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Lipschitz Continuous Selectors Part I: Linear Selectors*

Krzysztof Przesławski
Instytut Matematyki, Politechnika Zielonogórska,
ul. Podgórna 50, 65-246 Zielona Góra, Poland.
e-mail: K.Przeslawski@im.pz.zgora.pl

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We study various properties of Lipschitz continuous linear selectors on the family of all convex, nonempty and compact subsets of $\mathbb{R}^n$. In particular, it is shown that if s is such a selector then the Lipschitz constant of s can be estimated from below by the norm of $s(B^n)$, where B^n is the unit ball. A notion of a parametric representation of convex bodies is introduced and illustrated with examples.

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1. Introduction

In 1838, Steiner introduced the *Krümmungsschwerpunkt* of a convex curve. His definition extends to higher dimensions leading to the well-known Steiner point of a convex body. It can be shown that the mapping which subordinates to each convex body its Steiner point is a Lipschitz continuous selector. This mapping is also additive with respect to the Minkowski addition. Such selectors are the subject of this work.

In the 1980's, Lipschitz continuous selectors attracted a considerable interest in connection with applications to differential inclusions (see e.g. [1, 2, 3]). The Steiner selector played, often implicitly, a main role in these investigations. New selectors were derived from it by certain nonlinear procedures (see [24] and references therein). In a few papers, Minkowski additive selectors other than that of Steiner were described [11, 23, 30]. (However, selectors in [30] are not continuous.) In [6], such selectors appeared implicitly. On the other hand, it was shown [23] that uniformly continuous selectors cannot exist on the family of all convex, compact and nonempty subsets of any infinite-dimensional Banach space (a special case was published earlier in [8]). Thus the dimension of a space is a natural limitation for such investigations.

In Section 3, we describe a family of selectors that are sufficient for all our further purposes. An important notion of a parametric representation of convex bodies is introduced and an instance of such a representation is discussed in Section 4. Other examples are presented in the next section.

It can be seen that the Lipschitz constants of selectors which belong to the range of a parametric representation cannot be commonly bounded. We address this and related problems in Section 6.

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One can observe that each of the considered selectors is associated with a unique $\mathbb{R}^n$-valued measure on S^{n-1}. Some facts concerning these measures are collected in Section 7.

Section 8 is an introduction to the second part of this work where selectors will be allowed to range over unbounded sets.

It is rather clear that the presented results can be applied to differential inclusions, multimeasures and metric projections [5]. A recent observation made by Gromov [9] suggests that they should also be useful in the context of mixed volumes. These problems will be discussed elsewhere.

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2. Notation and definitions

Let $\mathcal{D}^n$ be the family of all convex, closed and nonempty subsets of $\mathbb{R}^n$, and let $\mathcal{K}^n = \{A \in \mathcal{D}^n : A \text{ is bounded}\}$. By the *Hausdorff distance* on $\mathcal{D}^n$, we mean the mapping $H\colon \mathcal{D}^n \times \mathcal{D}^n \to [0, \infty]$ defined as follows: $H(A, B) = \inf\{\rho : A \subset B + \rho B^n, B \subset A + \rho B^n\}$, where B^n denotes the Euclidean unit ball centred at the origin. (Conventionally, it is assumed that $\inf \emptyset = \infty$.) Clearly, H is a metric on $\mathcal{K}^n$; if we admit ∞ as a possible distance, then H is also a metric in the case of $\mathcal{D}^n$.

We define the *support function* h_A of $A \in \mathcal{D}^n$ by the formula $h_A(x) := \sup_{a \in A}\langle a, x\rangle$. We assume that the reader is familiar with this notion. However, let us recall that h_A is positive homogeneous of degree 1 and subadditive. Moreover, for every $A, B \in \mathcal{D}^n$ one has $h_{A+B} = h_A + h_B$; also $h_A \le h_B \Leftrightarrow A \subset B$, in particular, $\langle x, \cdot\rangle \le h_B \Leftrightarrow x \in B$. As is well-known (see e.g. [10]), the Hausdorff distance can be expressed by using support functions

$$H(A, B) = \sup\{|(h_A - h_B)(x)| : \|x\| = 1\} \tag{2.1}$$

If $A \in \mathcal{D}^n$, then we define the *face of A at direction $x \in \mathbb{R}^n$* as follows

$$V_x(A) := \{a \in A : \langle a, x\rangle = h_A(x)\}.$$

In general, $V_x(A)$ may be empty. However, this cannot happen if $A \in \mathcal{K}^n$. Simple calculations show that

$$V_x(A) = \partial h_A(x), \tag{2.2}$$

where ∂h_A denotes the subdifferential of h_A.

Let ∇f be the gradient of a convex function f. Recall that if the *effective domain* $\operatorname{dom} f$ of f has nonempty interior, then

$$\{\nabla f\} = \partial f \tag{2.3}$$

almost everywhere [25]. For this reason we shall often switch from ∂f to ∇f without any further comments.

Let $\mathcal{F} \subset \mathcal{D}^n$. Each mapping $s\colon \mathcal{F} \to \mathbb{R}^n$ such that $s(A) \in A$ is said to be a *selector* on $\mathcal{F}$. The selector s is *linear* if $s(A+B) = s(A) + s(B)$ whenever $A + B \in \mathcal{F}$. If $s(A \cup B) + s(A \cap B) = s(A) + s(B)$ whenever $A \cup B$ and $A \cap B$ belong to $\mathcal{F}$, then s is said to be a valuation. The map s is *Lipschitz continuous* if there exists a real L such that $\|s(A) - s(B)\| \le LH(A, B)$ for all A, B. The smallest admissible L is said to be the *Lipschitz constant* of s. The family of all Lipschitz continuous linear selectors on $\mathcal{K}^n$ will be denoted by $\mathcal{S}^n$.

3. A selection scheme

Probably, the first known example of a Lipschitz continuous selector on $\mathcal{K}^n$ is the *Steiner selector*

$$s_0(A) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_A(u) u \, d\sigma(u),$$

where S^{n-1} is the unit sphere, σ the usual surface area measure, and κ_n the volume of the unit ball in $\mathbb{R}^n$. At first sight it is not apparent that $s_0(A) \in A$. If A is a body with C^2-smooth boundary ∂A, then this can be deduced from the fact that $s_0(A)$ is the barycentre of the Gaussian curvature of ∂A (see e.g. [4, 26]). A standard approximation argument yields now the result for all A. Another reasoning has been proposed in Shephard [28]. As we will see below, this reasoning can be applied in order to define a whole family of selectors.

Suppose that $M \subset \mathbb{R}^n$ is a compact set with nonempty interior and sufficiently smooth boundary ∂M. Then for any sufficiently regular function g one has the following Stokes type formula

$$\int_{\partial M} g(u) n(u) \, d\sigma(u) = \int_M \nabla g(u) \, du, \tag{3.1}$$

where σ is the surface area measure and $n(u)$ is the outward normal at u.

