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The Distribution of Unbounded Random Sets and the Multivalued Strong Law of Large Numbers in Nonreflexive Banach Spaces

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In the first part, we introduce appropriate tools concerning the distribution of random sets. We study the relation between the distribution of a random set, whose values are closed subsets of a Banach space, and the set of distributions of its measurable selections. Also, criteria for two random sets to be equidistributed are given, along with applications to the multivalued integral. In the second part, in combination with other arguments involving convex analysis and topological properties of hyperspaces (i.e., spaces of subsets), the results of the first part are exploited to prove a multivalued strong law of large numbers for closed (possibly unbounded) valued random sets, when the space of all closed sets is endowed, either with the Wijsman topology or the 'slice topology' introduced by G. Beer. The main results extend others of the same type in the literature, especially in the framework of non reflexive Banach space, or allow for shorter and self-contained proofs.

Keywords: distribution of random sets, multivalued strong law of large numbers, set convergence, measurable multifunctions, convex sets

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1. Introduction

An increasing number of mathematical models involve random sets (alias, measurable multifunctions, multivalued random variables, etc). In recent years, strong laws of large numbers (SLLN), for random sets whose values are not assumed to be bounded, gave rise to applications in several fields, such as Mathematical Economics, Stochastic Optimization, Statistics and related fields. The aim of this paper is to present two new versions of the SLLN for random sets. We shall assume that the image space, namely, the space of closed subsets of a Banach space, is equipped with topologies, that have received special attention only recently, for the purpose of theoretical developements and/or potential applications.

To explain the problem that we address, let us recall that a random set is a random variable whose values are sets and that, formally, a multivalued SLLN consists of the equality

$$\overline{\text{co}} \int_\Omega \Gamma \, dP = \tau - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Gamma_i(\omega) \quad \text{almost surely,} \tag{1.1}$$

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where Γ is a given closed valued random set defined on a probability space $(\Omega, \mathcal{A}, P)$ and $(\Gamma_i)_{i\geq 1}$ is a sequence of independent random sets having the same distribution as Γ. In the right-hand side of (1.1), τ represents a topology (or convergence) on the space of closed sets, whereas on the left-hand side, the integral of Γ over Ω is the Aumann integral and $\overline{\text{co}}$ stands for the closed convex hull operation. However, the random sets Γ_i are not assumed to be convex valued, which is an interesting feature of this type of result. When (1.1) holds, it is convenient to say that the multivalued SLLN holds *with respect to the topology* τ.

For random sets with closed (possibly unbounded) values[1] in a Banach space X, results of such type were first proved by Artstein and Hart [2] in the case where X is finite dimensional. Later, Hiaï [29] and Hess [20, 21] independently proved similar results for random sets taking their values in an infinite dimensional Banach space, with respect to the Mosco convergence. Moreover, using an extension of the SLLN for single-valued random variables due to Etemadi [15], it was shown in [20, 21] that the independence condition on the sequence (Γ_i) could be replaced by a weaker condition, namely the pairwise independence. Also, in [20, 21] a converse to the multivalued SLLN was proved in the case where the values of Γ do not contain any line (they can contain half-line only).

The above multivalued SLLN were established assuming that the set of all closed subsets of X, denoted by $\mathcal{C}(X)$, is endowed, either with the Painlevé-Kuratowski convergence, which corresponds to the Fell topology (see [8]), or with the Mosco convergence, an interesting infinite-dimensional extension of the Painlevé-Kuratowski convergence to reflexive spaces. The Mosco convergence was introduced for studying variational inequalities (see [34, 35]).

Although these results provide useful convergence properties for sequences of random sets, one may ask for multivalued SLLN with respect to a topology τ on $\mathcal{C}(X)$ that satisfies the following additional requirements:

(i) the topology τ involves more explicitly the norm of the space X by means of a distance between sets (which is useful to study the rates of convergence)

(ii) even in non reflexive Banach spaces, τ has "nice" properties, such as metrizability, second countability. We also mention the variational properties which, in the stochastic context, are most helpful for applications to stochastic optimization.

In order to fulfill the first requirement, we consider the case where $\mathcal{C}(X)$ is endowed with the Wijsman topology $\mathcal{T}_W$, namely the topology of pointwise convergence of distance functions. Indeed, when X is separable, $\mathcal{T}_W$ is metrizable and separable, and, unlike the Mosco topology, it has interesting properties even in the non reflexive case. The Wijsman topology was introduced in [41, 42]. It has been studied intensively during the last ten years and appeared as a building block in the construction of a lot of other interesting topologies on hyperspaces (see [8, 9] and Section 3 for an example connected to the multivalued SLLN). The Wijsman topology is also useful to study the convergence of best approximations (see [3, 40]). Let us remark that the Hausdorff metric topology is not a good candidate, since it is not well-suited to the case of random sets with unbounded values, the case we are interested in.

So, the first question we address is: in a general (separable) Banach space, does the SLLN

[1]We do not mention systematically the references concerning the multivalued SLLN for integrably bounded random sets (see, e.g., [2] and [29]).

for closed valued random sets hold when $\mathcal{C}(X)$ is endowed with $\mathcal{T}_W$? The main objective of the present paper is to show that, by combining arguments of various kinds (probabilistic, topological, convex analysis) an affirmative answer to this question can be given. This is done in Section 3 (Theorem 3.5). As in the previous versions of the multivalued SLLN existing in the literature, the convexification effect of the averaging process indicated in (1.1) can be observed.

On the other hand, in the special case of random sets with convex values, it is possible to prove the multivalued SLLN with respect to a topology τ, which is stronger than $\mathcal{T}_W$ and satisfies requirement (ii). Recently, Beer [7, 8] introduced the so-called "slice topology" on $\mathcal{C}_c(X)$, the space of all closed convex sets of X, which is a natural and most interesting extension of the Mosco topology to non reflexive Banach spaces. More precisely, Beer showed that, in a general Banach space, the slice topology on $\mathcal{C}_c(X)$, denoted by $\mathcal{T}_S$, is stronger than the Mosco topology, and that both topologies coincide if and only if X is reflexive. Also, like the Mosco topology, the slice-topology enjoys good variational properties. Consequently, keeping in mind requirement (ii), one is led to ask if the multivalued SLLN holds for $\mathcal{C}_c(X)$-valued random sets, with respect to the slice topology. At the end of Section 3, a positive answer to that question is given.

Concerning the probabilistic arguments, the reader will see that the proof of our main results heavily relies upon specific properties of the distribution of random sets with closed values in an infinite dimensional Banach space. In our opinion, it is not possible to prove the multivalued SLLN stated in the present paper without appealing to these properties. Moreover, some of them are essential when X is not finite dimensional. It is why we begin by providing suitable results along this line. These results, which are interesting in their own, extend previous ones of Hart and Kohlberg [17], and Artstein and Hart [2], and allow for shorter or self-contained proofs.

2. The distribution of random sets and of their measurable selections

This section is devoted to the presentation of fundamental properties of the distribution of random sets, whose values lie in a Banach space. For proving the multivalued SLLN for sequences (Γ_n) of random sets that are pairwise independent and have the same distribution as a given random set Γ, it will be necessary to consider measurable selections that are measurable with respect to the σ-fields $\mathcal{A}_{\Gamma_n}$ and $\mathcal{A}_\Gamma$ (i.e. generated by the given random sets). This naturally leads to the question of the relation between the sets of distributions of $\mathcal{A}_\Gamma$-measurable selections and those of $\mathcal{A}$-measurable selections where $\mathcal{A}$ is the σ-field given on Ω. The answer is given in Theorem 2.3 and in Corollary 2.5(i) for integrable selections. In view of application to the SLLN, the relation between the multivalued integrals is given in Corollary 2.5(ii). On the other hand, we shall also need to extract equidistributed and pairwise independent selections from random sets having similar properties. For this purpose, appropriate characterizations of equidistribution are proved at the end of this section (Proposition 2.6), as well as a connection with the multivalued integral.

