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Topological Properties of a New Graph Topology

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In [10, 11, 12] a new graph topology τ was introduced which is useful in applications to differential equations. In this paper we study topological properties of τ and relations between τ and other known topologies. For example, we find conditions under which τ coincides with Back's generalized compact-open topology successfully used for convergence of utility functions [2] and for convergence of dynamic programming models [19].

Keywords: Kuratowski convergence, τ-convergence, Back's convergence

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1. Introduction

Topologies and convergences of graph spaces (spaces of functions identified with their graphs or epigraphs) has been applied to different fields of mathematics, including differential equations, convex analysis, optimization, mathematical economics, programming models, calculus of variations etc. [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 17, 19, 20, 21, 24].

The definition of a new graph topology τ was motivated by concrete problems in the theory of hereditary differential equations.

Let (X, d) be a metric space, $CL(X)$ be the family of all non-empty closed subsets and $G = \{\Gamma(f, \Omega) : \Omega \in CL(X), f \in C(\Omega, \mathbb{R}^m)\}$ be the set of all graphs.

In [8, 9, 13], the authors studied a Cauchy problem (P) for ordinary differential equations with delay. By virtue of the generality of the hereditary structure, the solutions of problem (P) are elements of the graph set G (where $X = E$ is a closed interval of $\mathbb{R}$). To study problem (P) the authors introduced the topology τ in G [10]; it arose as a localization on compact sets of the Hausdorff metric topology; the connection between τ and Hausdorff metric topology is the same as that between the compact-open topology τ_{CO} and the uniform convergence topology in $C(E, \mathbb{R}^m)$.

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In [11, 12] the authors extended the τ-topology over the graphs of functions defined on subsets of a metric space X, preserving its main properties. The aim is to introduce a same general hereditary structure in the theory of partial differential equations.

The peculiar property of τ, which makes it useful in applications to hereditary differential equations, is the homeomorphism between the topological space (G, τ) and the quotient space $[(CL(X), \tau_F) \times (C(X, \mathbb{R}^m), \tau_{CO})]/\mathcal{R}$ with respect to a suitable equivalence relation, where τ_F is the Fell topology. In force of this homeomorphism, the theory of hereditary differential equations in G has been reduced to the classical theory in $C(X, \mathbb{R}^m)$.

In [10] ($X = E \subset \mathbb{R}$) the proof of the homeomorphic property was constructive, while in metric spaces (see [11, 12]), the existence of the homeomorphism was proved by using the Dugundji's continuous extension and Michael's continuous selection theorem.

We wish to point out that none of the already known hypertopologies has this homeomorphic property without involving heavy restrictions on the family $CL(X)$.

In the present paper we analyse topological properties of τ, and we compare it with other known topologies. By using of a similar idea as Back [2], we find a subbase for the topology generated by τ-convergence. As a consequence the topology τ is well defined for every topological space X and the τ-convergence doesn't depend on the choice of a metric in X.

We here summarise the main results obtained telling the properties valid only for convergence of sequences from the ones formulated for nets.

(1) τ-convergence of sequences and convergence in Back's generalized compact-open topology [2, 19] coincide; in the case of nets the same convergences coincide if and only if X is locally compact.
(2) τ-convergence of sequences implies Kuratowski [3] convergence of graphs and domains; we find additional conditions under which also opposite is true. For nets τ implies Fell convergence, while the Kuratowski is implied if and only if X is locally compact.
(3) If X is locally compact, then τ is uniformizable; if X is locally compact and separable, then τ is metrizable and (G, τ) is a Polish space.
(4) τ is finer than the Attouch-Wets topology [1] if and only if X is boundedly compact.

2. Notations and definitions

In what follows let (X, d) be a metric space. For basic notions and definitions the reader is referred to recent Beer's monograph [3]. Given two subsets A, B of a metric space (X, d), the excess or Hausdorff semi-distance of A over B is denoted by $e_d(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$ with the convention $e_d(A, \emptyset) = +\infty$ if $A \neq \emptyset$ and $e_d(\emptyset, B) = 0$.

It is well known that $H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}$ defines the Hausdorff distance between A and B.

Denote by $CL(X)$ the family of all non-empty closed subsets of X and by $K(X)$ the family of all compact sets in $CL(X)$.

The open (resp. closed) ball with center x and radius $r > 0$ will be denoted by $S(x, r)$ (resp. $B(x, r)$). The open (resp. closed) r-enlargement of A is the set $S(A, r) = \{x \in$

$X : d(x, A) < r\}$ ($B(A, r) = \{x \in X : d(x, A) \leq r\}$), where by $d(x, A)$ we mean $\inf\{d(x, a) : a \in A\}$.

For every $\Omega \in CL(X)$, $C(\Omega, \mathbb{R}^m)$ denotes, as usual, the space of all continuous functions $f : \Omega \to \mathbb{R}^m$. If $f \in C(\Omega, \mathbb{R}^m)$ we denote by $\Gamma(f, \Omega) = \{(\omega, f(\omega)) : \omega \in \Omega\}$ the graph of f. Let $G = \{\Gamma(f, \Omega) : \Omega \in CL(X), f \in C(\Omega, \mathbb{R}^m)\}$ denote the set of all graphs.

Let u be the metric in $\mathbb{R}^m$ induced by the natural norm. In what follows the excess e on closed subsets of $X \times \mathbb{R}^m$ is induced by the box metric D of d and u.