Let a function $f\colon \mathbb{R}^n \setminus 0 \to \mathbb{R}$ be homogeneous of degree α, that is, $f(\lambda x) = \lambda^\alpha f(x)$, for every $\lambda > 0$. If $\alpha > -n$ and f is integrable over the unit sphere, then

$$\int_{B^n} f(u) \, du = \frac{1}{\alpha + n} \int_{S^{n-1}} f(u) \, d\sigma(u). \tag{3.2}$$

Let $\phi\colon S^{n-1} \to [0, +\infty)$ be C^1-smooth and let $\int \phi \, d\sigma = n$. Let $\tilde{\phi}\colon \mathbb{R}^n \to [0, +\infty)$ be the homogeneous extension of ϕ of degree 0, that is, $\tilde{\phi}(u) = \phi\left(\frac{u}{\|u\|}\right)$, for $u \neq 0$, and $\tilde{\phi}(0) = 0$. It is clear that $\tilde{\phi}$ is C^1-smooth with the exception of 0. By (3.2), its restriction to B^n is a density of a probability measure. Define

$$f_A(y) := \int_{B^n} h_A(y + u) \tilde{\phi}(u) \, du.$$

It is easy to see that f_A is a convex function. As h_A is Lipschitz continuous and differentiable almost everywhere, one has

$$\nabla f_A(y) = \int_{B^n} \nabla h_A(y + u) \tilde{\phi}(u) \, du. \tag{3.3}$$

Moreover, since, by (2.2) and (2.3), $\nabla h_A(x) \in A$ for almost all $x \in \mathbb{R}^n$, it follows that $\nabla f_A(y) \in A$. On the other hand, by (3.1) (compare also [18]),

$$\int_{S^{n-1}} h_A(y+u)\phi(u)u\,d\sigma(u) = \nabla f_A(y) + \int_{B^n} h_A(y+u)\nabla\tilde{\phi}(u)\,du.$$

Consequently, we have the following

Theorem 3.1 (selection scheme). *Let $\phi\colon S^{n-1} \to [0,+\infty)$ be C^1-smooth and let $\int \phi d\sigma = n$. Let $\tilde{\phi}$ be the homogeneous extension of ϕ of degree 0. Then for each $y \in \mathbb{R}^n$ the mapping*

$$A \mapsto \nabla f_A(y) = \int_{S^{n-1}} h_A(y+u)\phi(u)u\,d\sigma(u) - \int_{B^n} h_A(y+u)\nabla\tilde{\phi}(u)\,du$$

is a linear Lipschitz continuous selector. If L is its Lipschitz constant, then

$$L \leq (\|y\|+1)\left(n + \frac{1}{n-1}\int_{S^{n-1}} \|\nabla\tilde{\phi}(u)\|\,d\sigma(u)\right).$$

Proof. It remains to establish the estimate for L. For A and $B \in \mathcal{K}^n$, assume $m = \|\nabla f_A(y) - \nabla f_B(y)\|$. Then

$$\begin{aligned} m &\leq \int_{S^{n-1}} |h_A(y+u) - h_B(y+u)|\phi(u)\|u\|\,d\sigma(u) \\ &+ \int_{B^n} |h_A(y+u) - h_B(y+u)|\|\nabla\tilde{\phi}(u)\|\,du. \end{aligned}$$

It follows from the homogeneity of support functions and (2.1) that for $u \in B^n$,

$$|h_A(y+u) - h_B(y+u)| \leq H(A,B)\|y+u\| \leq H(A,B)\left(\|y\|+1\right).$$

These estimates together with (2.1) and the fact that $\|\nabla\tilde{\phi}(u)\|$ is homogeneous of degree -1 show that

$$\begin{aligned} m &\leq H(A,B)\left(\|y\|+1\right)\left(\int \phi\,d\sigma + \int_{B^n} \|\nabla\tilde{\phi}\|\right) \\ &\leq H(A,B)\left(\|y\|+1\right)\left(n + \frac{1}{n-1}\int \|\nabla\tilde{\phi}\|\,d\sigma\right). \end{aligned}$$

□

Observe that if we assume $\phi = \frac{1}{\kappa_n}$ and $y = 0$ in the selection scheme, then we get the Steiner selector.

Sometimes it is convenient to replace B^n by another domain M. By much the same method as above, one can show that the mapping

$$A \mapsto \frac{1}{\kappa}\int_{\partial M} h_A(u)n(u)\,d\sigma(u)$$

is a Lipschitz continuous linear selector (compare [13]). (κ denotes here the volume of M.) Selectors of that kind are used in [23] in order to define an analogue of the Steiner selector for arbitrary Minkowski space X. (M is there the unit ball of the conjugate space X^*.) The case where M is a hypercube is considered in [11].

4. The harmonic representation of convex sets

By a *parametric representation* of $\mathcal{F} \subset \mathcal{D}^n$, we mean a mapping S from a set M to selectors of $\mathcal{F}$ such that for each $A \in \mathcal{F}$ the set $A_S = \{S(m)(A) : m \in M\}$ is dense in A. As a rule, we shall be concerned with representations which fill up A so that $\operatorname{ri} A \subset A_S$, where $\operatorname{ri} A$ is the *relative interior* of A. (The only minute exception is Corollary 6.3.) In this section, we discuss a particular parametric representation of $\mathcal{K}^n$ which we call the *harmonic representation.* Its basic properties are collected in Theorems 4.1 and 4.2.

Theorem 4.1. *There exists a unique mapping* $s_\bullet$, $\operatorname{int} B^n \ni x \mapsto s_x \in \mathcal{S}^n$, *which satisfies the following conditions:*

(i) *for every* $A \in \mathcal{K}^n$, *the mapping* $x \mapsto s_x(A)$ *is harmonic;*
(ii) *the multifunction* $B^n \ni x \mapsto t_x(A)$, *such that* $t_x(A) = s_x(A)$ *for* $x \in \operatorname{int} B^n$, *and* $t_x(A) = \partial h_A(x)$ *for* $x \in S^{n-1}$ *is upper semicontinuous.*

Proof. (i) simply says that $x \mapsto s_x(A)$ must be a solution of the Laplace equation, while (ii) describes the boundary condition that this solution must satisfy. As is well-known and easily deducible from the maximum principle [17], these facts imply the uniqueness of $s_\bullet$.

The theory of harmonic functions suggests that s has to be defined as follows

$$s_x(A) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} \nabla h_A(u) \frac{1 - \|x\|^2}{\|u - x\|^n} d\sigma(u),$$

where σ_{n-1} stands for the area of S^{n-1}. The fact that s_x is a selector follows easily from the observation that the Poisson kernel $P_x(u) = \frac{1-\|x\|^2}{\|u-x\|^n}$ is a density of a probability measure with respect to $\tilde{\sigma} := \frac{1}{\sigma_{n-1}}\sigma$. Therefore, it remains to establish the Lipschitz continuity of s_x.

