,. \end{aligned} \tag{3.6}$$

Now let $\varepsilon \in \mathcal{E}(h)$; from Lemma 2.2 and (2.2), we have $\varepsilon \leq \bar{\varepsilon}$ when

$$\|h\| \leq \rho := \min\{\frac{\bar{\varepsilon}}{2L}, \frac{l}{2}\}\,.$$

Hence, using the last inequality of (3.6) and the definition of $\mathcal{E}(h)$, for $\|h\| \leq \rho$, we have

$$f(x+h) - f(x) - f'(x;h) \leq \frac{1}{2}C\|h\|^2\,,$$

which gives directly (3.3). $\square$

The inequalities (3.3) and (3.4) will be called respectively the *second-order horizontal upper growth condition* and the *half-order vertical upper growth condition*. These conditions have a geometric counterpart.

Theorem 3.2. *The statements* (3.3) *and* (3.4) *of* Theorem 3.1 *are also equivalent to*

$$\exists \bar{\varepsilon}, C > 0 : \varepsilon \leq \bar{\varepsilon} \Rightarrow \partial_\varepsilon f(x) \subset \partial f(x) + B(0, \sqrt{2\varepsilon C})\,, \tag{3.7}$$

$$\exists \bar{\varepsilon}, C > 0 : \varepsilon \leq \bar{\varepsilon} \Rightarrow \Delta_H(\partial f(x), \partial_\varepsilon f(x)) \leq \sqrt{2\varepsilon C}\,, \tag{3.8}$$

$$\exists \delta, D > 0 : \|h\| \leq \delta \Rightarrow \partial f(x+h) \subset \partial f(x) + B(0, D\|h\|)\,. \tag{3.9}$$

Proof. Clearly, (3.4) means

$$f'_\varepsilon(x; d) - f'(x; d) \leq \sqrt{2\varepsilon C} \quad \text{for all} \quad d \in S\,, \tag{3.10}$$

which is (3.7) formulated with support functions. Noting that $\partial f(x)$ and $\partial_\varepsilon f(x)$ are nonempty compact convex sets (see e.g., [8, Theorem XI.1.1.4]) the equivalence with (3.8) is obtained by taking the maximum over $d \in S$ in (3.10). In [11, Corollary 3.5], it is proved that the second-order horizontal upper growth condition (3.3) at x is equivalent to (3.9). $\square$

Note that (3.9) can be written analytically

$$\exists \delta, D > 0 : \|h\| \leq \delta \Rightarrow f'(x+h; d) \leq f'(x; d) + D\|h\| \quad \text{for all} \quad d \in S\,.$$

The equivalence of this relation with (3.3) can be compared to the equivalence between the Dini and de la Vallée-Poussin second derivatives, presented in [4, § 3] or more recently in [2]. Furthermore (3.9) can be connected to a property of strong convexity of f^*, the conjugate function of f:

$$\mathbb{R}^m \ni s \mapsto f^*(s) := \sup\{f(x) - \langle s, x\rangle : x \in \mathbb{R}^m\}\,. \tag{3.11}$$

Proposition 3.3 ([11, Cor. 3.3], [1, Th. 3.4]—). *Assume f satisfies the horizontal upper growth condition* (3.3) *at $x \in \mathbb{R}^m$ with $C, \rho > 0$ and take $g \in \partial f(x)$. Then f^* satisfies the following horizontal lower growth condition:*

$$f^*(s) \geq f^*(g) + \langle x, s-g\rangle + \frac{1}{2C}[d(s, \partial f(x))]^2 \quad \textit{for all} \quad s \in B(g, \frac{\rho C}{2})\,,$$

where $d(s, \partial f(x)) := \min_{s' \in \partial f(x)} \|s' - s\|$.

Yet we only have here one implication; to get the converse, additional assumption is needed on f: if $\partial f(x)$ is "not too big" (for example a singleton) [1, Theorem 3.5] gives us the converse to Proposition 3.3.

3.2. Uniform lower growth conditions

In this section we show that a (uniform) vertical lower growth condition holds if and only if an horizontal lower growth condition is satisfied. We start with the "easy" part.

Proposition 3.4. *Assume f satisfies at $x \in \mathbb{R}^m$*

$$\exists \bar{\varepsilon}, c > 0 : \varepsilon \leq \bar{\varepsilon} \Rightarrow \text{ for all } d \in S,\ c \leq V(x, d, \varepsilon)\,. \tag{3.12}$$

Then we have for the same constant $c > 0$

$$\exists \rho : \|h\| \leq \rho \Rightarrow c \leq H(x, h)\,. \tag{3.13}$$

Proof. Suppose (3.12) holds. Take $h \in \mathbb{R}\setminus\{0\}$, $d = \frac{h}{\|h\|}$ and $t = \|h\|$. Taking the square root in (3.12), multipliying by t and subtracting ε we get

$$t\sqrt{2c\varepsilon} - \varepsilon \leq f'_\varepsilon(x; td) - f'(x; td) - \varepsilon \ \text{ for all } \ \varepsilon \in [0, \bar{\varepsilon}]\,. \tag{3.14}$$