The distribution of random sets was studied by several authors for various purposes (applications to mathematical economics, stochastic optimization, etc.). Let us mention, for example, Hart and Kohlberg [17], Hildenbrand [30], Artstein [1], Giné, Hahn and Zinn [16], Artstein and Hart [2], Hess [18, 19, 24], Salinetti and Wets [38], Lavie [33] and

Raynaud de Fitte [36].

We begin with some notation. Consider a probability space $(\Omega, \mathcal{A}, P)$ and a separable Banach space X, whose norm is denoted by $\|.\|$ and Borel σ-field by $\mathcal{B}(X)$. Also denote by $\mathcal{M}(X)$ (resp. $M(X)$) the set of all finite measures (resp. probability measures) on $(X, \mathcal{B}(X))$. The vector space $\mathcal{M}(X)$ is endowed with the *weak topology*, namely the topology $\sigma(\mathcal{M}(X), C_b(X))$, where $C_b(X)$ denotes the space of real-valued, bounded continuous functions on X. The weak topology on $\mathcal{M}(X)$ (in fact, a weak* topology) is also called the topology of *narrow convergence*. Recall that $M(X)$ is a closed convex subset of $\mathcal{M}(X)$. For each measurable function $f : \Omega \to X$, the distribution of f is denoted by μ_f and defined on $\mathcal{B}(X)$ by $\mu_f(B) = P\{f^{-1}(B)\}$ $(B \in \mathcal{B}(X))$.

Let $\mathcal{C}(X)$ be the set of all closed subsets of X and $\mathcal{E}$ the Effros σ-field on $\mathcal{C}(X)$. This σ-field is generated by the subsets $U^- = \{F \in \mathcal{C}(X) : F \cap U \neq \emptyset\}$, where U ranges over the open subsets of X. On the other hand, for any subset Y of X, the *distance function* of Y is defined by

$$d(x, Y) = \inf\{\|x - y\| : y \in Y\} \quad x \in X.$$

A map Γ from Ω into $\mathcal{C}(X)$ is also called a *multifunction* with closed values in X. The *domain* and the *graph* of Γ are respectively defined by

$$\mathrm{dom}(\Gamma) = \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\} \quad \text{and} \quad \mathrm{Gr}(\Gamma) = \{(\omega, x) \in \Omega \times X : x \in \Gamma(\omega)\}.$$

Γ is said to be *Effros measurable* or *weakly measurable* in the terminology of Himmelberg [31] (or simply "measurable") if for every B in $\mathcal{E}$, $\Gamma^{-1}(B)$ is a member of $\mathcal{A}$. From the definition of the Effros σ-field it follows that Γ is measurable if and only if, for any open subset U of X,

$$\Gamma^{-1}(U^-) = \{\omega \in \Omega : \Gamma(\omega) \cap U \neq \emptyset\}$$

is a member of $\mathcal{A}$ ($\Gamma^{-1}(U^-)$ is also denoted by $\Gamma^- U$). The sub-σ-field $\Gamma^{-1}(\mathcal{E})$ generated by Γ is denoted by $\mathcal{A}_\Gamma$.

A measurable multifunction defined on a probability space is also called a *random set* (r.s.). Like for real or vector valued random variables, the *distribution* μ_Γ of the measurable multifunction Γ can be defined on the measurable space $(\mathcal{C}(X), \mathcal{E})$ by

$$\mu_\Gamma(B) = P\{\Gamma^{-1}(B)\} \quad \forall B \in \mathcal{E}. \tag{2.1}$$

Further, two random sets Γ and Δ are said to be *independent* if the equality $\mu_{(\Gamma,\Delta)} = \mu_\Gamma \otimes \mu_\Delta$ holds on the product space $(\mathcal{C}(X) \times \mathcal{C}(X), \mathcal{E} \otimes \mathcal{E})$.

First, we recall a simple and useful criterion for two r.s. to have the same distribution. We give the short proof for convenience.

Proposition 2.1. *If Γ and Δ are two r.s. with values in $\mathcal{C}(X)$ then the following three statements are equivalent:*

(i) *Γ and Δ have the same distribution on $(\mathcal{C}(X), \mathcal{E})$.*
(ii) *For any open subset U of X, $P\{\Gamma^-(U)\} = P\{\Delta^-(U)\}$.*
(iii) *For any finite subset $Y = \{x_1, \ldots, x_k\}$ of X (or of some countable dense subset), the $\mathbb{R}^k$-valued random vectors $(d(x, \Gamma))_{x \in Y}$ and $(d(x, \Delta))_{x \in Y}$ have the same distribution.*

Proof. For every subset F of X, the subset F^+ of $\mathcal{C}(X)$ is defined by $F^+ = \{C \in \mathcal{C}(X) : C \subseteq F\}$. In order to prove the equivalence (i) $\Leftrightarrow$ (ii), first observe that, for every open subset U of X, one has

$$(U^-)^c = (U^c)^+ = \{C \in \mathcal{C}(X) : C \subseteq U^c\}$$

where U^c denotes the complement of U. This class generates $\mathcal{E}$ and is stable under finite intersection, which by a classical result yields the desired equivalence. For proving the equivalence (i) $\Leftrightarrow$ (iii), remember that the Effros σ-field is also generated by the family of distance functions

$$C \to d(x, C)$$

where x ranges over X (or over a countable dense subset). Like in the first part of the proof, it is enough to observe that the class

$$\{C \in \mathcal{C}(X) : d(x_i, C) < \alpha_i \quad \forall i = 1, \ldots, k\}$$

(where $k \in \mathbb{N}^*, x_i \in X$ and $\alpha_i \in \mathbb{R}$), generates $\mathcal{E}$ and is stable under finite intersection. $\square$

For every sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, consider the space $\mathcal{L}^0(\Omega, \mathcal{F}, P; X)$ of all measurable functions[2] from $(\Omega, \mathcal{F})$ into $(X, \mathcal{B}(X))$. Further, define the two following subsets associated with the multifunction Γ and which concern its measurable selections:

$$S(\Gamma, \mathcal{F}) = \{f \in \mathcal{L}^0(\Omega, \mathcal{F}, P; X) : f(\omega) \in \Gamma(\omega), \text{ for almost every } \omega \in \mathrm{dom}(\Gamma)\}$$
$$M(\Gamma, \mathcal{F}) = \{\mu = \mu_f \in M(X) : f \in S(\Gamma, \mathcal{F})\}.$$

So, $S(\Gamma, \mathcal{F})$ is the set of $\mathcal{F}$-measurable selections of Γ and $M(\Gamma, \mathcal{F})$ the set of all probability measures μ on $(X, \mathcal{B}(X))$ such that each $\mu \in M(\Gamma, \mathcal{F})$ is the distribution of some $\mathcal{F}$-measurable selection of Γ. By the Kuratowski Ryll-Nardzewski Theorem every Effros measurable multifunction, Γ admits at least one measurable selection. Moreover, Γ admits a *Castaing representation*, that is, a sequence (f_n) of measurable selections, such that for every $\omega \in \mathrm{dom}(\Gamma)$, $\Gamma(\omega)$ is equal to the closure of the countable subset $\{f_n(\omega) : n \geq 1\}$ (see e.g. [13, chapter III] or [37]).

