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Convergence of Unbounded Multivalued Supermartingales in the Mosco and Slice Topologies

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Our starting point is the Mosco-convergence result due to Hess ([18]) for integrable multivalued supermartingales whose values may be unbounded, but are majorized by a w-ball-compact-valued function. It is shown that the convergence takes place also in the slice topology. In the case when both the underlying space X and its dual X^* have the Radon-Nikodym property a weaker compactness assumption guarantees convergence of the multivalued supermartingales in the slice topology. This result implies convergence in the Mosco topology and gives an analogue of Hess' result in the case when X and X^* have the RNP. Finally the results are restated in terms of normal integrands.

1. Introduction

There is a substantial number of results concerning convergence of random sets. These results are interesting not only for probabilists. They can be also applied in stochastic optimization or mathematical economics. Relaxing the assumption of boundedness of random sets leads to new applications of convergence results. These results can be applied for example to normal integrands. In that case one considers the epigraphs of normal integrands, which are obviously unbounded. This has been done for example in [18], [20], [21], [10] and [12].

Recently Beer [4] introduced the so called slice topology which coincides with the Mosco topology in reflexive spaces but is strictly stronger in general Banach spaces. One of the most important features of the slice topology is the fact that when the dual of the underlying Banach space is separable, Effros measurable multifunctions are measurable with respect to the Borel σ-algebra on the power set of that space equipped with the slice topology. Hess [21] proved the SLLN for unbounded random sets in the slice topology. Ezzaki [17] studied convergence of conditional expectations in that setting.

Here we consider a sequence of multivalued integrable supermartingales (F_n) whose values are supposed to be closed convex subsets of a separable Banach space. The sequence is supposed to fulfill some compactness condition. In the case when nothing is assumed about X^* the sequence is supposed to be majorized by a w-ball-compact set valued function. In that case the result follows easily from the theorem 5.12 of [18] (see lemma 5.1 here). Namely in that case the Mosco topology and the slice topology coincide. Many convergence results for sequences majorized by a w-ball-compact-valued functions (see [2], [25]) have analogues in the following setting. If X and X^* have both the Radon-Nikodym

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property (thus X^* is separable) we require that the integrals of F_n over any element A of the underlying σ-algebra are contained in a w-ball-compact set dependent on the set A. The techniques used in the proof are inspired by the proof of the already mentioned Hess' result ([18]) and the proof of theorem 2.5 in [2]. Somewhat stronger condition, namely domination of the integrals by a w-compact set, appeared in [6], [7] and [8] where bounded multivalued functions were considered. However, when mutivalued functions whose values may be unbounded are considered, this condition is not suitable.

As an application we restate our results in terms of supermartingale integrands i.e. normal integrands whose epigraphs are multivalued (unbounded) supermartingales.

2. Preliminaries

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, $(\mathcal{A}_n)$ an increasing sequence of sub-σ-algebras of $\mathcal{A}$ such that $\mathcal{A} = \sigma(\bigcup_n \mathcal{A}_n)$. Let X be a separable Banach space with the norm $\|\cdot\|$. X^* will denote the dual of X and $\langle\cdot,\cdot\rangle$ the usual duality. The strong and weak topologies on X will be denoted by s and w respectively. Let $\mathcal{P}(X)$ be the family of all closed subsets of X, $\mathcal{P}_c(X)$ closed convex, $\mathcal{P}_{cb}(X)$ closed convex bounded. $\mathcal{P}_{wkc}(X)$ will be used for the family of w-compact convex subsets of X. Also subfamilies of those families will be considered. Given a set $Y \subset X$ the restriction of $\mathcal{P}_.(X)$ to Y will be denoted $\mathcal{P}_.(Y)$. A non empty closed convex subset of X will be called *w-ball-compact* if its intersection with any closed ball is w-compact. Denote the family of w-ball-compact sets by $\mathcal{R}_w(X)$. For any set $Y \subset X$ let $\operatorname{cl}\operatorname{co} Y$ denote a closed convex hull of Y.

The *support function* and the *radius* of the set C in $\mathcal{P}$ will be defined in the following way:

$$s(x^*, C) := \sup_{x\in C}\langle x, x^*\rangle,\ \ \|C\| := \sup_{x\in C}\|x\|.$$

Topology of convergence of support functions will be denoted $\mathcal{T}_{\text{scalar}}$. A sequence (C_n) $\mathcal{T}_{\text{scalar}}$-converges to some set C if $s(x^*, C_n) \to s(x^*, C)$ for all $x^* \in X^*$. The *distance functional* is a mapping $d\colon X \times \mathcal{P}(X) \to \mathbb{R}$ such that

$$d(x, C) := \inf\{\|x - c\| \colon c \in C\}.$$

The topology determined by the convergence of distance functionals is called the *Wijsman topology.* It will be denoted by $\mathcal{T}_{\text{Wijsman}}$. Let the *strong lower limit* (denoted by s-li) of a sequence (K_n) be the set of all $x \in X$ such that $x = s\text{-}\lim x_n$, where $x_n \in K_n$ and the *weak upper limit* (denoted by w-ls) be the set of all $x \in X$ such that $x = w\text{-}\lim x_k$, where $x_k \in K_{n_k}$. A sequence $(K_n) \in \mathcal{P}(X)$ converges in the sense of Mosco to $K \in \mathcal{P}(X)$ (notation: $K = \mathcal{T}_{\text{Mosco}}\text{-}\lim K_n$) if

$$K = s\text{-li } K_n = w\text{-ls } K_n.$$

This holds if and only if w-ls $K_n \subset K \subset s$-li K_n.