Now let $\varepsilon \in \mathcal{E}(h)$; from Lemma 2.2 and (2.2), we have $\varepsilon \leq \bar{\varepsilon}$ when

$$\|h\| \leq \min\{\frac{\bar{\varepsilon}}{2L}, \frac{l}{2}\}\,.$$

Besides, using (3.14), we obtain for all $h \in B(0, \frac{\bar{\varepsilon}}{2L})$

$$\|h\|\sqrt{2c\varepsilon} - \varepsilon \leq f(x+h) - f(x) - f'(x; h) \ \text{ for all } \ \varepsilon \in [0, \bar{\varepsilon}]\,. \tag{3.15}$$

The concave function $\varepsilon \to \|h\|\sqrt{2c\varepsilon} - \varepsilon$ attains it maximum at $\varepsilon = \frac{1}{2}c\|h\|^2$, which is smaller than $\bar{\varepsilon}$ for $\|h\| \leq \sqrt{\frac{2\bar{\varepsilon}}{c}}$. Setting,

$$\rho := \min\{\frac{l}{2}, \frac{\bar{\varepsilon}}{2L}, \sqrt{\frac{2\bar{\varepsilon}}{c}}\}$$

and taking the maximum in the lefthand side of (3.15), we get

$$\frac{1}{2}c\|h\|^2 \leq f(x+h) - f(x) - f'(x; h) \ \text{ for all } \ h \in B(0, \rho)\,.$$

Then, (3.13) follows. □

The inequalities (3.12) and (3.13) will be called respectively the (uniform) *half-order vertical lower growth condition* and *second-order horizontal lower growth condition.* Proposition 3.4 shows how a vertical lower condition implies an horizontal lower condition. To analyse the converse, we need two elementary lemmas.

Lemma 3.5. *Let $\psi : \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ and $\varphi : \mathbb{R}^m \to \mathbb{R}\cup\{+\infty\}$ be two convex functions. Assume that*

$$\varphi(h) \leq \psi(h) \ \text{ for all } \ h \in \mathbb{R}^m\,, \tag{3.16}$$

and $\psi(0) = \varphi(0)$. Then we have

$$\varphi'_\varepsilon(0; d) \leq \psi'_\varepsilon(0; d) \ \text{ for all } \ d \in S \ \text{ and all } \ \varepsilon \geq 0\,. \tag{3.17}$$

Proof. By assumption, we have for all $\varepsilon \geq 0$

$$\frac{\varphi(td) - \varphi(0) + \varepsilon}{t} \leq \frac{\psi(td) - \psi(0) + \varepsilon}{t} \text{ for all } d \in S \text{ and all } t > 0 .$$

Take the infima and use (2.1) to obtain (3.17). $\square$

Lemma 3.6. *Let $x \in \mathbb{R}^m$ and $\rho > 0$. Set*

$$\mathbb{R}^m \ni h \mapsto \psi(h) := f(x+h) + I_{B(0,\rho)}(h) .$$

Then we have

$$f'_\varepsilon(x;d) \leq \psi'_\varepsilon(0;d) \leq f'_\varepsilon(x;d) + \frac{\varepsilon}{\rho} \text{ for all } d \in S \text{ and all } \varepsilon \geq 0 . \tag{3.18}$$

In particular,

$$\psi'(0;d) = f'(x;d) \text{ for all } d \in S . \tag{3.19}$$

Proof. Use Lemma 3.5 with $\varphi(\cdot)$ replaced by $f(x+\cdot)$ to see that $\partial_\varepsilon f(x) \subset \partial_\varepsilon \psi(0)$. Also, from [8, Theorem XI.3.1.1], for all $\varepsilon \geq 0$

$$\partial_\varepsilon \psi(0) \subset \partial_\varepsilon f(x) + \partial_\varepsilon I_{B(0,\rho)}(0) = \partial_\varepsilon f(x) + N_{B(0,\rho),\varepsilon}(0) .$$

Use support functions and observe that the support function of $N_{B(0,\rho),\varepsilon}(0)$ is $\frac{\varepsilon}{\rho}$; this gives (3.18). To obtain (3.19), set $\varepsilon = 0$. $\square$

Here, using Lemmas 3.5 and 3.6, we specify to what extent a horizontal lower condition implies a vertical lower one.