It is known that $\mathcal{L}^0(\Omega, \mathcal{A}, P; X)$ $(= \mathcal{L}^0(X))$ endowed with the topology of convergence in probability is a metrizable topological vector space. Since a sequence converging in probability admits an almost sure converging subsequence it is clear that, for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the set $S(\Gamma, \mathcal{F})$ is closed in $\mathcal{L}^0(\Omega, \mathcal{F}, P; X)$.

Before stating the results on the sets of selections and their associated set of probability measures, an example is in order.

Example 2.2. Consider the case where $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}(\Omega)$, $P =$ Lebesgue's measure, $X = \mathbb{R}$ and $\Gamma \equiv \{0, 1\}$ (i.e. Γ is a constant multifunction). Clearly we have

$$\mathcal{A}_\Gamma = \{\emptyset, \Omega\}, \; S(\Gamma, \mathcal{A}_\Gamma) = \{f \equiv 0, f \equiv 1\} \; \text{ and } \; M(\Gamma, \mathcal{A}_\Gamma) = \{\delta_0, \delta_1\}$$

and also

$$S(\Gamma, \mathcal{A}) = \{f = \mathbf{1}(A) : A \in \mathcal{A}\}, \; M(\Gamma, \mathcal{A}) = \{p \; \delta_0 + (1 - p) \; \delta_1 : p \in [0, 1]\}$$

[2]In the present paper, we consider the prequotient setting but all the results could be restated in the quotient setup.

where $\mathbf{1}(A)$, the (probabilistic) *indicator function* of A, is defined by $\mathbf{1}(A)(\omega) = 1$ if $\omega \in A$, $\mathbf{1}(A)(\omega) = 0$ otherwise. This shows that the inclusions

$$S(\Gamma, \mathcal{A}_\Gamma) \subset S(\Gamma, \mathcal{A}) \quad \text{and} \quad M(\Gamma, \mathcal{A}_\Gamma) \subset M(\Gamma, \mathcal{A})$$

may be strict.

The following theorem is the main result of the present section. It was already stated in [19, Proposition 5], but the proof contained a gap[3], so that a new proof has to be worked out. This theorem provides a fundamental equality which is the starting point for the study of the distribution of multifunctions in connection with their measurable selections. Its consequences, especially Corollary 2.5(ii) concerning the multivalued integral, play an important role in the proof of the multivalued SLLN.

Theorem 2.3. *If Γ is a r.s. with closed values in X, then, in $M(X)$ endowed with the topology of weak convergence of measures, the following equality holds true*

$$\overline{\mathrm{co}}\, M(\Gamma, \mathcal{A}) = \overline{\mathrm{co}}\, M(\Gamma, \mathcal{A}_\Gamma) \tag{2.2}$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull operation.

Proof. If $\Gamma(\omega)$ is empty for each $\omega \in \Omega$, (2.2) is trivially satisfied. Otherwise, it is possible to assume that $\mathrm{dom}(\Gamma)$ is equal to Ω. The inclusion $\mathcal{A}_\Gamma \subseteq \mathcal{A}$, clearly shows that the right-hand side of (2.2) is contained in the left-hand side. To prove the converse inclusion we shall use the support function for a closed convex set contained in the vector space $\mathcal{M}(X)$, which is in duality with $C_b(X)$. In this framework, the *support function* of $M \subseteq \mathcal{M}(X)$ is denoted by $s(., M)$ and given by

$$s(\phi, M) = \sup\{\int_X \phi(x)\mu(dx) : \mu \in M\} \quad \forall \phi \in C_b(X).$$

So, we ought to show that for any ϕ in $C_b(X)$, the following inequality holds

$$s(\phi, \overline{\mathrm{co}}\, M(\Gamma, \mathcal{A})) \leq s(\phi, \overline{\mathrm{co}}\, M(\Gamma, \mathcal{A}_\Gamma)). \tag{2.3}$$

which, by the definitions, is implied by

$$\begin{aligned} &\forall \phi \in C_b(X), \ \ \forall f \in S(\Gamma, \mathcal{A}) \ \text{ and } \ \forall \alpha > 0, \exists g \in S(\Gamma, \mathcal{A}_\Gamma) : \\ &\qquad\qquad \int_X \phi(x)\mu_f(dx) \leq \int_X \phi(x)\mu_g(dx) + \alpha. \end{aligned} \tag{2.4}$$

By the change-of-variable formula the last inequality is equivalent to

$$\int_\Omega \phi(f(\omega))P(d\omega) \leq \int_\Omega \phi(g(\omega))P(d\omega) + \alpha. \tag{2.5}$$

Now, define the multifunction Δ on Ω by $\Delta(\omega) = \mathrm{cl}\, \phi(\Gamma(\omega))$, for every $\omega \in \Omega$, where cl denotes the closure in $\mathbb{R}$. Clearly, Δ has non-empty, closed bounded values in $\mathbb{R}$, because

[3] We are glad to think H. Ziat for having called our attention on this point.

ϕ is bounded. Further, assuming that the set of all closed subsets of $\mathbb{R}$ is endowed with its own Effros σ-field, it is readily seen that Δ is $\mathcal{A}_\Gamma$-measurable. Indeed, for any open subset O of $\mathbb{R}$, one has

$$\Delta^- O = \{\omega \in \Omega : \phi(\Gamma(\omega)) \cap O \neq \emptyset\} = \{\omega \in \Omega : \Gamma(\omega) \cap \phi^{-1}(O) \neq \emptyset\} \in \mathcal{A}.$$

Also consider the bounded real function b on Ω defined by

$$b(\omega) = \sup\{r \in \mathbb{R} : r \in \Delta(\omega)\}.$$

For every $u \in \mathbb{R}$, the following equivalences are clear

$$b(\omega) > u \;\Leftrightarrow\; \phi(\Gamma(\omega)) \cap (u, +\infty) \neq \emptyset \;\Leftrightarrow\; \Gamma(\omega) \cap \phi^{-1}((u, +\infty)) \neq \emptyset$$

and show that $b(.)$ is $\mathcal{A}_\Gamma$-measurable. Now, define the multifunctions Φ and Ψ respectively, by setting for each $\omega \in \Omega$

$$\Phi(\omega) = [b(\omega) - \alpha, b(\omega)] \quad \text{and} \quad \Psi(\omega) = \Delta(\omega) \cap \Phi(\omega).$$

The multifunction Φ is $\mathcal{A}_\Gamma$-measurable since for any open interval I of $\mathbb{R}$, one has $\Phi^- I = b^{-1}(B)$ for some suitable Borel subset of $\mathbb{R}$, and we have just proved the $\mathcal{A}_\Gamma$-measurability of $b(.)$. Therefore, Ψ being the intersection of two $\mathcal{A}_\Gamma$-measurable multifunctions with closed values in $\mathbb{R}$, it is $\mathcal{A}_\Gamma$-measurable too (see, for example [37]). Since it is nonempty valued by construction, it admits at least one $\mathcal{A}_\Gamma$-measurable selection ψ. Now, consider the closed valued multifunction Θ defined on Ω by $\Theta(\omega) = \Gamma(\omega) \cap \phi^{-1}(\psi(\omega))$. Its graph $\mathrm{Gr}(\Theta)$ satisfies

$$\begin{aligned}\mathrm{Gr}(\Theta) &= \{(\omega, x) \in \Omega \times X : x \in \Theta(\omega)\} = \mathrm{Gr}(\Gamma) \cap \{(\omega, x) \in \Omega \times X : x \in \phi^{-1}(\psi(\omega))\} \\ &= \mathrm{Gr}(\Gamma) \cap \{(\omega, x) \in \Omega \times X : \phi(x) = \psi(\omega)\} \in \mathcal{A}_\Gamma \otimes \mathcal{B}(X).\end{aligned}$$