As the function $u \mapsto \nabla h_A(u) P_x\left(\frac{u}{\|u\|}\right)$ is homogeneous of degree 0 and $n\kappa_n = \sigma_{n-1}$, we deduce from (3.2) that

$$s_x(A) = \frac{1}{\kappa_n} \int_{B^n} \nabla h_A(u) P_x\left(\frac{u}{\|u\|}\right) du.$$

Now, if we assume $y = 0$ and $\phi = \frac{1}{\kappa_n} P_x$ in the definition of f_A then, according to Theorem 3.1 and the preceding equality, we get

$$s_x(A) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_A(u) P_x\left(\frac{u}{\|u\|}\right) u d\sigma(u) - \frac{1}{\kappa_n} \int_{B^n} h_A(u) \nabla P_x\left(\frac{u}{\|u\|}\right) du.$$

This makes the Lipschitz continuity of s_x transparent. □

The above formula can be simplified. To this end, let us check the gradient in the second integrand and observe that this integrand is homogeneous of degree 0. Thus, again by (3.2),

$$s_x(A) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_A(u) P_x(u) \left(u + \frac{u\langle x, u\rangle - x}{\|u - x\|^2}\right) d\sigma(u). \tag{4.1}$$

Let us remark that by the harmonicity of $x \mapsto P_x(u)$, the mapping $x \mapsto Q_x(u) := P_x(u)\left(u + \frac{u\langle x,u\rangle - x}{\|u-x\|^2}\right)$ is also harmonic.

Expression (4.1) enables us to give a simple estimate for the Lipschitz constant L_x of s_x. By a similar reasoning as in the proof of Theorem 3.1, one has

$$L_x \leq n \int_{S^{n-1}} \left\| u + \frac{u\langle x,u\rangle - x}{\|u-x\|^2} \right\| P_x(u) d\tilde{\sigma}(u).$$

Since P_x is a density of probability measure, we obtain

$$L_x \leq n \sup\left\{ \left\| u + \frac{u\langle x,u\rangle - x}{\|u-x\|^2} \right\| : u \in S^{n-1} \right\} = n\sqrt{1 + \frac{\|x\|^2}{(1-\|x\|^2)^2}} \tag{4.2}$$

Note that this estimate is not sharp for any x; e.g. if $x = 0$, then s_0 is the Steiner selector; therefore, $\sqrt{\frac{2n}{\pi}} < L_0 < \sqrt{\frac{2(n+1)}{\pi}}$ ([19, 26, 32]).

Theorem 4.2. $\{s_x(A) : x \in \operatorname{int} B^n\} = \operatorname{ri} A$.

The proof requires the following

Lemma 4.3. *Let $T\colon B^n \to A \in \mathcal{K}^n$ be an upper semicontinuous multivalued mapping satisfying the following conditions:*

(i) *for every $x \in B^n$, $T(x)$ is a compact subset of A;*
(ii) *if $x, y \in S^{n-1}$ and $x \neq y$, then $T(x) \cap T(y) = \emptyset$;*
(iii) *$T_{|\operatorname{int} B^n}$ is single-valued;*
(iv) *$\bigcup T(S^{n-1}) = \partial A$.*

Then $\bigcup T(B^n) = A$

Proof. Define $S := T_{|_{S^{n-1}}}$. Let $S^{-1}(x) = \{b : x \in S(b)\}$. By (iv), S^{-1} is a well-defined mapping from ∂A onto S^{n-1}. By (ii), S^{-1} is single-valued. By the upper semicontinuity of T, it is also continuous. Suppose that $\bigcup T(B^n) \neq A$. This, together with (iv), implies the existence of an $a \in \operatorname{ri} A \setminus \bigcup T(B^n)$. Now, let $C : A \setminus \{a\} \to \partial A$ be the central projection with centre a, that is, $C(x)$ is a unique element of the intersection of the ray emanating from a and passing through x and the boundary ∂A. Thus the mapping $S^{-1} \circ C \circ T$ is a continuous retraction of B^n onto S^{n-1}, which is a contradiction. □

Proof of the theorem. Recall first that for every u we have $\partial h_A(u) = \{a \in A : \langle a, u\rangle = h_A(u)\}$. Define $A_\epsilon = A + \epsilon B^n$. Clearly, $\bigcup\{\partial h_{A_\epsilon}(u) : u \in S^{n-1}\} = \partial A_\epsilon$ and $\partial h_{A_\epsilon}(u) \cap \partial h_{A_\epsilon}(v) = \emptyset$ for different u and v in S^{n-1}. Hence $x \mapsto t_x(A_\epsilon)$ satisfies the assumptions of Lemma 4.3 and, consequently, $A_\epsilon = t_{B^n}(A_\epsilon)$. Observe that for every $x \in B^n$,

$$H(t_x(A_\epsilon), t_x(A)) = H(t_x(A) + t_x(\epsilon B^n), t_x(A)) \leq \epsilon.$$

Thus $A = t_{B^n}(A)$. It is easy to see now that the conclusion holds true if $\operatorname{int} A \neq \emptyset$. Indeed, $x \mapsto t_x(A)$ sends then S^{n-1} to ∂A and for every $x \in \operatorname{int} B^n$ we have $t_x(A) \in \operatorname{int} A$. Therefore it remains to consider the case $\operatorname{int} A = \emptyset$. Without loss of generality, we may assume that $0 \in A$. Let X be the smallest linear space containing A, and let D be the unit ball of its orthogonal complement, i.e. $D = \{x \in X^\perp : \|x\| \le 1\}$. It is obvious that

$$\operatorname{int}(A + D) = \operatorname{ri} A + \operatorname{ri} D \ne \emptyset.$$

Therefore, what has been stated above applies to $A + D$. Let π denote the orthogonal projection from $\mathbb{R}^n$ onto X. Now, the proof can be easily completed:

$$\begin{aligned} \operatorname{ri} A &= \pi(\operatorname{int}(A + D)) = \{\pi t_x(A + D) : x \in \operatorname{int} B^n\} \\ &= \{\pi t_x(A) + \pi t_x(D) : x \in \operatorname{int} B^n\} = \{t_x(A) : x \in \operatorname{int} B^n\}. \end{aligned}$$

□

For further purposes we shall need the following

Proposition 4.4. *$s_x(B^n) = x$ for every $x \in \operatorname{int} B^n$.*

Proof. Clearly, $\partial h_{B^n}(x) = x$ for $x \in S^{n-1}$. By Theorem 4.1, the mappings $\mathrm{id}_{\operatorname{int} B^n}$ and $s_\bullet(B^n)$ are both solutions of the Laplace equation that satisfy the same boundary condition. Therefore, they must coincide. □

5. Cyclically monotonic representations

A mapping g from $M \subset \mathbb{R}^n$ into $\mathbb{R}^n$ is *cyclically monotonic* if for each sequence $x_1, x_2, \dots,$ x_m in M we have

$$\langle g(x_1), x_2 - x_1\rangle + \langle g(x_2), x_3 - x_2\rangle + \cdots + \langle g(x_{m-1}), x_m - x_{m-1}\rangle + \langle g(x_m), x_1 - x_m\rangle \le 0.$$

Recall that if g is continuous, then there exists a differentiable, convex function $G\colon \mathbb{R}^n \to \mathbb{R}$ such that $\nabla G = g$. Conversely, for each differentiable and convex $G\colon \mathbb{R}^n \to \mathbb{R}$, its gradient ∇G is continuous and cyclically monotonic [25].