Proposition 3.7. *Assume (3.13) holds at $x \in \mathbb{R}^m$ for a given $c > 0$. Then we have*

$$\sqrt{2\varepsilon c} - \frac{\varepsilon}{\rho} \leq f'_\varepsilon(x;d) - f'(x;d) \text{ for all } d \in S \text{ and all } \varepsilon \geq 0 , \tag{3.20}$$

or equivalently

$$\partial_\varepsilon f(x) \supset \partial f(x) + B(0, \sqrt{2\varepsilon c} - \frac{\varepsilon}{\rho}) \text{ for all } \varepsilon \geq 0 . \tag{3.21}$$

Proof. Set $\mathbb{R}^m \ni h \mapsto \psi(h) := f(x+h) + I_{B(0,\rho)}(h)$ and $\mathbb{R}^m \ni h \mapsto \varphi(h) := f(x) + f'(x;h) + \frac{1}{2}c\|h\|^2$. From (3.13), we have

$$\varphi(h) \leq \psi(h) \text{ for all } h \in \mathbb{R}^m .$$

Then, apply Lemma 3.5 to get

$$\varphi'_\varepsilon(0;d) \leq \psi'_\varepsilon(0;d) \text{ for all } d \in S \text{ and all } \varepsilon \geq 0 . \tag{3.22}$$

Furthermore it is easy to see that

$$\varphi'(0;d) = f'(x;d) \text{ for all } d \in S ,$$

and

$$\varphi'_\varepsilon(0;d) - f'(x;d) = \inf_{t>0}\{\frac{1}{2}ct + \frac{\varepsilon}{t}\} = \sqrt{2c\varepsilon} \text{ for all } d \in S .$$

Then use (3.22) with Lemma 3.6: (3.20) follows, together with its geometric form (3.21). $\square$

Proposition 3.4 and 3.7 enable us to establish the following theorem of equivalence.

Theorem 3.8. *Let $x \in \mathbb{R}^m$ and $c > 0$. The following statements are equivalent:*

$$\forall c' < c, \exists \rho > 0 : \quad \|h\| \leq \rho \Rightarrow c' \leq H(x,h)\,, \tag{3.23}$$

$$\forall c' < c, \exists \bar{\varepsilon} > 0 : \quad \varepsilon \leq \bar{\varepsilon} \Rightarrow c' \leq V(x,d,\varepsilon) \ \text{ for all } \ d \in S\,, \tag{3.24}$$

$$\forall c' < c, \exists \bar{\varepsilon} > 0 : \quad \varepsilon \leq \bar{\varepsilon} \Rightarrow \partial_\varepsilon f(x) \supset \partial f(x) + B(0, \sqrt{2\varepsilon c'})\,, \tag{3.25}$$

$$\forall c' < c, \exists \bar{\varepsilon} > 0 : \quad \varepsilon \leq \bar{\varepsilon} \Rightarrow \Delta_H(\partial_\varepsilon f(x), \partial f(x)) \leq \sqrt{2\varepsilon c'}\,. \tag{3.26}$$

Proof. [(3.23) $\Rightarrow$ (3.24)]. Assume (3.23) holds and take $0 < c' < c$: there exists ρ such that

$$\|h\| \leq \rho \Rightarrow c' + \eta \leq H(x,h)\,,$$

where $\eta = \frac{c-c'}{2}$. Then, from Proposition 3.7,

$$\sqrt{2\varepsilon(c'+\eta)} - \frac{\varepsilon}{\rho} \leq f'_\varepsilon(x;d) - f'(x;d) \ \text{ for all } \ d \in S \ \text{ and all } \ \varepsilon \geq 0\,.$$

The lefthand side term is equivalent to $\sqrt{2\varepsilon(c'+\eta)}$ when ε tends to 0; hence it is greater than $\sqrt{2\varepsilon c'}$ for ε small enough. This implies (3.24).

[(3.24) $\Rightarrow$ (3.23)]. This sense is a direct consequence of Proposition 3.4. Finally (3.25) and (3.26) are only geometric forms of (3.24) . ☐

3.3. Directional lower growth conditions

In the previous subsection, we have considered bounds such as (3.20) and (3.24), which hold uniformly with respect to the direction d. They bring geometric counterparts such as (3.21) and (3.25) on the approximate subdifferentials. However, if we accept bounds depending on d, we can be more accurate. Specifically, the need of a $c' < c$ can be eliminated from Theorem 3.8.

For all $d \in S$, consider the function

$$\mathbb{R}_+ \ni \varepsilon \to t(\varepsilon, d) := \inf\{t \in \mathbb{R}_+ : \varepsilon \in \mathcal{E}(td)\}\,.$$

It has the following properties (see [4, §2] and [8, §XI.2.3]).

Proposition 3.9. *Assume that the growth condition* (3.13) *holds at $x \in \mathbb{R}^m$. Then, for all $d \in S$, we have*

$$t(\varepsilon, d) > 0 \ \textit{ for all } \ \varepsilon > 0$$

and

$$\lim_{\varepsilon \downarrow 0} t(\varepsilon, d) = 0\,.$$

Proof. Let $\varepsilon > 0$ and $d \in S$. For $t > 0$ small enough we have

$$t\frac{c}{2} + \frac{\varepsilon}{t} + f'(x;d) \leq \frac{f(x+td) - f(x) + \varepsilon}{t}.$$

Then $\lim_{t \downarrow 0} \frac{f(x+td)-f(x)+\varepsilon}{t} = +\infty$ and the infimum cannot be obtained for $t \downarrow 0$. Also, the function $\mathbb{R}_+ \ni t \mapsto f(x+td)$ is not affine in a neighborhood of 0 (in $\mathbb{R}_+$) and thus by [4, Theorem 2.8 (ii)] we obtain the desired result. □

Proposition 3.9, together with Proposition 3.4, leads us to the following directional result.