Thus, Aumann-Von Neumann's selection Theorem (see, e.g., Theorem III.18 in [13]) yields the existence of an $\mathcal{A}_\Gamma$-measurable selection g of Θ such that, for almost every $\omega \in \Omega, g(\omega) \in \Theta(\omega)$ whence, by the definition of Θ, $\phi(g(\omega)) = \psi(\omega)$. Consequently, returning to the definition of multifunction Ψ of which ψ is a selection, we obtain

$$\int_\Omega \phi(f(\omega))P(d\omega) \le \int_\Omega b(\omega)P(d\omega) \le \int_\Omega \psi(\omega)P(d\omega) + \alpha \le \int_\Omega \phi(g(\omega))P(d\omega) + \alpha$$

which proves (2.5) and finishes the proof. □

We denote by $\mathcal{L}^1(\Omega, \mathcal{A}, P; X)$ ($= \mathcal{L}^1(X)$) the subspace of $\mathcal{L}^0(X)$ whose members are Bochner-integrable and by $M^1(X)$ the subset of $M(X)$ whose members μ satisfy $\int_X \|x\| d\mu < +\infty$. Given a sub-$\sigma$-field $\mathcal{F}$ of $\mathcal{A}$ and a random set Γ, define the following subsets of $\mathcal{L}^1(\Omega, \mathcal{F}, P; X)$ and $M(X)$ respectively

$$\begin{aligned}&S^1(\Gamma, \mathcal{F}) = \{f \in \mathcal{L}^1(\Omega, \mathcal{F}, P; X) : f(\omega) \in \Gamma(\omega), \text{ for almost every } \omega \in \mathrm{dom}(\Gamma)\} \\ &M^1(\Gamma, \mathcal{F}) = \{\mu_f : f \in S^1(\Gamma, \mathcal{F})\}.\end{aligned}$$

Using standard measurable selection arguments, it is not hard to see that, when $\mathcal{A}_\Gamma \subseteq \mathcal{F} \subseteq \mathcal{A}$, the set $S^1(\Gamma, \mathcal{F})$ is non empty if and only if the positive function $d(0, \Gamma(.))$ is integrable. In such a situation, we shall say that the multifunction Γ is *integrable*. Observe that when Γ is integrable, $P(\mathrm{dom}(\Gamma)) = 1$. On the other hand, Γ is said to be *integrably bounded* if the function $\omega \to h(\Gamma(\omega)) = \sup\{\|x\| : x \in \Gamma(\omega)\}$ is integrable. In this case, we have $S^1(\Gamma, \mathcal{A}) = S(\Gamma, \mathcal{A})$. An integrably bounded multifunction is also integrable, but the converse implication is false. For any measurable multifunction Γ and any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$, the *multivalued integral* of Γ over Ω, with respect to $\mathcal{F}$, is defined by

$$I(\Gamma, \mathcal{F}) = \{\int_\Omega dP : f \in S^1(\Gamma, \mathcal{F})\}.$$

$I(\Gamma, \mathcal{A})$ is non empty if and only if Γ is integrable. Now, consider an integrable multifunction Γ. Obviously, the inclusion $M^1(\Gamma, \mathcal{F}) \subseteq M(\Gamma, \mathcal{F})$ holds for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$. The following simple lemma shows that, in $M(X)$, the closure of both sides are equal.

Lemma 2.4. *For any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and any integrable r.s. Γ whose values are members of $\mathcal{C}(X)$, the following equality holds true*

$$\mathrm{cl}\, M^1(\Gamma, \mathcal{F}) = \mathrm{cl}\, M(\Gamma, \mathcal{F}) \tag{2.6}$$

the closure being taken in $\mathcal{M}(X)$ (or $M(X)$) in the weak topology.

Proof. In (2.6) the inclusion of the left-hand side in the right-hand side is clear. Conversely, consider μ in $M(\Gamma, \mathcal{F})$. There exists f in $S(\Gamma, \mathcal{F})$ such that $\mu = \mu_f$. Further, Γ being integrable, one can find at least one g in $S^1(\Gamma, \mathcal{F})$. For every positive integer k, define f_k in $S^1(\Gamma, \mathcal{F})$ by

$$A_k = \{\omega \in \Omega : \|f\| \le k\} \quad \text{and} \quad f_k = f\mathbf{1}(A_k) + g\mathbf{1}(A_k^c)$$

where A^c denotes the complement of A. It is readily seen that the sequence $(f_k)_{k\ge1}$ converges to f in probability, whence in distribution, as k tends to infinity. This shows that f is a member of the left-hand side of (2.6) as required. □

From Theorem 2.3 and Lemma 2.4, we easily deduce the following corollary, whose first part makes more precise equality (2.2) when Γ is integrable. Part (ii) is an application to the multivalued integral. It has been given by Artstein and Hart [2, Theorem 2.2] when Γ is an integrable measurable multifunction with closed values in a finite dimensional space. Let us also note that Artstein and Hart's proof relies on a previous result of Hart and Kohlberg [17] valid for integrably bounded r.s., whereas our proof is self-contained.

Corollary 2.5.

(i) *For every integrable r.s. Γ whose values are in $\mathcal{C}(X)$, the following equality holds true*

$$\overline{\mathrm{co}}\, M^1(\Gamma, \mathcal{A}) = \overline{\mathrm{co}}\, M^1(\Gamma, \mathcal{A}_\Gamma) \tag{2.7}$$

the closure being taken in $\mathcal{M}(X)$ (or $M(X)$) in the weak topology.

(ii) *For any integrable r.s. Γ whose values lie in $\mathcal{C}(X)$, one has*

$$\overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}) = \overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}_\Gamma). \tag{2.8}$$

Proof. (i) Appeal to Theorem 2.3 and Lemma 2.4, and observe that two subsets of a topological vector space have the same closed convex hull as soon they have the same closure.

(ii) Obviously, the right-hand side is included in the left-hand side, so it only remains to prove the opposite inclusion. For this purpose, first observe that, for any $f \in \mathcal{L}^1(X)$, we have

$$S^1(\Gamma + f) = S^1(\Gamma) + f \quad \text{and} \quad I(\Gamma + f) = I(\Gamma) + \int_\Omega f \, dP.$$

Consequently, since $S^1(\Gamma, \mathcal{A}_\Gamma)$ is non empty it is possible to assume that 0 is a member of $\Gamma(\omega)$, for almost all ω. Now, first consider the special case where Γ is integrably bounded. In this case, the linear map $\Phi : \mu \to \int_X x \, d\mu$ from $M^1(X)$ into X, is continuous on $M^1(\Gamma, \mathcal{A})$. Indeed, for every $f \in S^1(\Gamma, \mathcal{A})$ and $\omega \in \Omega$, one has $\|f(\omega)\| \le h(\Gamma(\omega))$. Hence, by Theorem 5.4 in [10] Φ is continuous. This yields the following equalities

$$\mathrm{cl}\, \Phi(\overline{\mathrm{co}}\, M^1(\Gamma, \mathcal{F})) = \overline{\mathrm{co}}\, \Phi(M^1(\Gamma, \mathcal{F})) = \overline{\mathrm{co}}\, I(\Gamma, \mathcal{F})$$

that are valid for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ containing $\mathcal{A}_\Gamma$. This yields (2.8) when Γ is integrably bounded.

To prove (2.8) in the general case, define for every integer $k \ge 1$ the multifunction Γ_k by setting $\Gamma_k(\omega) = \mathrm{cl}\{\Gamma(\omega) \cap B(0, k)\}$ and observe that each Γ_k is $\mathcal{A}_\Gamma$-measurable, integrably bounded.