A parametric representation S of $\mathcal{F} \subset \mathcal{D}^n$ defined on $M \subset \mathbb{R}^n$ is said to be *cyclically monotonic* if $m \mapsto S(m)(A)$ is cyclically monotonic for each $A \in \mathcal{F}$.

Let us adopt the following notation. If $A \in \mathcal{K}^n$ and f_A is as defined in Section 3, then $f_x(A) := \nabla f_A(x)$ and $f(A) := \{f_x(A) : x \in \mathbb{R}^n\}$. Thus each f_x is an element of $\mathcal{S}^n$. Sometimes it will also be convenient to write $h(A, x)$ instead of $h_A(x)$.

Theorem 5.1. *The mapping $\mathbb{R}^n \ni x \mapsto f_x \in \mathcal{S}^n$ is a cyclically monotonic representation of $\mathcal{K}^n$.*

Proof. Let e be an exposed point of A, i.e. there exists $y \in \mathbb{R}^n \setminus 0$ such that $V_y(A) = \{e\}$. By (2.2), we have $\{e\} = \nabla h_A(y)$. As $x \mapsto \nabla h_A(x)$ is upper semicontinuous, there exists $\rho > 0$ such that

$$H(\{e\}, \nabla h_A(v)) < \epsilon \tag{5.1}$$

whenever $\|y - v\| < \rho$. Let τ be chosen so that $1/\tau \le \rho$. By (5.1), the definition of τ and the fact that ∇h_A is homogeneous of degree 0, we have for each $u \in B^n$

$$\|\nabla h_A(\tau y + u) - \nabla h_A(\tau y)\| = \left\|\nabla h_A\left(y + \frac{u}{\tau}\right) - \nabla h_A(y)\right\| < \epsilon.$$

From this expression and (3.3) we obtain

$$\|f_{\tau y}(A) - e\| = \left\| \int_{B^n} (\nabla h_A(\tau y - u) - \nabla h_A(\tau y))\, \tilde{\phi}(u) du \right\| < \epsilon. \tag{5.2}$$

It is shown in Rockafellar [25, §24] that for every convex and closed function g

$$\operatorname{ri}(\operatorname{dom} g^*) \subset \operatorname{range} \partial g := \bigcup \{\partial g(x) : x \in \mathbb{R}^n\} \subset \operatorname{dom} g^*, \tag{5.3}$$

where g^* is the conjugate of g. Thus $\operatorname{cl}(\operatorname{range} \partial g)$ is a convex set and its relative interior is contained in the range of ∂g. Since $x \mapsto f_x(A)$ is the subdifferential of f_A, the set $C := \operatorname{cl}\{f_x(A) : x \in \mathbb{R}^n\}$ is convex. Clearly, $C \subset A$. By (5.2), the set $\exp A$ of all exposed points of A is contained in C. By the Straszewicz theorem (see e.g. [10]), the convex hull of $\exp A$ coincides with A. Thus $C = A$. The left inclusion in (5.3) implies now that $\operatorname{ri} A \subset \{f_x(A) : x \in \mathbb{R}^n\}$. □

Theorem 5.2. *Suppose that the function ϕ which appears in the definition of $f_\bullet$ is even, that is, $\phi(u) = \phi(-u)$ for $u \in S^{n-1}$. Then for $A \in \mathcal{K}^n$ and x, $y \in \mathbb{R}^n$*

$$\lim_{t \to +\infty} f_{x+ty}(A) = f_x(V_y(A)).$$

Proof. By the subadditivity of support functions, we have

$$-h_A(-v) \le h_A(ty + v) - h_A(ty) \le h_A(v), \tag{5.4}$$

for any $v \in \mathbb{R}^n$.

If $g\colon \mathbb{R}^n \to \mathbb{R}$ is convex, then, as is well-known,

$$\lim_{s \to 0^+} \frac{g(x + sy) - g(x)}{s} = h(\partial g(x), y).$$

In particular, by (2.2) and the homogeneity of support functions,

$$\lim_{t \to +\infty} h_A(ty + v) - h_A(ty) = \lim_{t \to +\infty} \frac{h_A(y + \frac{1}{t}v) - h_A(y)}{\frac{1}{t}} = h(V_y(A), v) \tag{5.5}$$

Let μ and ν be two $\mathbb{R}^n$-valued measures on S^{n-1} and B^n, respectively, defined so that $d\mu(u) = \phi(u) u d\sigma(u)$ and $d\nu(u) = \nabla\tilde{\phi}(u) du$. Since ϕ is even, μ and ν are odd, that is, $d\mu(-u) = -d\mu(u)$ and $d\nu(-u) = -d\nu(u)$. Therefore,

$$\int_{S^{n-1}} d\mu(u) = \int_{B^n} d\nu(u) = 0.$$

Define $v(u) := x + u$. By the selection scheme and the above equality, we get

$$f_{x+ty}(A) = \int_{S^{n-1}} h_A(ty + v(u)) - h_A(ty) d\mu(u) - \int_{B^n} h_A(ty + v(u)) - h_A(ty) d\nu(u).$$

The convergence $f_{x+ty}(A) \to f_x(V_y(A))$ can now be easily drawn from (5.4), the Lebesgue bounded convergence theorem and (5.5). □

Let $\omega = (x_1, \dots, x_n)$ be an ordered basis in $\mathbb{R}^n$. Let us define $V_\omega = V_{x_n} \circ \dots \circ V_{x_1}$. It is easily seen that for each A the set $V_\omega(A)$ is a singleton. Its unique member, say $v_\omega(A)$, is an extremal point of A. The mapping v_ω is linear on $\mathcal{K}^n$. We call it a *lexicographical selector*. It is well-known that v_ω is measurable on $\mathcal{K}^n$. Our next theorem describes the regularity of v_ω in the language of Baire classes. For further information on lexicographical selections and their applications the reader is referred to [15, 20, 21, 22, 30, 33].

Let $\Phi_0(X, Y)$ consist of all continuous mappings from a metric space X to a metric space Y. For each $m \in \mathbb{N}$ we define $\Phi_m(X, Y)$ as the set of all these mappings $f : X \to Y$ that there exists a sequence $\{f_k : k \in \mathbb{N}\} \subset \Phi_{m-1}(X, Y)$ which is pointwise convergent to f. The set $\Phi_m(X, Y) \setminus \Phi_{m-1}(X, Y)$ consists of functions which are of the *m-th Baire class.*

Theorem 5.3. *For every ordered basis ω in $\mathbb{R}^n$ the lexicographical selector v_ω belongs to $\Phi_n(\mathcal{K}^n, \mathbb{R}^n)$.*

Proof. Let $(k_1, \dots, k_n) \in \mathbb{N}^n$ and $\omega = (x_1, \dots, x_n)$. By the preceding theorem, we have $\lim_{k_n \to \infty} \dots \lim_{k_1 \to \infty} f_{k_1 x_1 + \dots + k_n x_n} = v_\omega$. $\square$

For every $A \in \mathcal{K}^n$, the function f_A is the convolution of h_A with $\tilde{\phi}\mathbf{1}_{B^n}$, where $\mathbf{1}_{B^n}$ is the characteristic function of the unit ball. This suggests that convolutions with smooth functions can be used for producing Lipschitz continuous linear selectors on $\mathcal{K}^n$.