Theorem 3.10. *Let $x \in \mathbb{R}^m$ and $d \in S$. The following statements are equivalent:*

$$\exists c, \rho > 0 : t \in (0, \rho] \Rightarrow c \leq H(x, td), \tag{3.27}$$

and

$$\exists c, \bar{\varepsilon} > 0 : \varepsilon \leq \bar{\varepsilon} \Rightarrow c \leq V(x, d, \varepsilon). \tag{3.28}$$

The constant c is the same in (3.27) *and* (3.28).

Proof. [(3.27) $\Rightarrow$ (3.28)]. Suppose (3.27) holds. From Proposition 3.9, there exists $\bar{\varepsilon}$ such that

$$0 < t(\varepsilon, d) \leq \rho \ \text{ for all } \ \varepsilon \in (0, \bar{\varepsilon}].$$

Then set $t = t(\varepsilon, d)$ in (3.27), multiply by $t(\varepsilon, d) > 0$ and add $\frac{\varepsilon}{t(\varepsilon,d)}$ to obtain for all $\varepsilon \in (0, \bar{\varepsilon}]$

$$\frac{1}{2} c\, t(\varepsilon, d) + \frac{\varepsilon}{t(\varepsilon, d)} \leq \frac{f(x + t(\varepsilon, d)\, d) - f(x) + \varepsilon - t(\varepsilon, d) f'(x; d)}{t(\varepsilon, d)} = f'_\varepsilon(x; d) - f'(x; d).$$

Thus

$$\sqrt{2c\varepsilon} = \inf_{t>0} \{\frac{1}{2} ct + \frac{\varepsilon}{t}\} \leq f'_\varepsilon(x; d) - f'(x; d) \text{ for all } \varepsilon \in [0, \bar{\varepsilon}],$$

which is (3.28).

[(3.28) $\Rightarrow$ (3.27)]. This sense is direct from Proposition 3.4. □

4. Vertical and horizontal limits

In this section, we take a directional point of view and we show that the second-order horizontal and half-order vertical difference quotients have the same lim sup and lim inf when the horizontal and vertical decrements respectively tend to zero. Then we apply this result to the case of an existing limit: this limit exists if and only if f has a second-order directional derivative. When this limits is not zero we have a development of $f'_\varepsilon(x; d) - f'(x; d)$; when this limit is zero, we are in a particular case of *degeneracy* and further analysis is needed to get a development of $f'_\varepsilon(x; d) - f'(x; d)$.

Theorem 4.1. *Let $x \in \mathbb{R}^m$ and $d \in S$. Then, we have*

(i) *the quotients $H(x,td)$ and $V(x,d,\varepsilon)$ have the same lim sup in $\mathbb{R}_+ \cup \{+\infty\}$ when t and ε respectively tend to 0:*

$$\limsup_{t\downarrow 0} H(x,td) = \limsup_{\varepsilon\downarrow 0} V(x,d,\varepsilon)\,,$$

(ii) *the quotients $H(x,td)$ and $V(x,d,\varepsilon)$ have the same lim inf in $\mathbb{R}_+ \cup \{+\infty\}$ when t and ε respectively tend to 0:*

$$\liminf_{t\downarrow 0} H(x,td) = \liminf_{\varepsilon\downarrow 0} V(x,d,\varepsilon)\,.$$

Proof. [(i)] This case can be divided in two subcases (a) and (b):

(a) One of the quotients is not bounded from above. Then, via Theorem 3.1 (applied in the direction d) the other one is not bounded. This implies

$$\limsup_{t\downarrow 0} H(x,td) = \limsup_{\varepsilon\downarrow 0} V(x,d,\varepsilon) = +\infty\,.$$

(b) Both quotients are (nonnegative and) bounded from above respectively for small t and ε. Then the lim sup's exist in $\mathbb{R}_+$. Set for example

$$C := \limsup_{t\downarrow 0} H(x,td) \in \mathbb{R}_+\,.$$

Then, by definition of the lim sup, for all $\eta > 0$, there exist $\rho > 0$ such that

$$H(x,td) \leq C + \eta \ \text{ for all } \ t \in]0,\rho]\,.$$

Here we stress the fact that $C + \eta > 0$ which enables us to apply Theorem 3.1– [(3.3) $\Rightarrow$ (3.4)]:

$$\exists \bar{\varepsilon} > 0 : \ V(x,d,\varepsilon) \leq C + \eta \ \text{ for all } \ \varepsilon \in]0,\bar{\varepsilon}]\,.$$

This implies

$$C' := \limsup_{\varepsilon\downarrow 0} V(x,d,\varepsilon) \leq C + \eta\,.$$

This inequality can be obtained for any $\eta > 0$. Thus,

$$C' \leq C\,.$$

The converse is similar using Theorem 3.1–[(3.4) $\Rightarrow$ (3.3)]. Then $C = C'$ and (i) is proved.