Moreover, we have

$$\Gamma(\omega) = \bigcup_{k \ge 1} \Gamma_k(\omega) \quad \forall \omega \in \Omega,$$

whence for any sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ containing $\mathcal{A}_\Gamma$

$$\mathrm{cl}\, I(\Gamma, \mathcal{F}) = \bigcup_{k \ge 1} I(\Gamma_k, \mathcal{F}). \tag{2.9}$$

Equality (2.9) is valid for $\mathcal{F} = \mathcal{A}$ and $\mathcal{F} = \mathcal{A}_\Gamma$ as well. Thus, taking the closed convex hull of both sides and using the equality proved in the integrably bounded case, we obtain

$$\overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}) = \mathrm{cl}\{\bigcup_{k \ge 1} \overline{\mathrm{co}}\, I(\Gamma_k, \mathcal{A})\} = \mathrm{cl}\{\bigcup_{k \ge 1} \overline{\mathrm{co}}\, I(\Gamma_k, \mathcal{A}_\Gamma)\} = \overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}_\Gamma)$$

which finishes the proof. □

The following result states that two r.s. Γ and Δ have the same distribution, if and only if, the sets of distributions of selections that are measurable relatively to the σ-fields $\mathcal{A}_\Gamma$ and $\mathcal{A}_\Delta$ respectively, are equal. As we shall see, this property also plays a crucial part in the proof of the multivalued SLLN in an infinite dimensional Banach space. Considering the σ-fields generated by the random sets, instead of the σ-field $\mathcal{A}$ allows for more precise results. Indeed, unlike in [2], the equalities of statements (ii) and (iii) below do not involve any closure operation (see also Remark 3.7).

Proposition 2.6. *Let Γ and Δ be two random sets with closed values in X. Then, the two following statements are equivalent:*

(i) *Γ and Δ have the same distribution on the measurable space $(\mathcal{C}(X), \mathcal{E})$.*
(ii) *In $M(X)$, the following equality holds true*

$$M(\Gamma, \mathcal{A}_\Gamma) = M(\Delta, \mathcal{A}_\Delta).$$

Moreover, if Γ and Δ are integrable then each of the above statements is equivalent to

(iii) *In $M^1(X)$, the following equality holds true*

$$M^1(\Gamma, \mathcal{A}_\Gamma) = M^1(\Delta, \mathcal{A}_\Delta).$$

Consequently, if Γ and Δ have the same distribution, one has

$$I(\Gamma, \mathcal{A}_\Gamma) = I(\Delta, \mathcal{A}_\Delta).$$

Proof. (i) $\Rightarrow$ (ii). For any μ in $M(\Gamma, \mathcal{A}_\Gamma)$, there exists $f \in S(\Gamma, \mathcal{A}_\Gamma)$ such that $\mu = \mu_f$. Since f is $\mathcal{A}_\Gamma$-measurable and X is complete, one can find a map ϕ from $\mathcal{C}(X)$ into X, that is measurable with respect to the σ-fields $\mathcal{E}$ and $\mathcal{B}(X)$, and satisfies $f = \phi \circ \Gamma$ (see e.g. [14], p. 18). Now, define g from Ω to X by $g = \phi \circ \Delta$. Clearly, g is $\mathcal{A}_\Delta$-measurable and has the same distribution as f. Further, the positive function $F \to d(\phi \circ F, F)$ defined on $\mathcal{C}(X)$ is $\mathcal{E}$-measurable. Consequently, the map $\omega \to d(g(\omega), \Delta(\omega)) = d(\phi \circ \Delta(\omega), \Delta(\omega))$ has the same distribution as the map $\omega \to d(f(\omega), \Gamma(\omega)) = d(\phi \circ \Gamma(\omega), \Gamma(\omega))$ (Δ is replaced by Γ). This implies

$$P\{\omega \in \Omega : d(g(\omega), \Delta(\omega)) = 0\} = P\{\omega \in \Omega : d(f(\omega), \Gamma(\omega)) = 0\} = 1$$

and shows that g is a selection of Δ.

(ii) $\Rightarrow$ (i). Assume that Γ and Δ are not equidistributed. Then, according to Proposition 2.1, it is possible find a non void open subset U of X such that $P\{\Gamma^- U\} \neq P\{\Delta^- U\}$. Assume for instance, without loss of generality, that $P\{\Gamma^- U\} > P\{\Delta^- U\}$. Using a Castaing representation of Γ whose members are $\mathcal{A}_\Gamma$-measurable, it is not difficult to construct $f \in S(\Gamma, \mathcal{A}_\Gamma)$, verifying, for any $\omega \in \Gamma^- U$, $f(\omega) \in \Gamma(\omega) \cap U$. Consequently, for any $g \in S(\Delta, \mathcal{A}_\Delta)$ one has

$$P\{g^{-1}(U)\} \leq P\{\Delta^- U\} < P\{\Gamma^- U\} = P\{f^{-1}(U)\} \tag{2.10}$$

which shows that no g in $S(\Delta, \mathcal{A}_\Delta)$ can have the same distribution as f.

(i) $\Rightarrow$ (iii). The beginning of the proof is similar to that of implication (i) $\Rightarrow$ (ii). Thus, it only remains to show that if f is integrable, so is g. This fact is a consequence of the following equalities which are easily obtained by the change-of-variables formula

$$\int_\Omega \|g\| dP = \int_\Omega \|\phi \circ \Delta\| dP = \int_{\mathcal{C}(X)} \|\phi\| d\mu_\Delta = \int_{\mathcal{C}(X)} \|\phi\| d\mu_\Gamma = \int_\Omega \|f\| dP$$

(iii) $\Rightarrow$ (i). As in the proof of implication (ii) $\Rightarrow$ (i) assume the existence of an open subset U of X such that $P\{\Gamma^- U\} > P\{\Delta^- U\}$. Inspecting the proof of Proposition 2.1, it is readily seen that U may be assumed to be bounded, so that each measurable function with values in U is automatically Bochner-integrable. Consider a Castaing representation

$(f_n)_{n\geq 1}$ of Γ, whose members are $\mathcal{A}_\Gamma$-measurable. For every integer $n \geq 1$, we set $A_n = \{\omega \in \Omega : f_n(\omega) \in U\}$. We define the sequence $(B_n)_{n\geq 1}$ by

$$B_1 = A_1 \quad \text{and} \quad B_n = A_n \backslash (\bigcup_{j<n} A_j) \quad n \geq 2.$$

Pick f_0 in $S_1(\Gamma, \mathcal{A}_\Gamma)$ and define $f = \sum_{n\geq 1} \mathbf{1}(B_n) f_n + \mathbf{1}(A^c) f_0$. Clearly, f is an $\mathcal{A}_\Gamma$-measurable Bochner integrable selection of Γ and satisfies $f(\omega) \in U$ for every $\omega \in \Gamma^- U$. Therefore, for every $g \in S^1(\Delta, \mathcal{A}_\Delta)$, relationships (2.10) are still valid, which yields the same contradiction as in the proof of implication (ii) $\Rightarrow$ (i). Finally, using the definition of $I(\Gamma, \mathcal{A}_\Gamma)$, we easily obtain the second statement. □

Remark 2.7.

(i) An alternate proof of the last statement of Proposition 2.6(iii) was given by Hiaï [29, Lemma 3.1(2)] using the properties of the multivalued conditional expectation as defined by Hiaï and Umegaki (J. Mult. Anal. 7, 1977).

(ii) It is worthwhile to remark that Theorem 2.3 and Proposition 2.6 ((i) and (ii)) remain valid when X is only assumed to be a complete separable metric space.