Let $B(x_0, r)$ denote an open ball with the center at x_0 and radius r. The *slice topology* on $\mathcal{P}(X)$ is the initial topology $\mathcal{T}_{\text{slice}}$ determined by the family of gap functionals

$$\{D(C,\cdot)\colon C \text{ is a nonempty slice of a ball}\}$$

where $D(B,C) := \inf\{\|b-c\|\colon b \in B, c \in C\}$ and a *slice of a ball* is an intersection of $\overline{B}(x_0,r) \cap \{x\colon \langle x, x^*\rangle \le \alpha\}$ (provided it is not empty). The slice topology is generally stronger than the Mosco topology (and the Wijsman topology). It coincides with the Mosco topology if and only if X is reflexive ([4]). The slice topology restricted to $\mathcal{P}_c(X)$ is generated by the subfamily $\{D(B,\cdot)\colon B \in \mathcal{P}_{cb}(X)\}$.

A *multifunction* is any mapping $F\colon \Omega \to 2^X$. A multifunction F is said to be *(Effros) measurable* if the preimage $F^-U = \{\omega \in \Omega\colon F(\omega) \cap U \neq \emptyset\}$ lies in $\mathcal{A}$ for any s-open set $U \subset X$. Only measurable multivalued functions with closed values will be considered and the adjective will be often omitted. Measurable multivalued functions are often called *random sets*.

Given any topology on 2^X it is an interesting question when Effros measurability coincides with the measurability with respect to the Borel σ-algebra generated by that topology. The Wijsman topology fulfills this condition if X is a separable metric space (Hess' theorem, [5, theorem 6.5.14]. The topology generated by the Hausdorff distance works if multifunctions with compact values are considered, or, more generally, when multifunctions have values in a ρ_H-separable subspace of $\mathcal{P}(X)$[1] ([1, theorem 3.4]). Finally, as was mentioned in the introduction, the slice topology guarantees the equivalence in a general case provided that X^* is separable. Another important fact is that, when X is reflexive, then the slice topology restricted to $\mathcal{P}(X)$ reduces to the topology generated by the Mosco convergence.

$\mathcal{L}^1_{\mathbb{R}}$ will denote the family of real valued, integrable functions. Given a sub-σ-algebra $\mathcal{B}$ of $\mathcal{A}$ let $\mathcal{L}^1_X(\mathcal{B})$ denote the family of X-valued Bochner integrable $\mathcal{B}$-measurable functions. We will write $\mathcal{L}^1_X$ instead of $\mathcal{L}^1_X(\mathcal{A})$. Let $\mathcal{L}^1_{\mathcal{P}(X)}$ denote the space of all closed valued random sets F such that $\|F\| \in \mathcal{L}^1_{\mathbb{R}}$. Those functions will be called integrably bounded. $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ will denote the subspace of $\mathcal{P}_{wkc}(X)$-valued random sets in $\mathcal{L}^1_{\mathcal{P}(X)}$. We will say that F is integrable if $d(0, F(\cdot)) \in \mathcal{L}^1_{\mathbb{R}}$. A multifunction is integrable if and only if it admits at least one Bochner integrable selection. A set $\mathcal{H}$ in $\mathcal{L}^1_{\mathcal{P}(X)}$ is *bounded* if the set $\{\|F\| : F \in \mathcal{H}\}$ is bounded in $\mathcal{L}^1_{\mathbb{R}}$. Recall that a set $E \subset \mathcal{L}^1_{\mathbb{R}}$ is *uniformly integrable* (shortly *UI*) if $\lim_{t\to\infty} \sup_{f\in E} \int_{[|f|>t]} |f| dP = 0$.

Let $\mathcal{L}^1_F(\mathcal{B}) := \{f \in \mathcal{L}^1_X(\mathcal{B}) : f(\omega) \in F(\omega) \text{ a.e.}\}$ be the set of integrable $\mathcal{B}$-measurable selections of F. In particular $\mathcal{L}^1_F := \mathcal{L}^1_F(\mathcal{A})$. The integral of $\mathcal{P}(X)$-valued function is defined as $\int_\Omega F dP := \{\int_\Omega f dP : f \in \mathcal{L}^1_F\}$. For $A \in \mathcal{A}$, $\int_A F dP$ is the integral of $F|_A$. In general this set might be empty. Theorem 3.6(ii) of [24] yields that for $F \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ the set $\int_A F dP$ is non empty and belongs to $\mathcal{P}_{wkc}(X)$.

If F is a $\mathcal{P}(X)$-valued function with $\mathcal{L}^1_F \neq \emptyset$ then (by theorem 5.1 of [23]) there exists an almost surely unique $\mathcal{B}$-measurable $\mathcal{P}(X)$-valued function $E^{\mathcal{B}}F$ satisfying

$$\mathcal{L}^1_{E^{\mathcal{B}}F}(\mathcal{B}) = \mathrm{cl}\{E^{\mathcal{B}} f : f \in \mathcal{L}^1_F\}$$

where the closure is taken in $\mathcal{L}^1_X$. $E^{\mathcal{B}}F$ will be called the *conditional expectation* of F with respect to the σ-algebra $\mathcal{B}$.