Given two elements $\Gamma(f, \Omega)$, $\Gamma(g, \Delta)$ in G and a set $K \in \mathcal{K}(X)$, we define

$$\rho_K(\Gamma(f, \Omega), \Gamma(g, \Delta)) = \max\{e(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta)), e(\Gamma(g, \Delta \cap K), \Gamma(f, \Omega))\}.$$

Remark 2.1. Note that ρ_K is non decreasing with respect to K, i.e. if $K_1 \subset K_2$ then $\rho_{K_1}(\cdot, \cdot) \leq \rho_{K_2}(\cdot, \cdot)$. We also have that $\rho_K(\cdot, \cdot) \leq H(\cdot, \cdot)$.

Moreover, we have

$$e(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta)) = e(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta \cap B(K, r))) \tag{$\star$}$$

for every number $r > e(\Gamma(f, \Omega \cap K), \Gamma(g, \Delta))$.

Definition 2.2. Let (X, d) be a metric space. A net $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ in G is said to be τ-convergent to $\Gamma(f_0, \Omega_0)$ if for every $K \in \mathcal{K}(X)$ the numerical net $\{\rho_k(\Gamma(f_0, \Omega_0), \Gamma(f_\sigma, \Omega_\sigma)) : \sigma \in \Sigma\}$ converges to zero.

Of course Hausdorff metric convergence in G implies τ-convergence and if X is compact these two convergences coincide.

3. τ-convergence of sequences in G

In this part we introduce some facts valid for τ-convergence of sequences in G for general metric spaces. (Most of these results cannot be formulated for nets.) We will be interested particularly to its relation to Kuratowski convergence of sequences in G.

Kuratowski convergence of sequences of restricted graphs in $CL(X \times \mathbb{R}^m)$ as well as sequences of domains is convergence with respect to the Fell topology [3, Theorem 5.2.10]. As such, they can be also expressed in terms of excess as given by the following

Corollary 3.1 ([3, Corollary 5.1.7]). *Let (X, d) be a metric space, let $A \in CL(X)$, and let $\{A_\sigma : \sigma \in \Sigma\}$ a net in $CL(X)$. Then $\{A_\sigma : \sigma \in \Sigma\}$ converges in the Fell topology to A if and only if for each $K \in \mathcal{K}(X)$, we have both $\{e_d(A \cap K, A_\sigma) : \sigma \in \Sigma\}$, $\{e_d(A_\sigma \cap K, A) : \sigma \in \Sigma\}$ converge to zero.*

From this perspective, it is natural to try to link τ-convergence with Kuratowski convergence of restricted graphs and domains.

By $\operatorname{Lim} \Omega_n, \operatorname{Li} \Omega_n$ and $\operatorname{Ls} \Omega_n$ of the sequence $\{\Omega_n\}_n$ in $CL(X)$ we mean [3] the Kuratowski limit, a lower closed limit and an upper closed limit of $\{\Omega_n\}_n$, respectively.

Lemma 3.2. *Let (X, d) be a metric space. If $\{\Gamma(f_n, \Omega_n)\}_n$ τ-converges to $\Gamma(f, \Omega)$, then $\operatorname{Lim} \Gamma(f_n, \Omega_n) = \Gamma(f, \Omega)$ and $\operatorname{Lim} \Omega_n = \Omega$.*

Proof. Let $x \in \Omega$. We show that $x \in \mathrm{Li}\,\Omega_n$ and $(x, f(x)) \in \mathrm{Li}\,\Gamma(f_n, \Omega_n)$. There is $x_n \in \Omega_n$ such that

$$\max\{d(x, x_n), | f_n(x_n) - f(x) |\} < \rho_{\{x\}}(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega)) + \frac{1}{n}$$

for every $n \in Z^+$.

From $\rho_{\{x\}}(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega)) \to 0$, we have $x_n \to x$ and $f_n(x_n) \to f(x)$; done.

Now we prove that $\mathrm{Ls}\,\Gamma(f_n, \Omega_n) \subset \Gamma(f, \Omega)$; the proof of $\mathrm{Ls}\,\Omega_n \subset \Omega$ is similar.

Suppose that $(x, y) \in \mathrm{Ls}\,\Gamma(f_n, \Omega_n)$. By Lemma 5.2.8 in [3] there is a sequence $\{(x_{n_k}, f_{n_k}(x_{n_k}))\}_k$ such that $x_{n_k} \in \Omega_{n_k}$ for every $k \in Z^+$ and $(x_{n_k}, f_{n_k}(x_{n_k})) \to (x, y)$. Put $K = \{x_{n_k} : k \in Z^+\} \cup \{x\}$. For every $k \in Z^+$ there is $\omega_{n_k} \in \Omega$ such that

$$\max\{d(x_{n_k}, \omega_{n_k}), | f_{n_k}(x_{n_k}) - f(\omega_{n_k}) |\} < \rho_K(\Gamma(f_{n_k}, \Omega_{n_k}), \Gamma(f, \Omega)) + \frac{1}{n_k}.$$

Since $\omega_{n_k} \to x$, we have $x \in \Omega$. The continuity of f at x and $f_{n_k}(x_{n_k}) \to y$ implies that $y = f(x)$, done. $\square$

The following proposition gives an equivalent description of τ-convergence in the language of Langen's generalized continuous convergence [19].