[(ii)] We consider here three subcases (a), (b) and (c):

(a) one of the lim inf's is $+\infty$, say $\liminf_{t\downarrow 0} H(x,td) = +\infty$. This means $\lim_{t\downarrow 0} H(x,td) = +\infty$ or equivalently

$$\forall c > 0, \exists \rho > 0 : t \in (0,\rho] \Rightarrow H(x,td) \geq c\,.$$

Apply Theorem 3.10–[(3.27)⇒(3.28)], to get

$$\forall c > 0, \exists \bar{\varepsilon} > 0 : \varepsilon \in (0, \bar{\varepsilon}] \Rightarrow V(x, d, \varepsilon) \geq c\,,$$

which implies $\liminf_{\varepsilon \downarrow 0} V(x, d, \varepsilon) = \lim_{\varepsilon \downarrow 0} V(x, d, \varepsilon) = +\infty$. The converse is similar using Theorem 3.10–[(3.28)⇒(3.27)].

(b) Both lim inf's exist in $\mathbb{R}_+ \setminus \{0\}$. This case is very similar to (i)–(b) via Theorem 3.10.

(c) One of the lim inf's is zero, say $\liminf_{t \downarrow 0} H(x, td) = 0$. By contradiction, assume that $\liminf_{\varepsilon \downarrow 0} V(x, d, \varepsilon) > 0$ (possibly $+\infty$); then,

$$\exists c > 0, \bar{\varepsilon} > 0 : \varepsilon \in (0, \varepsilon] \Rightarrow V(x, d, \varepsilon) \geq c\,.$$

Finally, use Theorem 3.10 to prove that

$$\exists c > 0, \rho > 0 : t \in (0, \rho] \Rightarrow H(x, td) \geq c\,,$$

which contradicts our assumption. The converse is similar. ☐

Consider, when it exists, the second-order derivative of f at $x \in \mathbb{R}^m$ in the direction $d \in S$:

$$f''(x; d) := \lim_{t \downarrow 0} H(x, td)\,.$$

As a corollary of Theorem 4.1, we have the following characterization of the second-order directional derivative.

Corollary 4.2. *The function f has a second-order derivative in the direction $d \in S$ at $x \in \mathbb{R}^m$ if and only if $V(x, d, \varepsilon)$ has a limit when ε tends to 0. In this case we have*

$$f''(x; d) = \lim_{\varepsilon \downarrow 0} V(x, d, \varepsilon)\,.$$

Proof. This is straightforward from Theorem 4.1. ☐

Corollary 4.2 can also be found as Theorem 2.1 in [18].

When $f''(x; d) \neq 0$, we have an estimate of $f'_\varepsilon(x; d) - f'(x; d)$:

$$f'_\varepsilon(x; d) - f'(x; d) = \sqrt{2f''(x; d)\varepsilon} + o(\sqrt{\varepsilon})\,.$$

When $f''(x; d) = 0$, Corollary 4.2 still applies but we do not have any more an equivalent of $f'_\varepsilon(x; d) - f'(x; d)$ in $\varepsilon^{1/2}$. This case is a particular case of *degeneracy*: the graph of f "osculates" its tangent at x in the direction d,

$$\liminf_{t \downarrow 0} H(x, td) = 0\,.$$

The osculation can be isolated, for example for $f(x) = \frac{1}{\gamma} c\|x\|^\gamma$ with $\gamma > 2$: in this case f is still strictly convex but not strongly convex at 0 in any direction $d \in S$; it is shown in [12] that we have for this example

$$f'_\varepsilon(x; d) - f'(x; d) = (\gamma c \varepsilon)^{1/\gamma} + o(\varepsilon^{1/\gamma})\,.$$

The osculation can also hold on an interval, say

$$\mathbb{R}_+ \ni t \to f(x+td) \text{ is affine in a neighborhood of } 0 \text{ (in } \mathbb{R}_+). \tag{4.1}$$

This case is well studied in [4]. Introducing

$$t_d := \sup\{t \in \mathbb{R}_+ : f(x+td) = f(x) + tf'(x;d)\} \in \mathbb{R}_+ \cup \{+\infty\},$$

the degeneracy condition (4.1) is equivalent to say $t_d > 0$. With this notation we have the following horizontal/vertical result.

Theorem 4.3 ([4, Th. 2.8 (i)], [8, Prop. XI.2.3.4]). *Let $x \in \mathbb{R}^m$ and $d \in S$; then we have*

$$\lim_{\varepsilon \downarrow 0} \frac{f'_\varepsilon(x;d) - f'(x;d)}{\varepsilon} = \frac{1}{t_d} \in \mathbb{R}_+ \cup \{+\infty\}. \tag{4.2}$$

Thus this result tells us for example that $t_d = 0$ means $\varepsilon = o(f'_\varepsilon(x;d) - f'(x;d))$; on the other hand, $t_d = +\infty$ means $f'_\varepsilon(x;d) = f'(x;d)$ for all $\varepsilon \geq 0$.