A sequence of random sets (F_n) is called a *martingale*, (resp. *supermartingale, submartingale*) if for all $m < n$, $E^{\mathcal{A}_m} F_n = F_m$ (resp. $E^{\mathcal{A}_m} F_n \subset F_m$, $E^{\mathcal{A}_m} F_n \supset F_m$).

[1] I would like to thank Ch. Hess for pointing this out to me.

3. Results

Combining the result on the Mosco convergence of $\mathcal{P}_c(X)$-valued supermartingales ([18, theorem 5.12]) and suitable properties of the slice topology the following result, which is essentially due to Hess, can be obtained.

Theorem 3.1. *Let $(F_n, \mathcal{A}_n)$ be a $\mathcal{P}_c(X)$-valued supermartingale. Suppose that*

(i) $\sup_n Ed(0, F_n) < \infty$,

(ii) *for all $n \in \mathbb{N}$ and almost all $\omega \in \Omega$ the sets $F_n(\omega)$ are majorized by a w-ball-compact set R dependent on ω.*

Then there exists an integrable random set F_∞ with values in $\mathcal{R}_w$ such that $F_\infty = \mathcal{T}_{\text{slice}}\text{-}\lim F_n$ a.e. Moreover if (F_n) is bounded in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ then F_∞ is integrably bounded. If $(d(0, F_n))$ is uniformly integrable then $E^{\mathcal{A}_n} F_\infty \subset F_n$ a.e. for all $n \in \mathbb{N}$.

If the space X is reflexive then $X \in \mathcal{R}_w$, so condition (ii) is trivially satisfied (take $R := X$). At the same time, in reflexive spaces convergence in the slice topology and Mosco convergence are equivalent. Convergence of multivalued martingales in the Mosco topology in reflexive spaces has been investigated in [10, theorem 2.3].

Recall that X has the RNP with respect to $(\Omega, \mathcal{A}, P)$ if any P-absolutely continuous measure Q with bounded variation has a density $f \in \mathcal{L}^1_X$ with respect to P, that is $Q(A) = \int_A f dP$ for all $A \in \mathcal{A}$.

Let us make some observations. The strong law of large numbers in [21] is proved in two cases. In one case the sequence is majorized by a fixed w-ball-compact set. In the second case the space X^* is supposed to be strongly separable. Also many convergence results (see [2] or [25]) that hold for sequences majorized by w-ball-compact-valued functions have analogues in the case when both the space X and its dual X^* have both the RNP (thus also X^* is separable, see [15, Stegall's theorem p.195]). In that case a different compactness assumption of a more global nature is required. This leads to formulation of the following analogues of theorem 5.12 of [18] (see proposition 5.1 in this paper) and theorem 3.1.

Theorem 3.2. *Suppose X and X^* have both the RNP (thus X^* is separable). Let $(F_n, \mathcal{A}_n)$ be a $\mathcal{P}_c(X)$-valued supermartingale, such that*

(i) $\sup_n Ed(0, F_n) < \infty$,

(ii) *for all $A \in \mathcal{A}$ the sets $\operatorname{cl}\operatorname{co} \bigcup_n \int_A F_n dP$ are w-ball-compact.*

Then there exists an integrable random set F_∞ with values in $\mathcal{R}_w$ such that $F_\infty = \mathcal{T}_{\text{slice}}\text{-}\lim_n F_n$ a.e. Moreover, if (F_n) is bounded in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ then F_∞ is integrably bounded. If $(d(0, F_n))$ is uniformly integrable then $E^{\mathcal{A}_n} F_\infty \subset F_n$ a.s.

Obviously also $F_\infty = \mathcal{T}_{\text{Mosco}}\text{-}\lim_n F_n$ a.e. This provides an analogue of theorem 5.12 of [18] (see proposition 5.1 in the present paper). To prove theorem 3.2 we will prove the convergence in the Mosco topology first (proposition 5.5) and extend this result to the slice topology case.

Of course when the space X is reflexive then condition (ii) is trivially satisfied. Mosco convergence of $\mathcal{P}(X)$-valued martingales in reflexive spaces has been investigated in

[12, theorem 2.14]. However an extra assumption on the set of martingale selections $MS(F_n) := \{(f_n) : f_n = E^{\mathcal{A}_n} f_{n+1}, f_n \in \mathcal{L}^1_{F_n}(\mathcal{A}_n) \text{ for all } n \in \mathbb{N}\}$ was needed there. Namely it was required that there exists an $(f_n) \in MS(F_n)$ such that $\sup_{n\in\mathbb{N}} \|f_n\| \in \mathcal{L}^1_{\mathbb{R}}$. Theorem 4.4 of [18] yields that this condition is satisfied under condition (i) of theorem 3.2. In [10, theorem 2.3] this extra assumption has been removed.

As example 2.9 in [2] shows, condition (ii) in theorem 3.2 or proposition 5.5 does not imply (ii) in theorem 3.1.