Proposition 3.3. *Let (X, d) be a metric space. Then $\{\Gamma(f_n, \Omega_n)\}_n \xrightarrow{\tau} \Gamma(f, \Omega)$ if and only if*

(i) $\mathrm{Lim}\,\Omega_n = \Omega$;
(ii) *for any $x \in \Omega$ and any sequence $x_n \to x,\ x_n \in \Omega_n,\ n \in Z^+,\ f_n(x_n) \to f(x)$.*

Proof. Let $\Gamma(f, \Omega) \in G$ and $\{\Gamma(f_n, \Omega_n)\}_n$ be a sequence in G satisfying (i) and (ii). Suppose that $\{\Gamma(f_n, \Omega_n)\}_n$ fails to τ-converge to $\Gamma(f, \Omega)$. There is $K \in \mathcal{K}(X)$, $\epsilon > 0$ and an infinite subset I of Z^+ with

$$\rho_K(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega)) > \epsilon$$

for every $n \in I$.

We have two possibilities:

(a) there is an infinite subset I_1 of I with $e(\Gamma(f_n, \Omega_n \cap K), (\Gamma(f, \Omega)) > \epsilon$ for every $n \in I_1$;
(b) there is an infinite subset I_2 of I with $e(\Gamma(f, \Omega \cap K), (\Gamma(f_n, \Omega_n)) > \epsilon$ for every $n \in I_2$.

In the case (a) choose for every $n \in I_1, x_n \in \Omega_n \cap K$ with $D((x_n, f_n(x_n)), \Gamma(f, \Omega)) > \epsilon$. Let $x \in K$ be a cluster point of $\{x_n\}$; without loss of generality we can suppose that x is a limit point of $\{x_n\}$. By (i), $x \in \Omega$. There is a sequence $\{z_n\}$ with $\{z_n\} \to x, z_n \in \Omega_n$ for every $n \in Z^+$ and $\{x_n\}$ is a subsequence of $\{z_n\}$. By (ii) we must have $f_n(x_n) \to f(x)$, a contradiction.

In the case (b) choose for every $n \in I_2, x_n \in \Omega \cap K$ with $D((x_n, f(x_n)), \Gamma(f_n, \Omega)) > \epsilon$. Let $x \in K$ be a cluster point of $\{x_n\}$. By using of (i) and (ii) there is a sequence $\{z_n\} \to x$ such that $z_n \in \Omega_n$ for every $n \in Z^+$ and $f_n(z_n) \to f(x)$. The continuity of f at x gives a contradiction with $D((x_n, f(x_n)), \Gamma(f_n, \Omega)) > \epsilon$, for every $n \in I_2$.

To prove the opposite implication we use the similar argument as above. $\square$

From the previous proposition we can guarantee the coincidence of τ-convergence of sequences in G and convergence in Back's generalized compact-open topology.

In [2] Back introduced a generalized compact-open topology on the space of utility functions. A utility function is a pair (Ω, f) where $\Omega \in CL(X)$ and $f \in C(\Omega, \mathbb{R})$. In [18] this topology was extended on the class $\mathcal{P}$ of all partial maps, i.e. pairs (Ω, f) where $\Omega \in CL(X)$ and $f \in C(X, Y)$. Of course, we can identify every partial map (Ω, f) with its graph $\Gamma(f, \Omega)$.

For any open set $U \subset X$, $K \in \mathcal{K}(X)$ and open (possibly empty) $I \subset \mathbb{R}^m$, let $[U] = \{(\Omega, f) \in \mathcal{P} : \Omega \cap U \neq \emptyset\}$ and $[K : I] = \{(\Omega, f) \in \mathcal{P} : f(K \cap \Omega) \subset I\}$.

Denote by τ_c the topology on $\mathcal{P}$ which has as a subbase the family of sets $[U], [K : I]$. If $m = 1$ τ_c is just the Back's topology.

Corollary 3.4. *If (X, d) is a metric space, then τ-convergence and τ_c-convergence for sequences coincide.*

Proof. Follows immediately from Proposition 3.3 and Lemma 1 in [2] which works also for $\mathcal{P}$. □

Lemma 3.5. *Let (X, d) be a metric space. If $\{\Gamma(f_n, \Omega_n)\}_n$ τ-converges to $\Gamma(f, \Omega)$, then for every $K \in \mathcal{K}(X)$ the sequence $\{f_n(\Omega_n \cap K)\}_n$ is equibounded (i.e. there is a bounded set $B \subset \mathbb{R}^m$ such that $f_n(\Omega_n \cap K) \subset B$ for every n).*

Proof. Let $K \in \mathcal{K}(X)$. Let $\tilde{f}$ be any continuous extension of $f : \Omega \to \mathbb{R}^m$ to all X. There is $\delta > 0$ such that $| \tilde{f}(x) - \tilde{f}(y) | < 1$ for every $x, y \in X$ with $d(x, y) < \delta$ and $x \in K$ (a consequence of the uniform continuity of $\tilde{f}_{|K}$). So the set $L = \tilde{f}(S(K, \delta))$ is bounded in $\mathbb{R}^m$. We show that $f_n(\Omega_n \cap K) \subset S(L, \delta)$ eventually. There is $N_0 \in Z^+$ such that for every $n \geq N_0$, $\rho_K(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega)) < \frac{\delta}{2}$. Let $n \geq N_0$ and $x \in \Omega_n \cap K$. By the definition of $\rho_K(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega))$ there is $z \in \Omega$ such that

$$\max\{d(x, z), | f_n(x) - f(z) | < \rho_K(\Gamma(f_n, \Omega_n), \Gamma(f, \Omega)) + \frac{\delta}{2} < \delta.$$

Thus $z \in S(x, \delta) \subset S(K, \delta)$ and $f_n(x) \in S(f(z), \delta) \subset S(L, \delta)$, done. Put $B = S(L, \delta) \cup \bigcup_{n=1}^{N_0} f_n(\Omega_n \cap K)$. Then $f_n(\Omega_n \cap K) \subset B$ for every n, and of course B is bounded in $\mathbb{R}^m$. □

The following proposition gives another characterization of τ-convergence.