5. Application to the maximum eigenvalue function

We consider the convex function $\mathcal{S}_n \ni A \mapsto \lambda(A)$, where $\mathcal{S}_n$ is the space of symmetric $n \times n$ matrices and $\lambda(A)$ is the maximum eigenvalue of A. We prove that $\lambda(\cdot)$ satisfies uniform upper growth conditions and we derive more than first-order results for it. We introduce first the following notation.

- $\lambda(A) = \lambda_1(A) \geq \ldots \geq \lambda_n(A)$ are the eigenvalues of $A \in \mathcal{S}_n$
- $E(A) \subset \mathbb{R}^n$ is the eigenspace associated with $\lambda(A)$
- $F(A) = E(A)^\perp$ is the subspace orthogonal to $E(A)$
- $\mathrm{proj}_{E(A)}\, x$ is the orthogonal projection of $x \in \mathbb{R}^n$ onto the subspace $E(A)$
- I is the identity matrix
- $\mathcal{O}_n := \{U \in \mathbb{R}^{n\times n} : U^T U = I\}$ is the set of unitary matrices
- $A \bullet B := \mathrm{tr}\, AB$ standard Euclidean inner product of $A, B \in \mathcal{S}_n$
- $\|A\| := \sqrt{A \bullet A}$ Euclidean norm of $A \in \mathcal{S}_n$
- S is the unit sphere of $\mathcal{S}_n$ equipped with $\|\cdot\|$
- $\mathcal{S}_n^+ \subset \mathcal{S}_n$ is the cone of positive semidefinite matrices
- $A \preceq B$ means that $B - A \in \mathcal{S}_n^+$

We start with the following descriptions of the subdifferential and ε-subdifferential of λ at $A \in \mathcal{S}_n$.

Theorem 5.1. *The subdifferential of λ at $A \in \mathcal{S}_n$ is given by*

$$\partial\lambda(A) = \{V \in \mathcal{S}_n^+ : \mathrm{tr}\, V = 1,\ A \bullet V = \lambda(A)\}, \tag{5.1}$$

and for all $\varepsilon \geq 0$, its ε-subdifferential is

$$\partial_\varepsilon\lambda(A) = \{V \in \mathcal{S}_n^+ : \mathrm{tr}\, V = 1,\ A \bullet V \geq \lambda(A) - \varepsilon\}. \tag{5.2}$$

Proof. The proof of (5.1) can be found in [15, Theorem 3] and [7, Theorem 3.1]. The extension to the ε-subdifferential is proposed in [23]. □

Using a matrix notation, we establish the following linear algebra result.

Lemma 5.2. *Let $U \in \mathcal{O}_n$. Then, there exist $n \times n$ matrices (E, F, Σ, T) such that*

$$\left.\begin{array}{ll} \text{(a)} & \textit{the columns of } E \textit{ are unit vectors of } E(A) \\ \text{(b)} & \textit{the columns of } F \textit{ are unit vectors of } F(A) \\ \text{(c)} & \Sigma \textit{ and } T \textit{ are diagonal and positive semidefinite} \\ \text{(d)} & \Sigma^2 + T^2 = I \\ \text{(e)} & U = E\Sigma + FT \end{array}\right\} \tag{5.3}$$

Proof. Let $e \in E(A)$ and $f \in F(A)$. Set $U = [u_1, \dots, u_n]$ and decompose each vector u_i on $E(A) \oplus F(A) = \mathbb{R}^n$ for $i = 1, \dots, n$ as follows:

$$u_i = \sigma_i e_i + \tau_i f_i \,,$$

where

$$\begin{cases} \sigma_i = \| \operatorname{proj}_{E(A)} u_i \| \\ e_i = \begin{cases} \frac{\operatorname{proj}_{E(A)} u_i}{\sigma_i} & \text{if } \sigma_i > 0 \\ e & \text{otherwise} \end{cases} \end{cases} \quad \text{and} \quad \begin{cases} \tau_i = \| \operatorname{proj}_{F(A)} u_i \| \\ f_i = \begin{cases} \frac{\operatorname{proj}_{F(A)} u_i}{\tau_i} & \text{if } \tau_i > 0 \\ f & \text{otherwise} \end{cases} \end{cases}$$

Then verify that

$$\begin{array}{ll} E := [e_1, \dots, e_n]\,, & \Sigma := \operatorname{diag}(\sigma_1, \dots, \sigma_n) \\ F := [f_1, \dots, f_n]\,, & T := \operatorname{diag}(\tau_1, \dots, \tau_n) \end{array}$$

satisfy the desired properties. □

We will also use the following inequality.