3. The multivalued strong law of large numbers with respect to the Wijsman topology

As already mentioned, the Wijsman topology $\mathcal{T}_W$ on $\mathcal{C}(X)$ is the topology of pointwise convergence of distance functions. It was introduced in [41, 42] in view of applications to consistency problems in Statistics, in the finite dimensional setting. Recall that a net (C_α) of closed sets is said to *converge to $\mathcal{C}$ in the Wijsman topology* if, for every $x \in X$, one has

$$d(x, C) = \lim_\alpha d(x, C_\alpha).$$

When X is finite dimensional, $\mathcal{T}_W$ coincides with the Fell topology (which induces Painlevé-Kuratowski convergence). When the Banach space X is reflexive, $\mathcal{T}_W$ is in general weaker than the Mosco topology but is equivalent to it when the norm is assumed to be Fréchet differentiable (see [40, 8]). An other interesting feature of the Wijsman topology is that the space $(\mathcal{C}(X), \mathcal{T}_W)$ is metrizable and separable, and that the Borel σ-field of $\mathcal{T}_W$ is equal to $\mathcal{E}$ (see [18, 23]). Furthermore, Beer has shown that if X is Polish then $(\mathcal{C}(X), \mathcal{T}_W)$ is Polish too, and that the converse implication holds (see [5]). This permits one to consider Effros measurable multifunctions as ordinary measurable maps with values in a complete separable metric space.

First, we need a bit of notation. The closed unit ball of X^* is denoted by B^*. Given a subset C of X, the (convex analysis) *indicator function* $\chi(C)$ and the *support function* $s(., C)$ are defined on X (resp. on X^*) by

$$\chi(C)(x) = 0 \text{ if } x \in C, \ +\infty \text{ otherwise} \quad x \in X$$

and

$$s(z, C) = \sup\{\langle z, x\rangle : x \in C\} \quad z \in X^*.$$

It is known that $s(., C)$ is the conjugate function of $\chi(C)$ and that, whenever C is a nonempty member of $\mathcal{C}_c(X), \chi(C)$ and $s(., C)$ are conjugate to each other.

We begin with two preparatory lemmas of purely deterministic nature. They will be helpful for proving the "lim inf half" of the multivalued SLLN when $\mathcal{C}(X)$ is endowed with the Wijsman topology . They are needed to control the cardinality of the negligible subsets.

Lemma 3.1. *For any closed convex subset C of X, there exists a countable subset D^* of B^* verifying, for any $x \in X$,*

$$d(x,C) = \sup_{z \in D_*} \{\langle z,x\rangle - s(z,C)\}. \tag{3.1}$$

Proof. It is easy to check that $d(.,C)$ is the infimal convolution of $\chi(C)$ and $\|.\|$, which can be expressed by the equality $d(x,C) = (\chi(C)\nabla\|.\|)(x)$, for every $x \in X$. By conjugacy this yields

$$d(.,C)^*(z) = s(z,C) + \chi(B^*)(z) \quad \forall z \in X^*.$$

Since $d(.,C)$ a is positive, convex, lower semicontinuous function, the following equality holds

$$d(x,C) = (d(.,C))^{**}(x) = \sup\{\langle z,x\rangle - s(z,C) - \chi(B^*)(z)\} \quad x \in X,$$

whence

$$d(x,C) = \sup_{z \in B^*} \{\langle z,x\rangle - s(z,C)\}. \tag{3.2}$$

X being separable, the supremum of the continuous functionals $x \to \langle z,x\rangle - s(z,C)$, for z in B^*, can be rewritten as the supremum over some countable subset D^* of B^* (see [12, Proposition 3 p. IX.60]) which proves (3.1). □

Lemma 3.2. *Let $(C_n)_{n\geq 1}$ be a sequence in $\mathcal{C}(X)$. Also consider $C \in \mathcal{C}(X)$ and a countable dense subset D^* of B^* such that*

$$d(x,\overline{\mathrm{co}}\,C) = \sup_{z \in D^*} \{\langle z,x\rangle - s(z,\overline{\mathrm{co}}\,C)\} \quad x \in X \tag{3.3}$$

(which is possible by Lemma 3.1*). If, for every $z \in D^*$, one has*

$$\limsup_{n\to\infty} s(z,C_n) \leq s(z,C) \tag{3.4}$$

then, for every $x \in X$,

$$\liminf_{n\to\infty} d(x,C_n) \geq d(x,\overline{\mathrm{co}}\,C).$$

Proof. For every $x \in X$, we have by (3.2) and elementary calculations

$$\liminf_{n\to\infty} d(x,C_n) \geq \liminf_{n\to\infty} d(x,\overline{\mathrm{co}}\,C_n) \geq \sup_{z \in D^*} \liminf_{n\to\infty} [\langle z,x\rangle - s(z,C_n)]$$

whence,

$$\liminf_{n\to\infty} d(x,C_n) \geq \sup_{z \in D^*} [\langle z,x\rangle - s(z,\overline{\mathrm{co}}\,C)] = d(x,\overline{\mathrm{co}}\,C)$$

□

The crucial step for proving the multivalued SLLN with respect to the Wijsman topology consists of the following proposition, which is an adaptation to infinite dimensional spaces of a technique due to Artstein and Hart [2]. This adaptation is done via Proposition 2.6(iii) (see Remark 3.7).

Proposition 3.3. *Consider a separable Banach space X, an integrable r.s. Γ with values in $\mathcal{C}(X)$, a sequence $(\Gamma_n)_{n\geq 1}$ of pairwise independent random sets having the same distribution as Γ and the set C' of all convex combinations of $I(\Gamma, \mathcal{A}_\Gamma)$, with rational coefficients. Then, for each $y \in C'$, there exists a negligible subset $N(y)$ of Ω and a sequence $(g_n)_{n\geq 1}$ in $\mathcal{L}^1(X)$ verifying:*

(i) *for each $n \geq 1$, $g_n \in S^1(\Gamma_n, \mathcal{A}_{\Gamma_n})$*

(ii) *for any $\omega \in \Omega \backslash N(y)$,*

$$y = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g_i(\omega).$$

Proof. Consider $y \in C'$. From the definition of C' we have $y = \sum_{j=1}^{k} \lambda_j y_j$ where k is a positive integer, λ_j are positive rationals summing to one and where, for each $j \geq 1, y_j = \int_\Omega f_j dP$, for some $f_j \in S^1(\Gamma, \mathcal{A}_\Gamma)$. Obviously, for every $j = 1, ..., k$, it is possible to write $\lambda_j = \frac{m_j}{m}$ where m and the m_j are positive integers satisfying $m = \sum_{j=1}^{k} m_j$. Then, Proposition 2.6(iii) shows that for each $j \in \{1, ..., k\}$ and each $i \geq 1$, one can find a member f_j^i of $S^1(\Gamma_i, \mathcal{A}_{\Gamma_i})$ that has the same distribution as f_j. Thus, for every $j \in \{1, ..., k\}$, $(f_j^i)_{i\geq 1}$ is a pairwise independent sequence of X-valued random variables having he same distribution as f_j. Now, define the sequence $(g_n)_{n\geq 1}$ by setting, for every $n \geq 1$,

$$g_n = f_j^n \quad \text{if } j \text{ satisfies} \quad 1 + \sum_{p=1}^{j-1} m_p \leq n \leq \sum_{p=1}^{j} m_p \quad (\text{modulo } m). \tag{3.5}$$

Moreover, if $n \leq m_1$ (modulo m), we set $g_n = f_1^n$.