Proposition 3.6. *Let (X, d) be a metric space. Then $\{\Gamma(f_n, \Omega_n)\}_n \xrightarrow{\tau} \Gamma(f, \Omega)$ if and only if*

(1) $\operatorname{Lim} \Gamma(f_n, \Omega_n) = \Gamma(f, \Omega)$;
(2) *for every compact K the sequence $\{f_n(\Omega_n \cap K)\}_n$ is equibounded.*

Proof. Let $\{\Gamma(f_n, \Omega_n)\}_n$ τ-converges to $\Gamma(f, \Omega)$. Then by Lemmas 3.2 and 3.5 (1) and (2) are satisfied.

Suppose now that (1) and (2) are satisfied. We show that also (i) and (ii) from Proposition 3.3 hold. $\Omega \subset \operatorname{Li} \Omega_n$ is obvious from (1). Suppose now that $x \in \operatorname{Ls} \Omega_n$, i.e. there is a sequence $\{x_{n_k}\}_k, x_{n_k} \in \Omega_{n_k}$ for every k ($\{n_k\}$ is an increasing sequence in Z^+) such that

$x_{n_k} \to x$. Put $K = \{x_{n_k} : k \in Z^+\} \cup \{x\}$. K is compact, so by (2) $\{f_n(\Omega_n \cap K)\}_n$ is equibounded, so there is a cluster point $y \in \mathbb{R}^m$ of the sequence $\{f_{n_k}(x_{n_k})\}_k$. Then $(x,y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n)$ i.e. $(x,y) \in \Gamma(f,\Omega)$. Thus $x \in \Omega$.

To verify (ii) let $x \in \Omega$ and $\{x_n\}_n$ be a sequence such that $x_n \in \Omega_n$ for every n and $x_n \to x$. To prove $\{f_n(x_n)\}_n$ converges to $f(x)$ it is enough to show that every subsequence of $\{f_n(x_n)\}_n$ has a subsequence convergent to $f(x)$. So let I be an infinite subset of Z^+ and consider $\{f_n(x_n) : n \in I\}$. Since $x_n \to x$ by (2) $\{f_n(x_n) : n \in I\}$ has a cluster point $y \in \mathbb{R}^m$. By (1) $(x,y) \in \Gamma(f,\Omega)$ since $(x,y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n)$ i.e. $y = f(x)$. So $\{f_n(x_n) : n \in I\}$ has a subsequence convergent to $f(x)$. □

The following two propositions give some sufficient conditions under which τ and Kuratowski convergence of sequences in G coincide.

Proposition 3.7. *Let (X,d) be a closed convex set in a linear metrizable space and $\Omega, \Omega_n (n \in Z^+)$ be closed convex sets in X. TFAE:*

(1) $\operatorname{Lim}\Gamma(f_n,\Omega_n) = \Gamma(f,\Omega)$ *and* $\operatorname{Lim}\Omega_n = \Omega$;
(2) $\{\Gamma(f_n,\Omega_n)\}_n$ *τ-converges to* $\Gamma(f,\Omega)$.

Proof. (2) $\Rightarrow$ (1) by Lemma 3.2.

To prove (1) $\Rightarrow$ (2), by Proposition 3.6 it is sufficient to verify that for every compact set K the sequence $\{f_n(\Omega_n \cap K)\}_n$ is equibounded. Suppose no. So there is a compact set K for which $\{f_n(\Omega_n \cap K)\}_n$ is not equibounded. There is a sequence $\{x_n\}_n, x_n \in \Omega_n \cap K$ such that $\{f_n(x_n)\}_n$ is unbounded. Let $x \in K$ be a cluster point of $\{x_n\}_n$. So $x \in \operatorname{Ls}\Omega_n \subset \Omega$. There is a sequence $\{y_n\}_n, y_n \in \Omega_n$ such that $y_n \to x$ and $f_n(y_n) \to f(x)$ for every n. There is an increasing sequence $\{n_k\}$ of positive integers such that $| f_{n_k}(x_{n_k}) - f(x) |> \epsilon$, $| f_{n_k}(y_{n_k}) - f(x) |< \epsilon/2$ for every k and $x_{n_k} \to x, y_{n_k} \to x$. The convexity of Ω_{n_k} guarantees the existence of a point $z_{n_k} \in [x_{n_k}, y_{n_k}] \subset \Omega_{n_k}$ such that $f_{n_k}(z_{n_k}) \in Fr(B(f(x),\epsilon))$ for every k. The compactness of $Fr(B(f(x),\epsilon))$ implies the existence of a point y which is a cluster point of $\{f_{n_k}(z_{n_k})\}_k$. So $(x,y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n) \subset \Gamma(f,\Omega)$, a contradiction. □

Proposition 3.8. *Let (X,d) be a closed convex set in a normed reflexive space (where d is a metric induced from the norm) and $\Omega, \Omega_n (n \in Z^+)$ be closed convex sets in X. TFAE:*

(1) $\operatorname{Lim}\Gamma(f_n,\Omega_n) = \Gamma(f,\Omega)$;
(2) $\{\Gamma(f_n,\Omega_n)\}_n$ *τ-converges to* $\Gamma(f,\Omega)$.