Lemma 5.3. *Let Σ, T, $\Delta \in \mathcal{S}_n$ be positive semidefinite diagonal matrices such that $\Sigma^2 + T^2 = I$ and $\operatorname{tr} \Delta = 1$. Then, we have*

$$\operatorname{tr}(\Sigma \Delta T) \leq [\operatorname{tr}(T \Delta T)]^{1/2} \,. \tag{5.4}$$

Proof. Note that $\Sigma \preceq I_n$ and ΔT is (diagonal) positive semidefinite to get

$$\operatorname{tr}(\Sigma \Delta T) \leq \operatorname{tr}(\Delta T) \,. \tag{5.5}$$

Now use $\Delta = \operatorname{diag}(\delta_1, \dots, \delta_n) \succeq 0$, $\operatorname{tr} \Delta = 1$ and $T = \operatorname{diag}(t_1, \dots, t_n) \succeq 0$, together with the concavity of the square-root function, to obtain

$$\sum_{i=1}^{n} \delta_i t_i \leq (\sum_{i=1}^{n} \delta_i t_i^2)^{1/2} \,.$$

With a matrix notation, this means $\operatorname{tr}(\Delta T) \leq [\operatorname{tr}(T \Delta T)]^{1/2}$. Altogether with (5.5), this gives (5.4). □

Besides the introduced notation, we denote by r the multiplicity of $\lambda(A)$; then we have the following characterization of ε-subgradients.

Proposition 5.4. *Let $A \in \mathcal{S}_n$ and $\varepsilon \geq 0$. For all $V \in \partial_\varepsilon \lambda(A)$, there exists $G \in \partial\lambda(A)$ and five $n \times n$ matrices $(E, F, \Sigma, T, \Delta)$ such that*

$$\left.\begin{array}{ll} \text{(a)} & (E,F,\Sigma,T) \textit{ satisfy } (5.3)\,(a,b,c,d) \\ \text{(b)} & \Delta \textit{ is diagonal, positive semidefinite and } \operatorname{tr}\Delta = 1 \\ \text{(c)} & V = G + (E\Sigma\Delta T F^T + F\Sigma\Delta T E^T) + (FT\Delta T F^T - ET\Delta T E^T) \\ \text{(d)} & \operatorname{tr} T\Delta T \leq \frac{\varepsilon}{\lambda(A) - \lambda_{r+1}(A)} \end{array}\right\} \qquad (5.6)$$

Proof. Write the spectral decomposition of $V \in \partial_\varepsilon\lambda(A)$: there exists $U \in \mathcal{O}_n$ and a diagonal matrix Δ such that $V = U\Delta U^T$. In view of (5.2), we have $\Delta \succeq 0$ and $\operatorname{tr}\Delta = 1$. Then, apply Lemma 5.2: $U = E\Sigma + FT$ where (E, F, Σ, T) satisfy $(5.3)\,(a, b, c, d)$. Plugging this in the spectral decomposition of V, we obtain

$$V = E\Delta E^T + (E\Sigma\Delta T F^T + F\Sigma\Delta T E^T) + (FT\Delta T F^T - ET\Delta T E^T)\,. \qquad (5.7)$$

Furthermore,

$$\begin{aligned} E\Delta E^T \bullet A &= \Delta \bullet (E^T A E) &&= \lambda(A)\operatorname{tr}(\Delta E E^T) \\ &= \lambda(A)\textstyle\sum_{i=1}^n \delta_i \|e_i\|^2 &&= \lambda(A)\,, \end{aligned}$$

and $E\Delta E^T \succeq 0$ means, together with (5.1), that $G := E\Delta E^T \in \partial\lambda(A)$. Then (a, b, c) are satisfied;
we still have to prove (d): use $A \bullet V \geq \lambda(A) - \varepsilon$, together with (5.7), to obtain

$$A \bullet V = \lambda(A) + (T\Delta T) \bullet (F^T A F - \lambda(A) I) \geq \lambda(A) - \varepsilon\,.$$

Knowing that $f_i^T A f_i \leq \lambda_{r+1}(A)$, for $i = 1, \ldots, n$, we obtain (d) and the proof is complete. $\square$

We are now in a position to prove that λ satifies everywhere a half-order vertical upper growth condition uniform with respect to $D \in S$.

Theorem 5.5. *At $A \in \mathcal{S}_n$, λ satisfies the following half-order vertical upper growth condition (3.4): for all $D \in \mathcal{S}_n$ and all $\varepsilon \geq 0$*

$$\lambda'_\varepsilon(A; D) \leq \lambda'(A; D) + \left[(2C(A)\varepsilon)^{1/2} + 2C(A)\varepsilon\right]\, \|D\|\,. \qquad (5.8)$$

where $C(A) := \frac{1}{\lambda(A) - \lambda_{r+1}(A)}$.