Let γ be a nonnegative C^1-smooth function on $\mathbb{R}^n$ with the following three properties:

(i) $\int \gamma(u)du = 1$;
(ii) $\int \|u\|\gamma(u)du < \infty$;
(iii) $\int \|u\| \cdot \sup\{\|\nabla\gamma(v)\| : \|v\| \geq \|u\|\}du < \infty$.

For $A \in \mathcal{K}^n$, let g_A be the convolution $g_A = h_A * \gamma$. (We let the notational convention established for f_A remain valid for g_A.) Because $|h_A(x)| \leq \sup\{\|a\| : a \in A\}\|x\|$, it follows from (ii) that g_A is well-defined. The Lebesgue bounded convergence theorem together with (iii) imply

$$g_x(A) = h_A * \nabla\gamma(x).$$

Also,

$$g_x(A) = (\nabla h_A) * \gamma(x) = \gamma * \nabla h_A(x).$$

Clearly, the second of these expressions and (i) imply that g_x is a linear selector. On the other hand, by the first of them, (2.1) and the homogeneity of support functions we have

$$\|g_x(A) - g_x(B)\| = \|(h_A - h_B) * \nabla\gamma(x)\| = \left\| \int (h_A - h_B)(x - u)\nabla\gamma(u)du \right\|$$

$$= \left\| \int (h_A - h_B)\left(\frac{x-u}{\|x-u\|}\right) \|x - u\|\nabla\gamma(u)du \right\| \leq H(A, B) \int \|x - u\| \cdot \|\nabla\gamma(u)\|du.$$

Summing up, $g_x \in \mathcal{S}^n$ for every $x \in \mathbb{R}^n$. Its Lipschitz constant $\bar{L}_x$ does not exceed $\int \|x - u\| \cdot \|\nabla\gamma(u)\|du$. (Obviously, this integral is finite by (iii).)

A similar procedure has been used in [12] in order to define certain projections.

One can easily see that Theorems 5.1 and 5.2 remain valid for $g_\bullet$. However, in the second of them, we must assume that γ is even.

Suppose that γ is a radial function, i.e. $\gamma(x) = \gamma(x')$ whenever $\|x\| = \|x'\|$. Then g_0 is equivariant under orthogonal transformations, that is, $\nabla g_0(TA) = T\nabla g_0(A)$ whenever T is an orthogonal transformation. Meyer's theorem [14] (see also [16, 27]) implies that g_0 is the Steiner selector s_0. In this implicit way, s_0 appears in [6].

6. Lipschitz constants of linear selectors

It is not surprising that the Lipschitz constants of selectors belonging to $\mathcal{S}^n$ cannot be commonly bounded. The next theorem shows how the Lipschitz constant of $s \in \mathcal{S}^n$ depends on the distance of $s(B^n)$ from the origin. As a consequence, we deduce that for each parametric representation $S\colon M \to \mathcal{S}^n$ of $\mathcal{K}^n$ the set $\{S(m) : m \in M\}$ cannot be uniformly equicontinuous.

Theorem 6.1. *Let $s \in \mathcal{S}^n$, where $n \geq 2$, and L_s be the Lipschitz constant of s. If $\|s(B^n)\| = \beta$, then $L_s \geq \frac{\lambda}{\sqrt{1-\beta^2}}$, where λ is a positive constant depending only on n. Moreover, $\lambda \geq \frac{\sqrt{3}}{3}$.*

Proof. For $v \in S^{n-1}$ define τ_n as follows

$$\tau_n \int_{S^{n-1}} |\langle v, u\rangle| d\tilde{\sigma}(u) = 1. \tag{6.1}$$

Let A_v denote the line segment $[-\tau_n v, \tau_n v]$. Then $h_{A_v}(u) = \tau_n|\langle v, u\rangle|$. As τ_n does not depend on the choice of v, we have

$$B^n = \int_{S^{n-1}} A_v d\tilde{\sigma}(v).$$

(Note that the integral here is multivalued. It can be defined by using the usual Riemann procedure, for example.) Since s is linear and continuous, we deduce that

$$s(B^n) = \int_{S^{n-1}} s(A_v) d\tilde{\sigma}(v). \tag{6.2}$$

Let $x_0 \in S^{n-1}$ be chosen so that

$$\beta x_0 = s(B^n). \tag{6.3}$$

There exists a unique orthogonal transformation U such that $Ux_0 = x_0$ and $Uw = -w$ for every w orthogonal to x_0. Define $\bar{s} = \frac{1}{2}(s + U^{-1}sU)$. It is evident that $L_{\bar{s}} \leq L_s$. Moreover, $\bar{s}(B^n) = s(B^n)$. Thus we can further assume $\bar{s} = s$. This assumption implies that U commutes with s. Since $A_w = UA_w$ for every $w \perp x_0$, it follows that

$$s(A_w) = 0. \tag{6.4}$$

Let $s[v] = \langle s(A_v), x_0\rangle$ and $l = L_s$. By (6.2) and (6.3),

$$\int_{S^{n-1}} s[v] d\tilde{\sigma}(v) = \beta. \tag{6.5}$$

Observe now that for every $v, w \in S^{n-1}$,

$$H(A_v, A_w) = \tau_n \sqrt{1 - \langle v, w\rangle^2}. \tag{6.6}$$

Suppose as before that $w \perp x_0$ and that x_0, v and w are linearly dependent. Then by (6.4), (6.6) and the definition of l,

$$\|s(A_v)\| = \|s(A_v) - s(A_w)\| \le l\tau_n \sqrt{1 - \langle v, w\rangle^2} = l\tau_n \langle v, x_0\rangle.$$

Since $s(A_v)$ and v are collinear, we also get $|s[v]| = \|s(A_v)\||\langle v, x_0\rangle|$. Therefore,

$$|s[v]| \le l\tau_n \langle v, x_0\rangle^2. \tag{6.7}$$

On the other hand, $s(A_v) \in A_v$ implies that

$$|s[v]| \le \tau_n |\langle v, x_0\rangle|. \tag{6.8}$$

Let $M = \{v \in S^{n-1} : |\langle v, x_0\rangle| \le \frac{1}{l}\}$, and let M' be the complement of M in S^{n-1}. By (6.5), (6.7) and (6.8),

$$\beta \le \int_{S^{n-1}} |s[v]| d\tilde{\sigma}(y) \le \int_M l\tau_n |\langle v, x_0\rangle|^2 d\tilde{\sigma}(v) + \int_{M'} \tau_n |\langle v, x_0\rangle| d\tilde{\sigma}(v).$$

From the fact that for every bounded and measurable function $g\colon [-1, 1] \to \mathbb{R}$,

$$\int_{S^{n-1}} g(\langle v, x_0\rangle) d\tilde{\sigma}(v) = \frac{n-1}{2\tau_n} \int_{-1}^{1} g(t) k_n(t) dt,$$

where $k_n(t) = (1 - t^2)^{\frac{1}{2}(n-3)}$ (see e.g. [7] or [31]), we deduce that

$$\beta \le \frac{(n-1)l}{2} \int_{|t| \le 1/l} t^2 k_n(t) dt + \frac{n-1}{2} \int_{1 \ge |t| \ge 1/l} |t| k_n(t) dt.$$