Proof. By Proposition 3.7 it is sufficient to verify that $\operatorname{Lim}\Omega_n = \Omega$.

$\operatorname{Lim}\Gamma(f_n,\Omega_n) = \Gamma(f,\Omega)$ implies $\Omega \subset \operatorname{Li}\Omega_n$ so only $\operatorname{Ls}\Omega_n \subset \Omega$ remains to prove.

So let $x \in \operatorname{Ls}\Omega_n$ and suppose $x \notin \Omega$. There is a sequence $\{x_n : n \in J\}$ where J is an infinite subset of Z^+ such that $x_n \in \Omega_n$ for every $n \in J$ and $x_n \to x$. Of course $\{f_n(x_n) : n \in J\}$ converges to no point from $\mathbb{R}^m$.

Let $\eta = d(x,\Omega)$. Since Ω is a closed convex set in a reflexive space, there is $l \in \Omega$ such that $d(x,l) = d(x,\Omega)$. There is a sequence $\{y_n\}_n, y_n \in \Omega_n$ for every $n \in Z^+$ such that $y_n \to l$ and $f_n(y_n) \to f(l)$. There is $\delta > 0$ and an infinite subset J_1 of J such that $f_n(y_n) \in S(f(l),\frac{\delta}{2})$ and $f_n(x_n) \notin S(f(l),\delta)$, for every $n \in J_1$. For every $n \in J_1$ there is a point $z_n \in [x_n, y_n] \subset \Omega_n$ such that $f_n(z_n) \in Fr(B(f(l),\frac{\delta}{2}))$. Let y be a cluster point of $\{f_n(z_n) : n \in J_1\}$. For every $n \in J_1$ let $q_n \in (0,1)$ be such that $z_n =$

$q_nx_n + (1-q_n)y_n$. Without loss of generality we can suppose that $\{f_n(z_n) : n \in J_1\} \to y$ and $\{q_n : n \in J_1\} \to q \in [0,1]$ (otherwise we pass to subsequences). If $q = 1$ then $\{z_n : n \in J_1\} \to x$, $\{f_n(z_n) : n \in J_1\} \to y$, so $(x,y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n) \subset \Gamma(f,\Omega)$ i.e. $x \in \Omega$, a contradiction. Suppose now that $q = 0$. Then $\{z_n : n \in J_1\} \to l, \{f_n(z_n) : n \in J_1\} \to y$, i.e. $(l,y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n) \subset \Gamma(f,\Omega)$. Thus $y = f(l)$, a contradiction. Now let $q \in (0,1)$. Then $\{z_n : n \in J_1\} \to (qx+(1-q)l)$ and $\{f_n(z_n) : n \in J_1\} \to y$, i.e. $(qx+(1-q)l, y) \in \operatorname{Ls}\Gamma(f_n,\Omega_n) = \Gamma(f,\Omega)$. So $(qx+(1-q)l) \in \Omega$, but $\|x-(qx+(1-q)l)\| = = (1-q)\|x-l\| < \|x-l\| = d(x,\Omega)$, a contradiction. □

Corollary 3.9. *Let (X,d) be a closed convex set in $\mathbb{R}^k (k \in Z^+)$ and $\Omega, \Omega_n (n \in Z^+)$ be closed convex sets in X. TFAE:*

(1) $\operatorname{Lim}\Gamma(f_n,\Omega_n) = \Gamma(f,\Omega)$;
(2) $\{\Gamma(f_n,\Omega_n)\}_n$ *τ-converges to* $\Gamma(f,\Omega)$.

4. Topological properties of τ-convergence

In this part we will study topological properties of τ-convergence in G. At first we prove that this convergence is always topological by finding a topology which topologizes τ-convergence.

First some notations. In Section 3 we mentioned Back's topology defined by a subbase from sets of the form $[U]$ and $[K : I]$. Here we use the same notations also for sets in G (of course we can identify partial maps with their graphs). So for every open sets U, V in X, $K \in \mathcal{K}(X)$, open $J \subset \mathbb{R}^m$ and open (possibly empty) $I \subset \mathbb{R}^m$, let

$$[U] = \{\Gamma(f,\Omega) \in G : \Omega \cap U \neq \emptyset\},$$
$$[K : I] = \{\Gamma(f,\Omega) \in G : f(K \cap \Omega) \subset I\},$$
$$[V \times J] = \{\Gamma(f,\Omega) \in G : \Gamma(f,\Omega) \cap V \times J \neq \emptyset\}.$$

Theorem 4.1. *Let (X,d) be a metric space. Then the topology which has as a subbase all sets of the form $[U], [K : I], [V \times J]$ where U, V are open sets in X, $K \in \mathcal{K}(X)$ and I, J are open sets in $\mathbb{R}^m$ (I possibly empty) topologizes τ-convergence.*

Proof. Let $\{\Gamma(f_\sigma,\Omega_\sigma) : \sigma \in \Sigma\}$ be a net in G which τ-converges to $\Gamma(f,\Omega)$. It is an easy exercise to verify that this net converges also in the above defined topology.

Now suppose $\{\Gamma(f_\sigma,\Omega_\sigma) : \sigma \in \Sigma\}$ converges to $\Gamma(f,\Omega)$ in the above mentioned topology, but fails τ-converge. So there is $K \in \mathcal{K}(X)$, $\epsilon > 0$ and a cofinal subset Δ in Σ such that $\rho_K(\Gamma(f_\sigma,\Omega_\sigma),\Gamma(f,\Omega)) > \epsilon$ for every $\sigma \in \Delta$. There are two possibilities:

(a) there is a cofinal subset Δ_1 of Δ such that $e(\Gamma(f_\sigma,\Omega_\sigma \cap K),\Gamma(f,\Omega)) > \epsilon$ for every $\sigma \in \Delta_1$;
(b) there is a cofinal subset Δ_2 of Δ such that $e(\Gamma(f,\Omega \cap K),\Gamma(f_\sigma,\Omega_\sigma)) > \epsilon$ for every $\sigma \in \Delta_2$.