Proof. Let $D \in S$, $\varepsilon \geq 0$ and $V \in \partial_\varepsilon\lambda(A)$. Use Proposition 5.4, to write

$$V \bullet D = G \bullet D + (\Sigma\Delta T) \bullet (E^T D F + F^T D E) + (T\Delta T) \bullet (F^T D F - E^T D E)\,.$$

Let us bound from above each of the three terms. First, $G \in \partial\lambda(A)$ implies

$$G \bullet D \leq \lambda'(A; D)\,. \qquad (5.9)$$

Then, denoting $(\Sigma\Delta T) = \operatorname{diag}(\sigma_1\delta_1 t_1, \ldots, \sigma_n\delta_n t_n)$, we have

$$(\Sigma\Delta T) \bullet (E^T D F + F^T D E) = \sum_{i=1}^n \sigma_i\delta_i t_i\,[D \bullet (e_i f_i^T + f_i e_i^T)]\,.$$

Now, use Cauchy-Schwartz inequality, together with $\|e_i f_i^T + f_i e_i^T\| = \sqrt{2}$, to get

$$(\Sigma\Delta T) \bullet (E^T DF + F^T DE) \leq \sqrt{2}\|D\| \operatorname{tr}(\Sigma\Delta T)\,. \tag{5.10}$$

Similarly we have for the last term

$$\begin{array}{rcl}(T\Delta T) \bullet (F^T DF - E^T DE) & = & \sum_{i=1}^n \delta_i t_i^2 \,[D \bullet (f_i f_i^T - e_i e_i^T)] \\ & \leq & \operatorname{tr}(T\Delta T)\,\|D\|\,(\|f_i f_i^T\| + \|e_i e_i^T\|) \\ & \leq & 2\operatorname{tr}(T\Delta T)\,\|D\|\,, \end{array} \tag{5.11}$$

since $\|f_i f_i^T\| = \|e_i e_i^T\| = 1$. Putting together (5.9), (5.10), (5.4) and (5.11), we get

$$V \bullet D \leq \lambda'(A; D) + \sqrt{2}\|D\|\,(\operatorname{tr}(T\Delta T))^{1/2} + 2\|D\|\,\operatorname{tr}(T\Delta T)\,. \tag{5.12}$$

Altogether with $(5.6)_{(\mathrm{d})}$, we obtain (5.8). □

Applying now the equivalence of § 3.1 between horizontal and vertical developments, we obtain the following result.

Corollary 5.6. *We have*

(i) λ *satisfies the half-order vertical upper growth condition* (3.4)*;*
(ii) λ *satisfies the second-order horizontal upper growth condition* (3.3)*;*
(iii) $\partial\lambda(\cdot)$ *is radially Lipschitz.*

Proof. Consider $C(A)$ from (5.8) and set $\bar{\varepsilon} := \frac{1}{2C(A)}$. Then for all $\varepsilon \in [0, \bar{\varepsilon}]$ and $D \in S$, we have

$$\lambda'_\varepsilon(A; D) \leq \lambda'(A; D) + \sqrt{2\varepsilon C'(A)}\,,$$

where $C'(A) = 4C(A)$. Hence (i) holds. Then apply respectively Theorem 3.1 and Theorem 3.2 to obtain (ii) and (iii). □

The fact that $\partial\lambda(\cdot)$ is radially Lipschitz was obtained by A. Shapiro using perturbation theory techniques [19, § 5]. Using recent results [20] we could have obtained directly the second-order horizontal upper growth condition by bounding from above the second-order directional derivative of λ.

6. Conclusion

This paper was motivated by the interest of the ε-subdifferential itself [8], [22], [5], [14]. Our aim was to give a better understanding of vertical developments. In particular we have shown in § 3 that bounding $V(x, \cdot, \varepsilon)$ for some $\varepsilon > 0$ allows us to bound $H(x, \cdot)$ in a whole neighborhood of $h = 0$ or equivalently to obtain a Lipschitz-like property of $\partial f(x + \cdot)$.

In addition to bounding difference quotients, one can also pass to the limit (§ 4). This somehow reveals a limitation of the vertical approach. Indeed, when it exists, $\lim_{\varepsilon\downarrow 0} V(x, d, \varepsilon) = \lim_{t\downarrow 0} H(x, td)$ but such a limit may not be so useful: it gives second-order information along the half-line $x + \mathbb{R}_+ d$ only, and it is not even continuous in d [8, Remark X.4.2.11]. Alternatively, the convergence cannot be uniform with respect to

$d \in S$. This explains the need for more sophisticated objects such as second-order epi-derivatives [17]. According to [10], it is also relevant to consider $\lim_{t\downarrow 0, d_t \to d} H(x, td_t)$ for a *particular* convergence law $t \mapsto d_t$.

Another comment concerns $\lambda(\cdot)$: using techniques from sensitivity analysis [9], directional second-order horizontal developments of $\lambda(\cdot)$ can be obtained as in [20] and [21]. Our contribution here, via the vertical approach, is to give uniform bounds on the second-order difference quotients without the perturbation theory needed in [20] and [21]: this is Corollary 5.6.

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