4. Application. Convergence of supermartingale integrands

As was mentioned in the introduction convergence results for unbounded random sets provide a convenient tool for investigating convergence of normal integrands. In this section theorem 3.1 and theorem 3.2 will be restated in terms of integrands. Let us first recall (see [18] and [21]) some basic definitions and facts.

Given a function $u\colon X \to \overline{\mathbb{R}}$ its *epigraph* is defined in the following way

$$\text{epi } u := \{(x, \alpha) \in X \times \mathbb{R}\colon u(x) \le \alpha\}$$

u is *proper* if it is not the constant $+\infty$ and does not attain value $-\infty$. Given functions $u_n, u : X \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$ we say that the sequence (u_n) $\mathcal{T}_{slice}$*-converges* to a function u if epi u_n $\mathcal{T}_{\text{slice}}$-converges in $X \times \mathbb{R}$ to epi u. A function $\varphi\colon \Omega \times X \to \overline{\mathbb{R}}$ is called a *normal integrand* if

(i) $\varphi(\omega, \cdot)$ is l.s.c. for almost all $\omega \in \Omega$,
(ii) the multifunction $\omega \mapsto$ epi $\varphi(\omega, \cdot)$ is measurable.

The multifunction that appears in condition (ii) is called the *epigraphical multifunction* of φ. Sometimes the term *random lower semicontinuous function* is used for a normal integrand. The above definition of a normal integrand is due to Rockafellar ([27]). Some authors (see [16]) define a normal integrand differently. Instead of conditions (ii) it is assumed that $\varphi(\cdot, \cdot)$ is $\mathcal{A} \otimes \mathcal{B}(X)$-measurable. However, if $(X, \mathcal{A}, P)$ is complete both notions coincide ([9, VII.1]). We say that a normal integrand φ is *integrable* if its epigraph is an integrable multifunction i.e. $\omega \mapsto d(0, \text{epi } \varphi(\omega, \cdot))$ is integrable. The last function will be denoted simply $d(\varphi)$.

The conditional expectation of a normal integrand is also defined via its epigraph. Namely, the *conditional expectation of a normal integrand* φ with respect to the σ-algebra $\mathcal{B}$ is the normal integrand γ, whose epigraphical multifunction is the conditional expectation of $\omega \mapsto$ epi $\varphi(\omega, \cdot)$ ([9, ch. VIII, §9]). When $\mathcal{B} = \{\emptyset, \Omega\}$ the integrand γ can be expressed as $\gamma = \text{cl}\, \eta$, where cl denotes the l.s.c. regularization operation and where η is given by the continuous infimum convolution

$$\eta(x) = \inf\left\{\int_\Omega \gamma(\omega, f(\omega))dP(\omega) : f \in \mathcal{L}^1_X, \int_\Omega f(\omega)dP(\omega) = x\right\}, \quad x \in X.$$

(see formula (4.7) in [21]).

A sequence (φ_n) of convex normal integrands is a *martingale (submartingale, supermartingale) integrand* if the sequence of its epigraphical multifunctions is a martingale (submartingale, supermartingale). The results of the previous section can be restated in terms of normal integrands in the following way.

Theorem 4.1. *Let X be a separable Banach space and (φ_n) a supermartingale integrand such that* $\sup_n Ed(\varphi_n) < \infty$ *and one of the following conditions holds:*

(i) *for almost all $\omega \in \Omega$ there exists a w-ball-compact set $R(\omega)$ such that* $\operatorname{epi} \varphi_n(\omega, \cdot) \subset R(\omega) \times \mathbb{R}$.

(ii) *in the case when both X and X^* have the RNP and for any $A \in \mathcal{A}$ there exists a w-ball-compact set R_A such that* $\bigcup_n \int_A \operatorname{epi} \varphi_n(\omega, \cdot) dP(\omega) \subset R_A \times \mathbb{R}$.

Then there exists an integrable convex normal integrand φ_∞ such that for almost all $\omega \in \Omega$, $\varphi_\infty(\omega, \cdot) = \mathcal{T}_{\text{slice}}\mathcal{T}_{\text{slice}}\text{-}\lim \varphi_n(\omega, \cdot)$.

Remark 4.2. Obviously, we have also that $\operatorname{epi} \varphi_n \overset{\mathcal{T}_{\text{Mosco}}}{\to} \operatorname{epi} \varphi_\infty$ almost surely. In case (i) this is exactly theorem 6.3 of [18]. Case (ii) is an analogue of that theorem when X and X^* both have the RNP. In reflexive spaces both conditions are trivially satisfied. Convergence results for martingale integrands in the Mosco tolopology in this setting have been given in [10, theorem 3.1].

Remark 4.3. It follows from the proofs of theorems 3.1 and 3.2 that theorem 4.1 still holds if instead of each of the compactness assumptions (i) and (ii) imposed on $\operatorname{epi} \varphi_n$ conditions (i') and (ii') are used respectively. Namely, suppose that there exists a random l.s.c. minorant ψ of φ_n such that

(i') for any $\alpha, \beta \in \mathbb{R}$ and for almost all $\omega \in \Omega$ the set $\{x \in X\colon \|x\| \le \alpha, \psi(\omega, x) \le \beta\}$ is w-compact,

(ii') in the case when both X and X^* have the RNP for any $A \in \mathcal{A}$ and any $\alpha, \beta \in \mathbb{R}$ the set $\{x \in X\colon \|x\| \le \alpha, \inf\left\{\int_A \psi(\omega, f(\omega)) dP(\omega) : f \in \mathcal{L}^1_X, \int_A f(\omega) dP(\omega) = x\right\} \le \beta\}$ is w-compact.