In (a) let $x_\sigma \in \Omega_\sigma \cap K$ for every $\sigma \in \Delta_1$ such that $(x_\sigma, f_\sigma(x_\sigma)) \notin S(\Gamma(f,\Omega),\epsilon)$. Let x be a cluster point of $\{x_\sigma : \sigma \in \Delta_1\}$. It is easy to verify that $x \in \Omega$. Let $0 < \delta < \epsilon$ be such that $f(S(x,\delta) \cap \Omega) \subset S(f(x),\epsilon)$. Put $C = K \cap B(x,\frac{\delta}{2})$, then $\Gamma(f,\Omega) \in [C : S(f(x),\epsilon)]$, so $\Gamma(f_\sigma,\Omega_\sigma) \in [C : S(f(x),\epsilon)]$ eventually, a contradiction.

In (b) let $x_\sigma \in \Omega \cap K$ for every $\sigma \in \Delta_2$ such that $(x_\sigma, f_\sigma(x_\sigma)) \notin S(\Gamma(f,\Omega),\epsilon)$. Let x be a cluster point of $\{x_\sigma : \sigma \in \Delta_2\}$. Then $x \in \Omega \cap K$. Let $0 < \delta < \epsilon$ be such that $f(S(x,\delta) \cap \Omega) \subset S(f(x), \frac{\epsilon}{2})$.

$S(x, \frac{\delta}{2}) \times S(f(x), \frac{\epsilon}{2}) \cap \Gamma(f_\sigma, \Omega_\sigma) \neq \emptyset$ eventually, a contradiction. □

In what follows denote by τ also the topology described in the previous theorem. From the presentation of τ-convergence by using of the above mentioned topology τ we see that this convergence doesn't depend on the choice of a metric in domain X. The topology τ is even well defined for every topological space X.

From the presentation of Back's topology τ_c given in Section 3 we see that $\tau_c \subset \tau$. We can characterize the local compactness of X by a coincidence of τ and τ_c.

Theorem 4.2. *Let (X,d) be a metric space. TFAE:*

(1) *X is locally compact;*

(2) *τ-convergence of nets in G and τ_c-convergence coincide.*

Proof. (2) $\Rightarrow$ (1). Suppose X is not locally compact. There is a point $x \in X$ which has no compact neighbourhood of x. Denote by $\mathcal{U}(x)$ the family of all open neighbourhoods of x. For every $K \in \mathcal{K}(X)$ and $U \in \mathcal{U}(x)$ there is a point $x_{K,U} \in U \setminus K$. Consider the following directions on $\mathcal{K}(X)$ and $\mathcal{U}(x)$: if $B, C \in \mathcal{K}(X)$ then $B \geq C \Leftrightarrow B \supset C$ and if $U, V \in \mathcal{U}(x)$ then $U \geq V \Leftrightarrow U \subset V$. Let $\mathcal{K}(X) \times \mathcal{U}(x)$ be equipped with the natural direction induced by the above ones. For every $(K,U) \in \mathcal{K}(X) \times \mathcal{U}(x)$ put $\Omega_{K,U} = \{x_{K,U}\}$ and define $u_{K,U}$ on $\Omega_{K,U}$ by $u_{K,U}(x_{K,U}) = 1$. Let further $\Omega = \{x\}$ and u be the function defined on Ω by $u(x) = 0$. The net $\{\Gamma(u_{K,U}, \Omega_{K,U}) : (K,U) \in \mathcal{K}(X) \times \mathcal{U}(x)\}$ τ_c-converges to $\Gamma(U, \Omega)$. Let V be an open set in X such that $\Gamma(u,\Omega) \in [V]$. Then $\Omega \cap V \neq \emptyset$, i.e. $x \in V$, so $V \in \mathcal{U}(x)$. Choose a set $K \in \mathcal{K}(X)$. For every $(B,U) \in \mathcal{K}(X) \times \mathcal{U}(x)$ such that $(B,U) \geq (K,V)$ we have $\Gamma(u_{B,U}, \Omega_{B,U}) \in [V]$. Now take $C \in \mathcal{K}(X)$ and open $I \subset \mathbb{R}^m$ such that $\Gamma(u,\Omega) \in [C : I]$. Let $U \in \mathcal{U}(x)$ be arbitrary. Then for every $(B,V) \geq (C,U)$ we have $\Omega_{B,V} \cap C = \emptyset$, so $\Gamma(u_{B,V}, \Omega_{B,V}) \in [C : I]$ for every $(B,V) \geq (C,U)$.

It is easy to verify that the net $\{\Gamma(u_{B,U}, \Omega_{B,U}) : (B,U) \in \mathcal{K}(X) \times \mathcal{U}(x)\}$ does not τ-converge to $\Gamma(u,\Omega)$, since $\rho_{\{x\}}(\Gamma(u_{B,U}, \Omega_{B,U}), \Gamma(u,\Omega)) = 1$ eventually.