In the above relation, the integer j depends on n. Also define for every $j, n \geq 1$, the following set of integers

$$I(j, n) = \{i \leq n : g_i = f_j^i\}. \tag{3.6}$$

For every $n \geq 1$ and $\omega \in \Omega$, the following equalities hold

$$\frac{1}{n}\sum_{i=1}^{n} g_i(\omega) = \frac{1}{n}\sum_{j=1}^{k} \sum_{i\in I(j,n)} f_j^i(\omega) = \sum_{j=1}^{k} \frac{\#I(j,n)}{n} \cdot \frac{1}{\#I(j,n)} \sum_{i\in I(j,n)} f_j^i(\omega) \tag{3.7}$$

where # denotes the cardinality of a subset. On the other hand, for any $n \geq 1$, there exist two integers q_n and r_n satisfying

$$n = mq_n + r_n, \tag{3.8}$$

$$0 \leq r_n < m \quad \text{and} \quad q_n \geq 0, \tag{3.9}$$

from which we deduce that, for any $j \in \{1, ..., k\}$,

$$\#I(j,n) = m_j(q_n + \delta_n) \tag{3.10}$$

where δ_n may be equal to 0 or 1. Using (3.7) and (3.9) we obtain

$$\frac{\#I(j,n)}{n} = \frac{m_j(q_n + \delta_n)}{mq_n + r_n} = \frac{m_j(1 + \delta_n/q_n)}{m + r_n/q_n}. \tag{3.11}$$

From the above relationships, we deduce that the sequences (r_n) and (δ_n) are bounded, whereas from (3.8) and (3.9) we deduce

$$\lim_{n\to\infty} q_n = +\infty. \tag{3.12}$$

Therefore, for every $j \in \{1, ..., k\}$, equality (3.11) yields

$$\lambda_j = \frac{m_j}{m} = \lim_{n\to\infty} \frac{\#I(j,n)}{n}. \tag{3.13}$$

Further, looking at (3.10), we see that (3.12) also implies

$$\lim_{n\to\infty} \#I(j,n) = +\infty. \tag{3.14}$$

Then, returning to (3.7), using (3.13) and (3.14), and applying Etemadi's SLLN (see [15]) for vector-valued random variables to each sequence $(f_j^i)_{i\geq 1}$, for any $j \in \{1, ..., k\}$, one can easily deduce the existence of a negligible subset $N(y)$ such that for every $\omega \in \Omega \backslash N(y)$,

$$y = \sum_{j=1}^{k} \lambda_j y_j = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g_i(\omega).$$

$\square$

Given the closed-valued integrable random sets Γ and $\Gamma_n (n \geq 1)$, we set

$$C = \overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}). \tag{3.15}$$

and

$$Z_n(\omega) = \mathrm{cl}\, \frac{1}{n} \sum_{i=1}^{n} \Gamma_i(\omega) \quad n \geq 1 \quad \omega \in \Omega. \tag{3.16}$$

As already mentioned, the above result is the main tool to prove the most difficult "half" of the multivalued SLLN in the Wijsman topology (the easy "half" involves the "lim inf" operation instead of the "lim sup" one). This is the purpose of the next proposition.

Proposition 3.4. *Under the same hypotheses as in* Proposition 3.3*, there exists a negligible subset N such that, for any $\omega \in \Omega \backslash N$ and $x \in X$,*

$$\limsup_{n\to\infty} d(x, Z_n(\omega)) \leq d(x, \overline{\mathrm{co}}\, I(\Gamma, \mathcal{A})). \tag{3.17}$$

Proof. For every $\omega \in \Omega$ and $n \geq 1$, let C and $Z_n(\omega)$ be as in (3.15) and (3.16) respectively. From Corollary 2.5(ii) we know that $\overline{\text{co}}\, I(\Gamma, \mathcal{A}) = \overline{\text{co}}\, I(\Gamma, \mathcal{A}_\Gamma)$. Further, let D' be a countable dense subset of $I(\Gamma, \mathcal{A}_\Gamma)$ satisfying $\text{cl}\, D' = \text{cl}\, I(\Gamma, \mathcal{A}_\Gamma)$, and consider the set C' of all rational convex combinations of members of D'. On the other hand, consider a countable dense subset D of X and observe that it suffices to prove (3.17) for all x in D. Indeed, each side of (3.17) defines a Lipschitz function of x (with Lipschitz constant 1). So, consider $x \in D$ and an integer $p \geq 1$. One can find $y' = y'(x,p) \in C'$, depending on x and p, such that

$$\|x - y'\| \leq d(x, \overline{\text{co}}\, I(\Gamma, \mathcal{A})) + 1/p.$$

Further, Proposition 3.3 applied to y', yields the existence of a negligible subset $N(x,p)$ and of a sequence $(g_n)_{n\geq 1}$ verifying properties (i) and (ii). Then, define the negligible subset N as the union of the $N(x,p)$ where $x \in D$ and $p \geq 1$, and consider $\omega \in \Omega \backslash N$. For every $x \in D$, we have

$$\limsup_{n\to\infty} d(x, Z_n(\omega)) \leq \lim_{n\to\infty} \|x - \frac{1}{n}\sum_{i=1}^{n} g_i(\omega)\| = \|x - y\| \leq d(x, \overline{\text{co}}\, I(\Gamma, \mathcal{A})) + \frac{1}{p},$$

whence, by the arbitrariness of p, yields the desired conclusion. □

Now, we can state and prove the main result of the present paper, namely, the multivalued SLLN when $\mathcal{C}(X)$ is endowed with the Wijsman topology.

Theorem 3.5. *Consider a separable Banach space X, an integrable r.s. Γ with values in $\mathcal{C}(X)$ and a sequence $(\Gamma_n)_{n\geq 1}$ of pairwise independent r.s. having the same distribution as Γ. Then, there exists a negligible subset N such that, for any $\omega \in \Omega\backslash N$,*

$$\overline{\text{co}}\, I(\Gamma, \mathcal{A}) = \mathcal{T}_W - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \Gamma_i(\omega)$$

that is, for any $x \in X$,

$$d(x, \overline{\text{co}}\, I(\Gamma, \mathcal{A})) = \lim_{n\to\infty} d(x, \frac{1}{n}\sum_{i=1}^{n} \Gamma_i(\omega)).$$

Proof. We begin by choosing a countable subset D^* of B^*, satisfying (3.1) relatively to the subset $\overline{\text{co}}\, C = I(\Gamma, \mathcal{A})$, that is

$$d(x, C) = \sup_{z\in D^*} [\langle z, x\rangle - s(z, C)] \quad \forall x \in X.$$

Let z be fixed in D^*. Since the map $F \to s(z, F)$ is Effros measurable from $\mathcal{C}(X)$ in $\bar{\mathbb{R}}$, we deduce that $(s(z, \Gamma_i))_{i\geq 1}$ is a sequence of $\bar{\mathbb{R}}$-valued, pairwise independent random variables having the same distribution as $s(z, \Gamma)$. Further, by the equality $d(0, \Gamma(\omega)) = \sup\{-s(z, \Gamma(\omega)) : z \in B^*\}$ and the integrability of Γ, it is possible to apply the SLLN for $\bar{\mathbb{R}}$-valued random variables to the sequence $(s(z, \Gamma_i))$. This yields the existence of a negligible subset $N(z)$ of Ω verifying, for every $\omega \in \Omega\backslash N(z)$,

$$s(z, C) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} s(z, \Gamma_i(\omega)) = \lim_{n\to\infty} s(z, Z_n(\omega)).$$

Now, defining the negligible subset N_1 as the union of the $N(z)$, for $z \in D^*$, we deduce that

$$s(z, C) = \lim_{n\to\infty} s(z, Z_n(\omega)) \quad \forall z \in D^* \quad \forall \omega \in \Omega \backslash N_1.$$

The above equality and Lemma 3.2 entail

$$\liminf_{n\to\infty} d(x, Z_n(\omega)) \geq d(x, C) \quad \forall x \in X \quad \forall \omega \in \Omega \backslash N_1. \tag{3.18}$$