Since $-(n-1)tk_n(t) = \frac{d}{dt} k_{n+2}(t)$, we obtain

$$\beta \le -l \int_0^{1/l} t \frac{d}{dt} k_{n+2}(t) dt + k_{n+2}(1/l).$$

Integration by parts yields

$$\beta \le l \int_0^{1/l} k_{n+2}(t) dt =: b_n(l).$$

It is easy to see that b_n is strictly increasing and maps $[1, +\infty)$ onto $[n/\tau_{n+1}, 1)$. Furthermore, for every n we have $b_n \ge b_{n+1}$. In particular, $\beta \le b_2(l)$. By the Schwarz inequality, we get

$$\beta \le b_2(l) = l \int_0^{1/l} \sqrt{1 - t^2} dt \le l \left(\int_0^{1/l} dt \right)^{1/2} \left(\int_0^{1/l} 1 - t^2 dt \right)^{1/2} = \left(1 - \frac{1}{3l^2} \right)^{1/2}.$$

Hence

$$l \geq \frac{\sqrt{3}/3}{\sqrt{1-\beta^2}}.$$

□

Let us remark that our theorem cannot be generalized to valuations. More precisely, for each $x \in B^n$ there exists an s which is a valuation and a Lipschitz continuous selector on $\mathcal{K}^n$ such that its Lipschitz constant is not greater than $2L_0$. This result will be proved in Part II of this work.

Corollary 6.2. *The set $\mathcal{L}^n := \{L_s : s \in \mathcal{S}^n\}$ is unbounded.*

Proof. By Proposition 4.4, we know that $s_x(B^n) = x$ for $x \in \operatorname{int} B^n$. By our theorem,

$$L_x \geq \frac{\lambda}{\sqrt{1-\|x\|^2}}.$$

Thus $\{L_x : x \in \operatorname{int} B^n\}$ is unbounded and so is $\mathcal{L}^n$. □

Corollary 6.3. *If $S\colon M \to \mathcal{S}^n$ is a parametric representation of $\mathcal{K}^n$, then the set of all Lipschitz constants L_m of selectors $S(m)$ is unbounded.*

Proof. Since $\{S(m)(B^n) : m \in M\}$ is dense in B^n, for every $\beta \in [0,1)$ there exists an $m \in M$ such that $\|S(m)(B^n)\| \geq \beta$. For such an m we have $L_m \geq \lambda\sqrt{1-\beta^2}$. □

Let $l_n(\beta) = \inf\{L_s : s \in \mathcal{S}^n,\ \|s(B^n)\| = \beta\}$, and let $\lambda_n(\beta) = l_n(\beta)\sqrt{1-\beta^2}$. As we have shown in Theorem 6.1, $\lambda_n(\beta) \geq \sqrt{3}/3$. A certain estimate from the above for λ_n can be easily obtained by using (4.2). This estimate, however, is far from being optimal. In search of a better result we have to find an appropriate, for this purpose, family of selectors.

Assume $\phi = n/\sigma_{n-1}$ in the selection scheme, and define

$$k_x(A) := n\int_{S^{n-1}} h_A(x+u)u\,d\tilde{\sigma}(u). \tag{6.9}$$

By Theorem 5.1, $k_\bullet$ is a parametric representation of $\mathcal{K}^n$. Let L'_x be the Lipschitz constant of k_x. Since k_0 is the Steiner selector, $L'_0 = L_0$.

Proposition 6.4. $L'_x \leq L_0\sqrt{\|x\|^2+1}$.

Proof. Fix elements A and B in $\mathcal{K}^n$ such that $H(A,B) \leq 1$. Pick $v \in S^{n-1}$ so that

$$N := \|k_x(A) - k_x(B)\| = \langle k_x(A) - k_x(B), v\rangle.$$

Then

$$\begin{aligned} N &= n\int_{S^{n-1}} (h_A(x+u) - h_B(x+u))\langle u, v\rangle d\tilde{\sigma}(u) \\ &\leq n\int_{S^{n-1}} \left| h_A\left(\frac{x+u}{\|x+u\|}\right) - h_B\left(\frac{x+u}{\|x+u\|}\right)\right| \|x+u\||\langle u,v\rangle| d\tilde{\sigma}(u) \\ &\leq n\int_{S^{n-1}} \|x+u\||\langle u,v\rangle| d\tilde{\sigma}(u). \end{aligned}$$

The change of variable $u \mapsto -u$ gives us

$$N \leq \frac{n}{2} \int_{S^{n-1}} (\|x+u\| + \|x-u\|) |\langle u, v\rangle| d\tilde{\sigma}(u).$$

By the obvious inequality $\|x+u\| + \|x-u\| \leq 2\sqrt{\|x\|^2 + \|u\|^2}$ and the fact that $\|u\| = 1$, we obtain

$$N \leq n \int_{S^{n-1}} \sqrt{\|x\|^2 + 1} |\langle u, v\rangle| d\tilde{\sigma}(u) = L_0 \sqrt{\|x\|^2 + 1},$$

where we have used the formula $L_0 = n \int |\langle u, v\rangle| d\tilde{\sigma}(y)$, which can be found in [32] or [26] (compare also [23, Th. 6.1]). □

Proposition 6.5. *Given any $\beta \in [0, 1)$, suppose that $\|k_x(B^n)\| = \beta$. Then $\|x\| \leq \frac{\beta}{\sqrt{1-\beta^2}}$.*

Proof. Define $x_0 = k_x(B^n)$. Let $v \in S^{n-1}$ be chosen so that $\langle x_0, v\rangle = \beta$. Since $h_{B^n}(x + u) = \|x + u\|$,

$$\beta = n \int_{S^{n-1}} \|x+u\| \langle u, v\rangle d\tilde{\sigma}(u).$$

Let U be any orthogonal transformation which preserves x, i.e. $Ux = x$. As σ is invariant under the action of U, it follows that

$$\beta = n \int_{S^{n-1}} \|x+u\| \langle u, U^* v\rangle d\tilde{\sigma}(u).$$

This implies that v and x are collinear. (For $\beta \neq 0$, x_0 and v are collinear as well.) Similarly as in the proof of the previous proposition, by the change of variable $u \mapsto -u$, we get

$$\begin{aligned} \beta &= \frac{n}{2} \int_{S^{n-1}} (\|x+u\| - \|x-u\|) \langle u, v\rangle d\tilde{\sigma}(u) \\ &= \frac{n}{2} \int_{S^{n-1}} \frac{\|x+u\|^2 - \|x-u\|^2}{\|x+u\| + \|x-u\|} \langle u, v\rangle d\tilde{\sigma}(u). \end{aligned}$$

Thus

$$\beta = n \int_{S^{n-1}} \frac{2\langle x, u\rangle \langle u, v\rangle}{\|x+u\| + \|x-u\|} d\tilde{\sigma}(u),$$

and $v = \frac{x}{\|x\|}$ if $x \neq 0$. Consequently,

$$\beta \geq n \int_{S^{n-1}} \frac{\|x\| \langle u, v\rangle^2}{\sqrt{\|x\|^2 + 1}} d\tilde{\sigma}(u) = \frac{\|x\|}{\sqrt{\|x\|^2 + 1}}.$$

□

It follows immediately from these two propositions that

Theorem 6.6. $\lambda_n(\beta) \leq L_0$.