(1) $\Rightarrow$ (2). Let $\{\Gamma(f_\sigma, \Omega_\sigma) : \sigma \in \Sigma\}$ be a net in G τ_c-convergent to $\Gamma(f,\Omega)$. By Theorem 4.1 and the description of Back's topology it is sufficient to verify that if $\Gamma(f,\Omega) \in [V \times I]$, where V, I are open sets in $X, \mathbb{R}^m$ respectively, then also $\Gamma(f_\sigma, \Omega_\sigma) \in [V \times I]$ eventually. □

In the following part of our paper we are interested in uniformizability and metrizability of the topology τ.

Proposition 4.3. *If (X,d) is a locally compact metric space, then τ is uniformizable.*

Proof. For every $K \in \mathcal{K}(X)$ and $\epsilon > 0$ put

$$U_{K,\epsilon} = \{(\Gamma(f,\Omega), \Gamma(g,\Delta)) \in G \times G : \rho_K(\Gamma(f,\Omega), \Gamma(g,\Delta)) < \epsilon\}.$$

We show that the family $\{U_{K,\epsilon} : K \in \mathcal{K}(X) \text{ and } \epsilon > 0\}$ is a base for a uniformity U on G. Let $K \in \mathcal{K}(X)$ and $\epsilon > 0$. Then $U_{K,\epsilon} = U^{-1}_{K,\epsilon}$ and $U_{K,\epsilon}$ contains the diagonal in $G \times G$. Now, we show that there is $C \in \mathcal{K}(X)$ and $\delta > 0$ such that $U_{C,\delta} \circ U_{C,\delta} \subset U_{K,\epsilon}$. The local compactness of X implies that there is $\delta > 0$, $\delta < \frac{\epsilon}{2}$ such that $B(K,\delta)$ is compact. Put $C = B(K,\delta)$. Let $(\Gamma(f,\Omega),\Gamma(g,\Delta)) \in U_{C,\delta} \circ U_{C,\delta}$. There is $\Gamma(h,\Sigma) \in G$ such that $(\Gamma(f,\Omega),\Gamma(h,\Sigma)) \in U_{C,\delta}$ and $(\Gamma(h,\Sigma),\Gamma(g,\Delta)) \in U_{C,\delta}$. We have

$$e(\Gamma(f,\Omega\cap K),\Gamma(g,\Delta)) \leq e(\Gamma(f,\Omega\cap K),\Gamma(h,\Sigma\cap B(K,\delta)))+$$

$$+e(\Gamma(h,\Sigma\cap B(K,\delta)),\Gamma(g,\Delta)) < \delta+\delta < \epsilon$$

where the first estimation is implied by the inclusion $U_{C,\delta} \subset U_{K,\delta}$ and the property $(\star)$.

Finally, let $K_1, K_2 \in \mathcal{K}(X)$ and $\epsilon_1, \epsilon_2 > 0$. Put $K = K_1 \cup K_2$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $U_{K,\epsilon} \subset U_{K_1,\epsilon_1} \cap U_{K_2,\epsilon_2}$. So the family $\{U_{K,\epsilon} : K \in \mathcal{K}(X)$ and $\epsilon > 0\}$ forms a base for a uniformity U on G. It is easy to verify that the convergence in the topology of the uniformity U is just τ-convergence in G. □

Proposition 4.4. *If (X,d) is a locally compact and separable metric space, then τ is metrizable.*

Proof. For a metric space (X,d) the locally compactness and separability is equivalent to hemicompactness, that is the existence of an increasing sequence $\{K_n\}_n$ of compact sets such that for every compact $K \in \mathcal{K}(X)$ there is $n \in Z^+$ with $K \subset K_n$. X is locally compact [15], so by Proposition 4.3 the family $\{U_{K,\epsilon} : K \in \mathcal{K}(X), \epsilon > 0\}$ is a base for a uniformity U on G. It is easy to verify that the family $\{U_{K_n,\frac{1}{n}} : n \in Z^+\}$ is a countable base of U. □

Proposition 4.5. *If (X,d) is a locally compact and separable metric space, then (G,τ) is a Polish space.*

Proof. By Theorem 4.2 τ and τ_c coincide on G. By Theorem 3.4 in [18], as X is hemicompact, then (G,τ) is a Polish space. □

Of course to find a complete metric which describes τ-convergence in G for a locally compact and separable metric space (X,d) would be useful. We solve this problem at least for a closed interval X in $\mathbb{R}$.

Let X be a closed interval in $\mathbb{R}$ equipped with the natural metric. In [10] for every $f \in C(\Omega,\mathbb{R}^m)$ the extension $\tilde{f} \in C(X,\mathbb{R}^m)$ of f was defined such that the mapping $\Psi : (G,\tau) \to (C(X,\mathbb{R}^m),\tau_{CO}), \Psi(\Gamma(f,\Omega)) = \tilde{f}$ is continuous (τ_{CO} is the compact-open topology).

Denote by τ_F the Fell topology on $CL(X)$ [3, 16] and define the mapping $\Phi : (G,\tau) \to (CL(X),\tau_F) \times (C(X,\mathbb{R}^m),\tau_{CO})$ by $\Phi(\Gamma(f,\Omega)) = (\Omega,\tilde{f})$. By Theorem 4.2 in [18] Φ is a homeomorphism and $\Phi(G)$ is a closed subspace of $(CL(X),\tau_F) \times (C(X,\mathbb{R}^m),\tau_{CO})$. On $CL(X)$ the Fell topology coincides with the Attouch-Wets topology [1, 3], for which the well-known expression of a complete metric is

$$d_1(A,B) = \sum_{i=1}^{\infty} 2^{-i}\min\{1, \sup_{d(x,x_0)<i} \mid d(x,A) - d(x,B) \mid\}.$$

Also the compact-open topology τ_{CO} on $C(X,\mathbb{R}^m)$ can be described by the following complete metric

$$d_2(f,g) = \sum_{i=1}^{\infty} 2^{-i} \min\{1, \sup_{d(x,x_0)<i} \mid f(x) - g(x) \mid\}.$$

Thus if we define

$$L(\Gamma(f,\Omega),\Gamma(g,\Delta)) = d_1(\Omega,\Delta) + d_2(\tilde{f},\tilde{g})$$

we obtain a compatible complete metric for (G,τ).