5. Proofs

Before the proof of our results some additional results will be needed. As was mentioned before, the starting point is the following proposition.

Proposition 5.1 (Theorem 5.12, [18]). *Let (F_n) be a $\mathcal{P}(X)$-valued supermartingale satisfying the following conditions:*

(i) $\sup_n Ed(0, F_n) < \infty$,

(ii) *for almost all $\omega \in \Omega$,* $\operatorname{cl} \operatorname{co} \bigcup_{n=1}^\infty F_n(\omega)$ *is a subset of some w-ball-compact set dependent on ω.*

Then there exists an integrable random set F_∞ with values in $\mathcal{R}_w$ such that

$$F_\infty(\omega) = \mathcal{T}_{\text{Mosco}}\text{-}\lim F_n(\omega) \text{ for almost all } \omega \in \Omega.$$

Moreover if (F_n) is bounded in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ then F_∞ is integrably bounded. If (F_n) is uniformly integrable then $E^{\mathcal{A}_n} F_\infty \subset F_n$ a.s. If $(d(0, F_n))$ is UI then $E^{\mathcal{A}_n} F_\infty \subset F_n$.

The following lemmas characterize the slice topology:

Lemma 5.2 (Theorem 5.3, [3]). *Let X be a normed linear space. Then the slice topol-*

ogy on $\mathcal{P}_c(X)$ has as a subbase all sets of the form

$$V^- = \{A \in \mathcal{P}(X)\colon A \cap V \neq \emptyset\},\ V \text{ is } s\text{-open},$$
$$(B^c)^{++} = \{A \in \mathcal{P}(X)\colon D(B,A) > 0\},\ B \in \mathcal{P}_{cb}(X).$$

Lemma 5.3 (Proposition 3.3, [21]). *Let X be a Banach space and R_0 a given member of $\mathcal{R}_w(X)$. Define the subset $\mathcal{R}_w(R_0)$ of $\mathcal{P}_c(X)$ by $\mathcal{R}_w(R_0) := \{C \in \mathcal{P}_c(X)\colon C \subset R_0\}$. Then the restriction of $\mathcal{T}_{\text{slice}}$ to $\mathcal{R}_w(R_0)$ has the following properties:*

(i) *it is generated by the families*

$$\mathcal{F}_1 := \{V^-\colon V \text{ is } s\text{-open in } X\} \text{ and}$$
$$\mathcal{F}_3 := \{(K^c)^+\colon K \in \mathcal{P}_{wkc}(R_0)\}$$

where $W^+ := \{C \in 2^X\colon C \subset W\}$ and $\mathcal{P}_{wkc}(R_0) := \{K \in \mathcal{P}_{wkc}(X)\colon K \subset R_0\}$,

(ii) *it is the weak topology generated by the gap functionals $D(\cdot, K)$ where $K \in \mathcal{P}_{wkc}(R_0)$.*

Note that this means that in $\mathcal{R}_w(R_0)$ the slice topology coincides with the Mosco topology. This fact provides the following:

Proof of Theorem 3.1. By proposition 5.1 there exists an integrable random set F_∞ with values in $\mathcal{R}_w$ such that $F_n \overset{\mathcal{T}_{\text{Mosco}}}{\to} F_\infty$ a.s. If (F_n) is bounded in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ then F_∞ is integrably bounded. If $(d(0,F_n))$ is UI then $E^{\mathcal{A}_n}F_\infty \subset F_n$ a.s. for all $n \in \mathbb{N}$. Recall (the definition after corollary 5.5 in [3]) that the Mosco topology is generated by the families $\{V^-\colon V$ is s-open in $X\}$ and $\{(K^c)^+\colon K \in \mathcal{P}_{wkc}(X)\}$. Invoking lemma 5.3 the result follows. □

Let us recall the multivalued Radon-Nikodym theorem for multimeasures which plays a crucial role in the proof of theorem 3.2 (or, more precisely, in the proof of proposition 5.5, which is a weaker version of theorem 3.2).

Lemma 5.4 ([13, Théorème 3], [14, Théorème 8, p.III.31]). *Suppose that X and X^* both have the Radon-Nikodym property. Let M be a weak multimeasure of bounded variation with values in $\mathcal{P}_{wkc}(X)$. If M is P-absolutely continuous, there exists a multifunction $\Gamma \in \mathcal{L}^1_{\mathcal{P}_{wkc}}(X)$, (a version of) the Radon-Nikodym derivative of M with respect to P, such that*

$$M(A) = \int_A \Gamma\, dP \text{ for all } A \in \mathcal{A}.$$

Before the proof of the main theorem let us state a weaker result. It concerns the Mosco convergence of the $\mathcal{P}_c(X)$-valued supermartingales and is an analogue of the proposition 5.1 in the case when both the space X and its dual X^* have the RNP.