7. Selection measures

We begin with listing some standard facts which are well-known and explained in detail in [23], for example. The set $K^n = \{h_A : S^{n-1} \to \mathbb{R}^n : A \in \mathcal{K}^n\}$ is a closed cone in $C(S^{n-1})$. The mapping $\mathcal{K}^n \ni A \mapsto h_A$ is an isometry between $\mathcal{K}^n$ and K^n equipped with the Hausdorff and the supremum metrics, respectively. It is also an algebraic isomorphism of cones. ($\mathcal{K}^n$ possesses a natural structure of a cone with usual addition of sets and multiplication of sets by nonnegative scalars.) The set $K^n - K^n = \{f - g : f, g \in K^n\}$ is a dense subset of $C(S^{n-1})$.

These facts imply that for every Lipschitz continuous linear selector s on $\mathcal{K}^n$ there exists exactly one continuous and linear mapping $\tilde{s} \colon C(S^{n-1}) \to \mathbb{R}^n$ such that $s(A) = \tilde{s}(h_A)$. It should also be clear that the mapping $s \mapsto \tilde{s}$ is an isometric isomorphism onto $\tilde{\mathcal{S}}^n := \{\tilde{s} : s \in \mathcal{S}^n\} \subset L(C(S^{n-1}), \mathbb{R}^n)$, i.e. for every two selectors s_1 and s_2 in $\mathcal{S}^n$ the operator norm $\|\tilde{s}_1 - \tilde{s}_2\|$ coincides with $L_{s_1 - s_2}$. In this way, $\mathcal{S}^n$ is a natural example of a convex unbounded set in a Banach space whose recession cone is 0, something that cannot happen in finite dimensions [25]:

Proposition 7.1. *If $n \geq 2$, then $\tilde{\mathcal{S}}^n$ is convex, unbounded and does not contain any half-line.*

Proof. By Corollary 6.2, $\tilde{\mathcal{S}}^n$ is unbounded. Clearly, it is also convex. Moreover, if there would exist a half-line ℓ in $\tilde{\mathcal{S}}^n$, then taking two different elements $\tilde{s}_1, \tilde{s}_2$ in ℓ and $A \in \mathcal{K}^n$ such that $\tilde{s}_1(A) \neq \tilde{s}_2(A)$ one could deduce $\{\tilde{s}(A) : s \in \ell\}$ be a half-line contained in A, which is impossible. $\square$

It is not clear to the author whether $\mathcal{S}^n$ contains any extremal point (compare [33, 20, 21], where the case of linear but not necessarily continuous selectors is discussed).

It follows from the Riesz representation theorem that for every $s \in \mathcal{S}^n$ there exists a unique $\mathbb{R}^n$-valued Borel measure μ_s on S^{n-1} such that $\tilde{s}$ has an integral representation with respect to μ_s. Thus $\mathcal{S}^n$ can also be identified with the set $\mathcal{M}^n$ of all measures μ_s. The elements of $\mathcal{M}^n$ will be called *selection measures*. It would be interesting to have effective methods that enable one to determine whether a given measure on S^{n-1} corresponds to a selector. A little appears to be known in this direction even in the case of $n = 2$. The following properties of selection measures have been established in a joint work of G. Rote and the present author.

Theorem 7.2.

(i) *If $s \in \mathcal{S}^2$ and μ_s is singular with respect to the arc length measure, then $\mu_s(-B) = -\mu_s(B)$ for every Borel set $B \subset S^{n-1}$ (equivalently, $s(-A) = -s(A)$ for every $A \in \mathcal{K}^2$).*

(ii) *If $n \geq 3$, $\mu \in \mathcal{M}^n$, then for every $n-2$-dimensional subspace X of $\mathbb{R}^n$,*

$$|\mu|(S^{n-1} \cap X) = 0,$$

where $|\mu|$ denotes the variation of μ.

Details will be published elsewhere.

Observe that these selection measures which correspond to selections s_x that appear in the harmonic representation have their densities, with respect to the surface area measure σ, determined by (4.1). To provide further examples, we shall need the following

Lemma 7.3. *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be continuous and homogeneous of degree 1, $\Phi\colon S^{n-1} \to \mathbb{R}^k$ be measurable and bounded, and let R be a linear isomorphism of $\mathbb{R}^n$. For any w and z in $\mathbb{R}^n$ put $(w|z) = \langle R^{-1}w, R^{-1}z\rangle$ and $|w| = \|R^{-1}w\|$. Let*

$$F(y) = \int_{S^{n-1}} \varphi(y + Ru)\Phi(u)d\sigma(u).$$

Let

$$\kappa_- = \frac{(v|y) - \tau}{|v|^2}$$

and

$$\kappa_+ = \frac{(v|y) + \tau}{|v|^2},$$

where $v \in S^{n-1}$, $y \in \mathbb{R}^n$ and $\tau = \sqrt{(v|y)^2 + |v|^2(1 - |y|^2)}$.

Then the function F can be expressed in the following manner:

(i) *If $|y| \le 1$, then*

$$F(y) = \int_M \varphi(v)\Phi \circ R^{-1}(\kappa_+ v - y)\kappa_+^n |\det R|^{-1}\tau^{-1}d\sigma(v),$$

where $M = S^{n-1}$ if $|y| < 1$, and $M = \{v \in S^{n-1} : (v|y) > 0\}$ otherwise.

(ii) *If $|y| > 1$, then*

$$F(y) = \int_M \varphi(v)\left(\kappa_-^n \Phi \circ R^{-1}(\kappa_- v - y) + \kappa_+^n \Phi \circ R^{-1}(\kappa_+ v - y)\right) |\det R|^{-1}\tau^{-1}d\sigma(v),$$

where $M = \{v \in S^{n-1} : (\frac{v}{|v|}|y) > \sqrt{|y|^2 - 1}\}$.