On the other hand, Proposition 3.4 yields the existence of a negligible subset N_2 such that

$$\limsup_{n\to\infty} d(x, Z_n(\omega)) \leq d(x, C) \quad \forall x \in X \quad \forall \omega \in \Omega \backslash N_2. \tag{3.19}$$

Finish the proof by defining the negligible subset $N = N_1 \cup N_2$ and by combining inequalities (3.18) and (3.19). $\square$

Remark 3.6. If X is assumed to be finite dimensional, Theorem 3.5 is comparable to Theorem 3.2 of [2], except that we only assume pairwise independence and we do not assume the completeness of $(\Omega, \mathcal{A}, P)$. As to the infinite dimensional case, if X is reflexive then Mosco convergence implies Wijsman convergence. As already mentioned, the converse is true when X is suitably renormed, so that in this case Mosco and Wijsman topologies are equivalent, and Theorem 3.5 appears as a variant of [20, 21]. However, when X is not reflexive, the situation is quite different because the Mosco convergence has no longer nice properties. Indeed, the Mosco topology, as defined in [7], is not even Hausdorff. So, it is important to observe that Theorem 3.5 continues to hold in the framework of non reflexive Banach spaces, where $\mathcal{T}_W$ still enjoys nice properties.

Remark 3.7. As the reader has noticed, the key ingredient in the proof of Theorem 3.5 is Proposition 3.3 whose proof is an adaptation of a technique due to Artstein and Hart [2]. Let us explain why an adaptation of their proof is necessary. In Artstein and Hart's proof, the main argument consists of an appeal to the SLLN for independent sequences of vector-valued random variables (they are not assumed to be identically distributed). This result holds in finite dimensional spaces but, in infinite dimensions, it remains valid only in Hilbert spaces or in B-convex Banach spaces (see [4]). In our proof, we only need the SLLN for i.i.d. sequences, which is known to hold in every Banach space, whatever be the norm.

Now, we shall deduce from Theorem 3.5 and from a recent result of Beer, a version of the SLLN in the slice topology for r.s. with closed convex values. One of the main advantages of the slice topology is that, as the Mosco topology in the reflexive setting, it makes the polarity and, when applied to convex functions via the epigraphs, the Young-Fenchel transform both bicontinuous. The slice topology was introduced by Sonntag and Zalinescu [39] and Beer [7], and intensively studied by Beer in [6, 7, 8]. First, we recall the definition of this topology along with a characterization involving the Wijsman topology.

The *gap* between two nonempty subsets B and C of X is denoted $D(B, C)$, and defined by

$$D(B, C) = \inf\{\|x - y\| : x \in B, y \in C\}.$$

A *slice of a ball* is the intersection of a closed ball $\bar{B}(x_0, r)$ (where $x_0 \in X$ and $r > 0$) and of a closed half space $F(z, \alpha) = \{x \in X : \langle z, x\rangle \geq \alpha\}$ (where $z \in X^*, z \neq 0$ and $\alpha \in \mathbb{R}$). Moreover, it is assumed that $F(z, \alpha)$ passes through the interior of $\bar{B}(x_0, r)$. This occurs if and only if

$$\alpha - \langle z, x_0\rangle < r\|z\|.$$

Definition 3.8. The *slice topology* on $\mathcal{C}(X)$ is the weak (or initial) topology determined by the family of gap functionals $\{D(B, .) :\ B \text{ is a nonempty slice of a ball}\}$. It is denoted by $\mathcal{T}_s$.

From Theorem 5.2 in [7], we know that the slice topology restricted to $\mathcal{C}_c(X)$ is the weak topology determined by the family $\{D(B, .) :\ B \text{ closed convex bounded}\}$. Our proof of the multivalued SLLN with respect the slice topology is based on the next result (Theorem 3.1 and Lemma 3.3 in [6]).

Proposition 3.9. *Let (X, p_0) be a normed linear space and Π_0 the family of all norms equivalent to p_0. Then, the slice topology $\mathcal{T}_s$ on $\mathcal{C}_c(X)$ is the weak topology determined by the family of distance functions*

$$\{d_p(x, .) : x \in X, p \in \Pi_0\}$$

where d_p stands for the distance function associated with the norm p, that is, for any subset C of X and for any $x \in X$,

$$d_p(x, C) = \inf_{y \in C} p(x - y).$$

Furthermore, if X has a strongly separable dual, the slice topology on $\mathcal{C}_c(X)$ is determined by the countable family

$$\{d_p(x, .) : x \in D, p \in \Pi_1\}$$

where D is a countable subset of X and Π_1 a countable subfamily of Π_0.

In other words, the topology $\mathcal{T}_s$ on $\mathcal{C}_c(X)$ is the supremum of the topologies $\mathcal{T}_{W(p)}$, where $W(p)$ denotes the Wijsman topology associated with the norm p and where p ranges over Π_0. If X^* is separable then p needs only range over the countable subfamily Π_1, which is decisive for our approach. Here is a multivalued version of the SLLN with respect to the slice topology.

Theorem 3.10. *Let X be a Banach space with strongly separable dual X^* and Γ an integrable r.s. with values in $\mathcal{C}_c(X)$. If $(G_n)_{n\geq 1}$ is a sequence of pairwise independent r.s. having the same distribution as Γ then one has for almost all $\omega \in \Omega$,*

$$\overline{\mathrm{co}}\, I(\Gamma, \mathcal{A}) = \mathcal{T}_s - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Gamma_i(\omega). \tag{3.20}$$

Proof. Let C and $Z_n(\omega)$ be defined as in (3.15) and (3.16). Consider the countable subfamily Π_1 as above. From Theorem 3.5, we know that, for every norm $p \in \Pi_1$, there exists a negligible subset $N(p)$ such that for every $\omega \in \Omega \backslash N(p)$, one has

$$C = \mathcal{T}_{W(p)} - \lim_{n\to\infty} Z_n(\omega). \tag{3.21}$$

Thus, if we define the negligible subset N as the union of the $N(p)$, where $p \in \Pi_1$, we can see that, for every $\omega \in \Omega \backslash N$ and $p \in \Pi_1$, equality (3.21) will hold. By Proposition 3.9, this yields the desired conclusion. □

Remark 3.11. (i) Another proof of Theorem 3.10 can be given starting, as above, from the multivalued SLLN with respect to the Wijsman topology (Theorem 3.5), but using the definition of the slice topology in terms of gap functionals, instead of appealing to Proposition 3.9. This alternate approach allows one for proving the SLLN in the slice topology for random sets with closed, possibly non convex, values (so that the convexification effect of the averaging operation can be observed once more). This was done recently in [27] but, there, the arguments involving the slice topology are more involved. In the present paper, mainly devoted to the SLLN with respect to the Wijsman topology, we have chosen presenting a self-contained treatment and avoiding too much technicalities about the slice topology.

(ii) On the other hand, we remark that natural applications of Theorems 3.5 and 3.10 to integrands can be proved via the epigraphical multifunctions. For example, it was shown in [27, Theorem 4.1] that the discrete stochastic inf-convolution of a sequence of integrands $\mathcal{T}_s$-epi-converges to a continuous deterministic infimal convolution. This is a useful property in stochastic optimization. Similar results concerning economic models had been proved in [2] in the context of finite dimensional spaces.

(iii) The results of the present paper suggest that other strong limit theorems (such as martingale convergence or ergodic theorems) for random sets with respect to the Wijsman or slice topology could be true. As already mentioned, the case of random sets with unbounded values is the most interesting in view of applications. In fact, the case of multivalued martingales has received special attention recently. For example, one can consult the papers by Hess [25, 26, 28] and Krupa [32], were further references can be found.

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