Proposition 5.5. *Suppose X and X^* both have the RNP (thus X^* is separable). Let $(F_n, \mathcal{A}_n)$ be a $\mathcal{P}_c(X)$-valued supermartingale such that*

(i) $\sup_n Ed(0, F_n(\omega)) < \infty$,

(ii) *for all $A \in \mathcal{A}$ the sets* $\operatorname{cl}\operatorname{co} \bigcup_n \int_A F_n\, dP$ *are w-ball-compact.*

Then there exists an integrable random set F_∞ *with values in* $\mathcal{R}_w$ *such that* $F_\infty = \mathcal{T}_{\mathrm{Mosco}}\text{-}\lim_n F_n$ *a.e. Moreover, if* $sup_n E\|F_n\| < \infty$ *then* F_∞ *is integrably bounded. If* $(d(0, F_n))$ *is uniformly integrable then* $E^{\mathcal{A}_n} F_\infty \subset F_n$ *a.s.*

Recall the following result which will be used in the proof of proposition 5.5.

Lemma 5.6 (Lemma 5.11, [18]). *Let* (C_n) *be a sequence in* 2^X *and* (r_k) *an increasing sequence of positive real numbers such that* $\lim_k r_k = +\infty$*. Assume that for every* $k \in \mathbb{N}$ *that sequence* $(C_n \cap \overline{B}(0, r_k))_n$ *has a Mosco limit* C^k*. If we set* $C := \bigcup_{k\in\mathbb{N}} C^k$ *then* $C = \mathcal{T}_{\mathrm{Mosco}}\text{-}\lim_n C_n$*. (In particular* C *is closed).*

Proof of proposition 5.5. Observe (see lemma 4.3 of [18]) that for any σ-algebra $\mathcal{B} \subset \mathcal{A}$ and all $x \in X$ we have

$$E^{\mathcal{B}} d(x, F) \geq d(x, E^{\mathcal{B}} F).$$

Since (F_n) is a supermartingale

$$v_n^k(\omega) := d(0, F_n(\omega)) + k, \ \omega \in \Omega$$

defines a positive submartingale for any $k \in \mathbb{N}$. By Krickeberg's decomposition there exist a positive integrable martingale (r_n^k) and a positive integrable supermartingale (s_n^k) such that $v_n^k = r_n^k - s_n^k$. Define the following supermartingale

$$F_n^k(\omega) := F_n(\omega) \cap \overline{B}(0, r_n^k(\omega)), \ \omega \in \Omega.$$

Let D^* denote the countable, dense subset of X^*. Notice that $E\|F_n^k\| \leq Er_n^k < \infty$, thus by proposition IV-1-2 of [26] there exist $\psi^k \in \mathcal{L}^1_{\mathbb{R}}$ and for any $x^* \in D^*$, $\varphi^k_{x^*} \in \mathcal{L}^1_{\mathbb{R}}$ such that

$$s(x^*, F_n^k(\omega)) \to \varphi^k_{x^*}(\omega) \text{ for a.e. } \omega \in \Omega \text{ and all } k \in \mathbb{N}, \tag{5.1}$$

$$r_n^k(\omega) \to \psi^k(\omega) \text{ for a.e. } \omega \in \Omega, \text{ all } k \in \mathbb{N}. \tag{5.2}$$

Fix $k \in \mathbb{N}$. Notice that (r_n^k) is a martingale, thus $Er_n^k = E\psi^k$ for any $n \in \mathbb{N}$. Notice that for any $A \in \mathcal{A}$ w-ball-compactness of the set $\mathrm{cl\,co} \bigcup_n \int_A F_n$ implies that the set $\mathrm{cl\,co} \bigcup_n \int_A F_n^k$ is w-compact. Indeed,

$$\begin{aligned}
\mathrm{cl\,co} \bigcup_n \int_A F_n^k &= \mathrm{cl\,co} \bigcup_n \int_A (F_n \cap \overline{B}(0, r_n^k)) \subset \mathrm{cl\,co} \bigcup_n \left(\int_A F_n \cap \int_A \overline{B}(0, r_n^k) \right) \\
&\subset \ \mathrm{cl\,co} \bigcup_n \left(\int_A F_n \cap \overline{B}(0, \int_A r_n^k) \right) \subset \mathrm{cl\,co} \bigcup_n \left(\int_A F_n \cap \overline{B}(0, Er_n^k) \right) \\
&= \ \mathrm{cl\,co} \bigcup_n \left(\int_A F_n \cap \overline{B}(0, c) \right) = \mathrm{cl\,co} \bigcup_n \int_A F_n \cap \overline{B}(0, c).
\end{aligned}$$