Proof. Let $S_- = \{u \in S^{n-1} : \langle R^{-1}y, u\rangle < -1\}$, $S_0 = \{u \in S^{n-1} : \langle R^{-1}y, u\rangle = -1\}$ and $S_+ = \{u \in S^{n-1} : \langle R^{-1}y, u\rangle > -1\}$. Let

$$F_-(y) = \int_{S_-} \varphi(y + Ru)\Phi(u)d\sigma(u)$$

and

$$F_+(y) = \int_{S_+} \varphi(y + Ru)\Phi(u)d\sigma(u).$$

Since $\sigma(S_0) = 0$, it follows that

$$F(y) = F_-(y) + F_+(y).$$

For each $u \in S_- \cup S_+$, define $\Psi(u) = \frac{y+Ru}{\|y+Ru\|}$. Now, let $e_1, \ldots, e_{n-1}$ be an orthonormal basis in T_uS^{n-1}. Let $v = \Psi(u)$, and $f_i = \Psi_*(e_i)$, where $i = 1, \ldots, n-1$. If for each $p \in S^{n-1}$ we identify T_pS^{n-1} with $p^\perp$, then elementary calculations show that

$$f_i = \frac{1}{\kappa}\left(Re_i - \frac{\langle Re_i, v\rangle v}{\kappa}\right),$$

where $\kappa = \|y + Ru\|$. Since $\kappa v = y + Ru$, we have

$$u = \kappa R^{-1}v - R^{-1}y. \tag{7.1}$$

This implies

$$1 = \kappa^2|v|^2 - 2\kappa(v|y) + |y|^2.$$

It is obvious that κ_- and κ_+ are the only solutions of this equation. It is also easily seen that if $u \in S_-$, then $\kappa = \kappa_-$; otherwise $\kappa = \kappa_+$. Moreover, from (7.1) and the expressions for κ_- and κ_+ it follows immediately that $\tau = |\langle R^{-1}v, u\rangle|$. Let $\|f_1 \wedge \cdots \wedge f_{n-1}\|$ be the volume of the parallelepiped spanned by $f_1, \dots, f_{n-1}$. Then

$$\begin{aligned}
\|f_1 \wedge \ldots \wedge f_{n-1}\| &= \|f_1 \wedge \cdots \wedge f_{n-1} \wedge v\| \\
&= \frac{1}{\kappa^{n-1}}\|Re_1 - \frac{\langle Re_1, v\rangle v}{\kappa} \wedge \cdots \wedge Re_{n-1} - \frac{\langle Re_{n-1}, v\rangle v}{\kappa} \wedge v\| \\
&= \frac{1}{\kappa^{n-1}}\|Re_1 \wedge \cdots \wedge Re_{n-1} \wedge v\| \\
&= \frac{|\det R|}{\kappa^{n-1}}\|e_1 \wedge \cdots \wedge e_{n-1} \wedge R^{-1}v\| = \frac{|\det R|\tau}{\kappa^{n-1}}.
\end{aligned}$$

Suppose now that $S_- \neq \emptyset$. Then Ψ is a bijection from S_- onto M, and by a change of variables, the homogeneity of φ and the above formulae we get

$$\begin{aligned}
F_-(y) &= \int_{S_-} \varphi\left(\frac{y + Ru}{\kappa}\right) \kappa\Phi(u)d\sigma(u) \\
&= \int_M \varphi(v)\Phi \circ R^{-1}(\kappa_- v - y)\kappa_- \|f_1 \wedge \ldots \wedge f_{n-1}\|^{-1}d\sigma(v) \\
&= \int_M \varphi(v)\Phi \circ R^{-1}(\kappa_- v - y)\kappa_-^n |\det R|^{-1}\tau^{-1}d\sigma(v).
\end{aligned}$$

By the same reasoning, we obtain the appropriate formula for F_+. To complete the proof, it remains to observe that S_+ is nonempty for all y, while S_- only for those of them for which $|y| > 1$. □

From the formal point of view, the mapping $k_\bullet$, defined by (6.9), is the simplest cyclically monotonic representation, and as such deserves to have their selection measures described.

Theorem 7.4. *Let*

$$\kappa_- = \langle v, x\rangle - \tau, \quad \kappa_+ = \langle v, x\rangle + \tau,$$

where $v \in S^{n-1}$, $x \in \mathbb{R}^n$ *and* $\tau = \sqrt{\langle v, x\rangle^2 + 1 - \|y\|^2}$.

Then the selector k_x *can be expressed in the following manner:*

(i) *If* $\|x\| \leq 1$, *then*

$$k_x(A) = n\int_{S^{n-1}} h_A(v)\mathbf{1}_M(v)(\kappa_+ v - x)\kappa_+^n \tau^{-1}d\tilde{\sigma}(v),$$

where $M = S^{n-1}$ *if* $\|x\| < 1$, *and* $M = \{v \in S^{n-1} : \langle v, x\rangle > 0\}$ *otherwise.*

(ii) *If $\|x\| > 1$, then*

$$k_x(A) = n \int_{S^{n-1}} h_A(v)\mathbf{1}_M(v) \left(\kappa_-^n(\kappa_- v - x) + \kappa_+^n(\kappa_+ v - x)\right) \tau^{-1} d\tilde{\sigma}(v),$$

where $M = \{v \in S^{n-1} : \langle v, x\rangle > \sqrt{\|x\|^2 - 1}\}$.

Proof. It suffices to assume in Lemma 7.3 $y = x$, $\varphi = h_A$, $R = \mathrm{id}$ and $\Phi(u) = u/\kappa_n$. □

Suppose that we are given another scalar product, say $(\cdot,\cdot)$, on $\mathbb{R}^n$. Then we may define the Steiner selector, say ${s'}_0$, corresponding to this product. Now we determine the selection measure of ${s'}_0$.

Proposition 7.5. *Let a linear isomorphism T of $\mathbb{R}^n$ be chosen so that $(x, y) = \langle Tx, Ty\rangle$ for any x and y. Then*

$${s'}_0(A) = T^{-1} \circ s_0 \circ T(A) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_A(u) \frac{(T^*T)^{-1}u}{\|(T^*)^{-1}u\|^{n+2} |\det T|} d\sigma(u),$$

where T^* is adjoint to T.

Proof. Since

$${s'}_0(A) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_{T(A)}(u) T^{-1} u d\sigma(u) = \frac{1}{\kappa_n} \int_{S^{n-1}} h_A(T^*u) T^{-1} u d\sigma(u),$$

it suffices to use Lemma 7.3 assuming $R = T^*$, $y = 0$, $\varphi = h_A$ and $\Phi(u) = (1/\kappa_n)T^{-1}u$. □

8. Conclusion

We conclude with remarks concerning selections on $\mathcal{D}^n$. It is clear that there exist sets $A, B \in \mathcal{D}^n$ such that $A + B$ is not closed. This enforces us to define the Minkowski addition on $\mathcal{D}^n$ as follows

$$A \oplus B = \mathrm{cl}(A + B).$$

Obviously, $(\mathcal{D}^n, \oplus)$ is a semigroup and one can ask if there exists a linear selection s on it. The answer is in the negative in a trivial way. Nevertheless, it seems to be of considerable interest that Lipschitz continuous selections which are valuations do exist on $\mathcal{D}^n$. It can be a bit surprising for those who know how strongly these two classes of selections are related to each other in the case of $\mathcal{K}^n$.

It follows from the equation

$$(A \cup B) + (A \cap B) = A + B$$

whenever $A \cup B, A, B \in \mathcal{K}^n$ that each linear selection on $\mathcal{K}^n$ is a valuation. On the other hand, by a result of Spiegel [29], if s is a Lipschitz continuous valuation, and if for every $x \in \mathbb{R}^n$ and $A \in \mathcal{K}^n$ we have $s(2A) = 2s(A)$, and $s(A + x) = s(A) + x$, then s is linear.

Selectors which are valuations will be investigated in detail in Part II of this work.

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