5. Comparison with hyperspace topologies

In the last part of our paper we would like to present some results concerning relations between τ-topology and some hyperspace topologies induced on G from $CL(X\times\mathbb{R}^m)$. Hyperspace topologies on graphs of continuous functions with common domain were studied in many papers. We quote at least some [4, 5, 6, 7, 17, 20, 21].

Denote further by $\tau_F, \tau_{AW}, \tau_H$ the Fell [16], Attouch-Wets [1] and Hausdorff metric topology [3, 23, 24], respectively. We are going to show that $\tau_F \subset \tau$ and if X is boundedly compact, then also $\tau_{AW} \subset \tau$. Of course $\tau \subset \tau_H$ in G.

From Lemma 3.2 we know that τ-convergence of sequences implies τ_F-convergence (since it coincides with Kuratowski convergence). Now we show that also τ-convergence of nets implies τ_F-convergence.

Proposition 5.1. *Let (X,d) be a metric space. Then $\tau_F \subset \tau$ in G.*

Proof. Let $\{\Gamma(f_\sigma,\Omega_\sigma) : \sigma\in\Sigma\}$ τ-converges to $\Gamma(f,\Omega)$. It is sufficient to show that if K is a compact set in $X\times\mathbb{R}^m$ such that $\Gamma(f,\Omega)\cap K=\emptyset$, then also $\Gamma(f_\sigma,\Omega_\sigma)\cap K=\emptyset$ eventually.

Suppose there is a cofinal subset Δ of Σ such that $\Gamma(f_\sigma,\Omega_\sigma)\cap K\neq\emptyset$ for every $\sigma\in\Delta$. Since $\Pi_1(K)$ is compact in X (where $\Pi_1(K)$ is the projection of K to X) we have $\rho_{\Pi_1(K)}(\{\Gamma(f_\sigma,\Omega_\sigma),\Gamma(f,\Omega)) \to 0$. Therefore for every $\epsilon>0$ $S(K,\epsilon)\cap\Gamma(f,\Omega)\neq\emptyset$ (where by $S(K,\epsilon)$ we mean an enlargement induced by the box metric of d and u). So also $K\cap\Gamma(f,\Omega)\neq\emptyset$, a contradiction. □

The following proposition shows that Lemma 3.2 does not work in general for nets. Of course, if X is locally compact, then from previous propositon we see that τ-convergence of nets implies Kuratowski convergence (τ_F topologizes Kuratowski convergence in locally compact spaces).

Proposition 5.2. *Let (X,d) be a metric space. TFAE:*

(1) *X is locally compact;*

(2) *τ-convergence of nets implies Kuratowski convergence in G.*

Proof. Only (2) ⇒ (1) needs a proof. Suppose X is not locally compact. There is a point $x\in X$ which has no compact neighbourhood. We use the same notation as in the proof of Theorem 4.2. For every $(K,U) \in \mathcal{K}(X) \times \mathcal{U}(x)$ put $\Omega_{K,U} = \{x_{K,U}, y\}$, where $x_{K,U} \in U \setminus K$ and y is a point in X different from x, and define $u_{K,U}$ on $\Omega_{K,U}$ as $u_{K,U}(y) = u_{K,U}(x_{K,U}) = 0$. Let further $\Omega = \{y\}$ and u is defined on Ω as $u(y) = 0$. Then $\{\Gamma(u_{K,U}, \Omega_{K,U}) : (K,U) \in \mathcal{K}(X) \times \mathcal{U}(x)\}$ τ-converges to $\Gamma(u,\Omega)$, but not Kuratowski since $(x,0) \in \mathrm{Li}\,\Gamma(u_{K,U}, \Omega_{K,U}) \setminus \Gamma(u,\Omega)$. □

Proposition 5.3. *Let (X,d) be a metric space. TFAE:*

(1) *X is boundedly compact (i.e. every closed bounded set is compact);*
(2) *$\tau_{AW} \subset \tau$ on G.*

Proof. (1) ⇒ (2) The proof in this direction uses the same idea as in [10].

(2) ⇒ (1) Suppose X is not boundedly compact. So there is a closed bounded set B which is not compact. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence in B which has no cluster point. Put $\Omega = \{x_1\}$ and $\Omega_n = \{x_1, x_n\}$ for every $n \in Z^+$. Let $u, u_n (n \in Z^+)$ be restrictions of the zero function on Ω, Ω_n, respectively. Then it is easy to verify that $\{\Gamma(u_n, \Omega_n)\}_n$ τ-converges to $\Gamma(u,\Omega)$, but fails τ_{AW}-converge to $\Gamma(u,\Omega)$. □

We finish with the following proposition.

Proposition 5.4. *Let (X,d) be a metric space. TFAE:*

(1) *X is compact;*
(2) *$\tau_H = \tau$ on G.*

Proof. For (2) ⇒ (1) we can use a similar proof as in the previous proposition. □

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