Since (F_n^k) are a.e. bounded $(s(\cdot, F_n^k(\omega)))$ is equicontinuous on X^*. Hence (5.1) holds for any $x^* \in X^*$. Define a sublinear function $\zeta_A^k \colon X^* \to \mathbb{R}$

$$\zeta_A^k(x^*) := \int_A \varphi^k_{x^*} dP.$$

For all $x^* \in X^*$,

$$\zeta_A^k(x^*) \leq s(x^*, R_A^k) \tag{5.3}$$

where $R_A^k = \operatorname{cl}\operatorname{co}\bigcup_n \int_A F_n^k$. This set is w-compact, therefore ζ_A^k is Mackey continuous on X^* and is also w^*-l.s.c. Theorem II.16 of [9] implies that there exists a nonempty closed convex set $M^k(A) \subset X$ such that $\zeta_A^k(\cdot) = s(\cdot, M^k(A))$. Obviously $s(x^*, M^k(A)) = \int_A \varphi_{x^*}^k \leq E\psi^k$ thus

$$\|M^k(A)\| = \sup_{\|x^*\|_* \leq 1} s(x^*, M^k(A)) \leq E\psi^k < \infty.$$

By (5.3) $M^k(A) \subset R_A^k$, thus is w-compact. Now following the proof of theorem 2.5 in [2] it can be showed that the function $M^k\colon \mathcal{A} \to \mathcal{P}_{wkc}(X)$ is additive, absolutely continuous with respect to P, has bounded variation and $s(x^*, M^k(\cdot))$ is σ-additive for all $x^* \in X^*$. Therefore the multivalued Radon-Nikodym theorem (lemma 5.4) can be applied. It implies the existence of a multivalued function $F_\infty^k : \Omega \to \mathcal{P}_{wkc}$ such that

$$M^k(A) = \int_A F_\infty^k \text{ for all } A \in \mathcal{A}.$$

This multifunction is determined up to a null set. By the properties of support functions and multivalued integrals $s(x^*, \int_A F_\infty^k) = \int_A s(x^*, F_\infty^k)$ for all $A \in \mathcal{A}$. Thus

$$F_n^k \overset{\mathcal{T}_{\text{scalar}}}{\to} F_\infty^k \text{ a.e.} \tag{5.4}$$

Recall that for all $x \in X$

$$d(x, F_n^k(\omega)) = \sup\{\langle x, x^*\rangle - s(x^*, F_n^k(\omega))\colon x^* \in D^*, \|x^*\|_* \leq 1\}.$$

For all $x \in X$, $(d(x, F_n^k))_n$ is a submartingale. Applying lemma V-2-9 of [26] we conclude that for all x in a countable dense subset of X

$$d(x, F_n^k(\omega)) \to d(x, F_\infty^k(\omega)) \text{ for a.a. } \omega \in \Omega. \tag{5.5}$$

Since distance functionals are Lipschitz with the constant 1, (5.5) holds for any $x \in X$. This and (5.4) imply that $F_\infty^k(\omega) = \mathcal{T}_{\text{Mosco}}\text{-}\lim F_n^k(\omega)$ for all $\omega \in \Omega$. Let N_k be a negligible set such that this convergence takes place for all $\omega \in N_k^c$. Let $N := \bigcup_k N_k$. Define

$$F_\infty(\omega) := \begin{cases} \bigcup_{k\in\mathbb{N}} F_\infty^k(\omega), & \omega \in N^c, \\ \{0\}, & \omega \in N. \end{cases}$$

Since $F_\infty^k(\omega)$ are w-compact for all $\omega \in N^c$, $F_\infty(\omega)$ are w-ball-compact for any $\omega \in \Omega$. Lemma 5.6 yields $F_\infty(\omega) = \mathcal{T}_{\text{Mosco}}\text{-}\lim_n F_n(\omega)$ for all $\omega \in N^c$.

If (F_n) is bounded in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ then

$$\begin{aligned} \int \|F_\infty\| &= \int \sup_{\|x^*\|_* \leq 1} s(x^*, F_\infty) = \int \sup_{\|x^*\|_* \leq 1} \sup_k s(x^*, F_\infty^k) \\ &= \int \sup_{\|x^*\|_* \leq 1} \sup_k \lim_n s(x^*, F_n^k) \leq \int \liminf_n \sup_{\|x^*\|_* \leq 1} \sup_k s(x^*, F_n^k) \\ &\leq \liminf_n \int \sup_{\|x^*\|_* \leq 1} \sup_k s(x^*, F_n^k) \leq \liminf_n \int \sup_{\|x^*\|_* \leq 1} s(x^*, F_n) \\ &= \liminf_n \int \|F_n\| < \infty. \end{aligned}$$

If $(d(0, F_n))$ is UI then by proposition 5.1 and theorem 2.1 of [22]

$$E^{\mathcal{A}_n} F_\infty = E^{\mathcal{A}_n}(\bigcup_k F_\infty^k) = \mathrm{cl}\left(\bigcup_k E^{\mathcal{A}_n} F_\infty^k\right) \subset \mathrm{cl} \bigcup_k F_n^k = F_n.$$

$\square$

Before the proof of the main result let us formulate an analogue of lemma 5.6 for the case when convergence in the slice topology is considered.

Lemma 5.7. *Let (C_n) be a sequence in $\mathcal{P}_c(X)$ and (r_k) an increasing sequence of positive real numbers such that $\lim_k r_k = +\infty$. Assume that for every $k \in \mathbb{N}$ that sequence $(C_n \cap \overline{B}(0, r_k))_n$ has a limit C^k in the slice topology. If we set $C := \bigcup_{k\in\mathbb{N}} C^k$ then $C = \mathcal{T}_{\text{slice}}\text{-}\lim_n C_n$. (In particular C is closed).*

Proof. Denote $C_n \cap \overline{B}(0, r_k)$ by C_n^k. By the assumptions $C^k = \mathcal{T}_{\text{slice}}\text{-}\lim_n C_n^k$ also $C^k = \mathcal{T}_{\text{Mosco}}\text{-}\lim_n C_n^k$. Lemma 5.6 yields that C_n Mosco-converges to C, thus $C = s\text{-li } C_n$. Let (y_i) be a countable dense subset of C. Obviously $y_i \in s\text{-li } C_n$ for all $i \in \mathbb{N}$. Thus for any $x \in X$ and $i \in \mathbb{N}$, $d(x, y_i) \geq \limsup_n d(x, C_n)$. Taking the infimum over all integers i we obtain

$$\limsup_n d(x, C_n) \leq d(x, C) \text{ for all } x \in X. \tag{5.6}$$

Recalling that the slice topology has as a subbase the sets V^- (where V is norm open) and $(B^c)^{++}$ (where B is closed, bounded and convex) we have by (5.6) that if the intersection of C and V is not empty then for sufficiently large n the intersection $C_n \cap V$ is also non-empty.

Now it has to be shown that for any $B \in \mathcal{P}_{cb}(X)$ if $D(B, C) > 0$ then for sufficiently large n also $D(B, C_n) > 0$. Choose k so large that $C \cap \overline{B}(0, r^k) \neq \emptyset$ and $B \subset \frac{1}{2}\overline{B}(0, r^k)$. Now choose $x^* \in X^*$ and $\beta \in \mathbb{R}$ such that

$$\sup_{x\in C}\langle x, x^*\rangle < \beta < \inf_{x\in B}\langle x, x^*\rangle. \tag{5.7}$$

The left inequality implies that $s(x^*, C \cap \overline{B}(0, r^k)) < \beta$. Since C_n^k converges to C^k in the slice topology, theorem 5.4 of [3] (or theorem 2.4.8 of [5]) implies that C^k is also a scalar limit of (C_n^k). Take a positive $\varepsilon < \beta - s(x^*, C \cap \overline{B}(O, r^k))$. For sufficiently large $n \in \mathbb{N}$

$$s(x^*, C_n^k) \leq s(x^*, C^k) + \varepsilon \leq s(x^*, \overline{B}(0, r^k) \cap C) + \varepsilon < \beta.$$

Thus $\sup_{x\in C_n^k}\langle x, x^*\rangle < \beta$ for sufficiently large $n \in \mathbb{N}$. Recalling the right inequality in (5.7) we conclude that $C_n \cap \overline{B}(0, r^k)$ and B can be strongly separated. Thus $D(B, C_n^k) > 0$ by closedness of B and C_n^k. Since $B \subset \frac{1}{2}B(0, r^k)$ we have $D(B, C_n) > 0$. $\square$

Now we are in a position to prove our main result.

Proof of theorem 3.2. This proof is a modification of the proof of proposition 5.5. As in the proof of that proposition, for any $k \in \mathbb{N}$ define a submartingale

$$v_n^k(\omega) := d(0, F_n(\omega)) + k.$$

It can be represented as a difference of a positive integrable martingale and a positive integrable supermartingale: $v_n^k(\omega) = r_n^k(\omega) - s_n^k(\omega)$. For any $k \in \mathbb{N}$ define a supermartingale

$$F_n^k := F_n \cap \overline{B}(0, r_n^k).$$

Proceeding as in the proof of proposition 5.5 one obtains the existence of a negligible set $N \subset \Omega$ and multifunctions F_∞^k such that for all $k \in \mathbb{N}$

$$\begin{aligned} F_n^k(\omega) &\overset{\mathcal{T}_{\text{scalar}}}{\rightarrow} F_\infty^k(\omega), \text{ for } \omega \in N^c, \\ F_n^k(\omega) &\overset{\mathcal{T}_{\text{Wijsman}}}{\rightarrow} F_\infty^k(\omega), \text{ for } \omega \in N^c. \end{aligned} \tag{5.8}$$

Define $F_\infty(\omega) = \bigcup_k F_\infty^k(\omega)$ for all $\omega \in N^c$ and $F_\infty(\omega) = \{0\}$ on N. Since X^* is separable lemmas 3.5 and 3.11 of [21] yield that $F_n^k(\omega) \overset{\mathcal{T}_{\text{slice}}}{\rightarrow} F_\infty^k(\omega)$ for $\omega \in N^c$. Now the result follows by lemma 5.7.

It has been shown in the proof of proposition 5.5 that if $\sup_n E\|F_n\| < \infty$ then F_∞ is integrably bounded and if $(d(0, F_n))$ is UI then $E^{\mathcal{A}_n} F_\infty \subset F_n$. $\square$

In view of the following lemma proof of theorem 4.1 is straightforward.

Lemma 5.8 (Lemma 6.2, [18]). *Let (φ_n) be an adapted sequence of convex normal integrands. Then the following conditions are equivalent:*

(i) $\sup_n Ed(\varphi_n) < \infty$,

(ii) *there exists an integrable adapted sequence u_n with* $\sup_n \int_\Omega \varphi_n(\omega, u_n(\omega))^+ dP < \infty$.

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