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Some Geometric Properties in Orlicz Sequence Spaces equipped with Orlicz Norm

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It is proved that for any reflexive Banach space X, both X and X^* are **CLUR** if and only if both X and X^* have property **H**. Criteria for rotundity, local uniform rotundity, compact local uniform rotundity and property **H** in Orlicz sequence spaces equipped with the Orlicz norm are given. Criteria for property **H**, rotundity and **LUR** were already known in the literature only for finitely valued Orlicz functions which vanish only at zero and are N-functions (i.e. they satisfy conditions (0_1) and (∞_1)). All our criteria except Corollary 2.15 are given for arbitrary Orlicz functions. Criteria for smoothness of l^0_Φ in Corollary 2.15 are given for any finitely valued Orlicz function satisfying condition (∞_1), extending the respective result of [2] proved only for Orlicz functions vanishing only at zero.

Keywords: Orlicz sequence space, rotundity, local uniform rotundity, compact local uniform rotundity, property **H**, smoothness, copy of l_∞

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1. Introduction

In the whole paper $\mathcal{N}$ and $\mathcal{R}$ stand for the sets of natural numbers and of real numbers, respectively. Let $(X, \|\cdot\|)$ be a real Banach space and $B(X)$ $(S(X))$ be the closed unit ball (the unit sphere) of X. By X^* denote the dual space of X. Clarkson [4] introduced the concept of uniform rotundity.

A Banach space X is said to be *uniformly rotund* (**UR** for short) if for every sequences (x_n)

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and (y_n) in $S(X)$ such that $\lim_{n\to\infty} \|x_n + y_n\| = 2$, there holds $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

A Banach space X is said to be *rotund* (**R** for short) if for any x and y in $S(X)$ with $\|x + y\| = 2$, we have $x = y$.

A Banach space X is called *locally uniformly rotund* (**LUR** for short) if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim_{n\to\infty} \|x_n + x\| = 2$, there holds $\lim_{n\to\infty} \|x_n - x\| = 0$.

A Banach space X is said to be *compactly locally uniformly rotund* (**CLUR** for short) if for each $x \in S(X)$ and each sequence (x_n) in $S(X)$ such that $\lim_{n\to\infty} \|x_n + x\| = 2$, it follows that the set $\{x_n : n \in \mathcal{N}\}$ is relatively compact in norm topology.

If in a Banach space X a partial order "$\leq$" is defined and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, then X is said to be a *Banach lattice*. If X is a Banach lattice and $\|x\| < \|y\|$ whenever $0 \leq x \leq y$ and $x \neq y$, then X is said to be *strictly monotone* (see [8]).

It is clear that a Banach space X is **LUR** if and only if it is **CLUR** and **R** (see [15]).

A Banach space X is said to have *property* **H** if on the unit sphere every weakly convergent sequence to a point on the sphere is convergent in norm.

For the definitions of these and many other geometric notions, some consequences and some relationships between them, see [5].

A map $\Phi : \mathcal{R} \to [0, \infty]$ is said to be an *Orlicz function* if it is even, convex, continuous and vanishing at 0 and $\Phi(u) \to \infty$ as $u \to \infty$.

We say that an Orlicz function Φ is an *N'-function* if it satisfies the following condition:

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty. \tag{∞_1}$$

An N'-function Φ is said to be an *N-function* if

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0. \tag{0_1}$$

We define the *Orlicz sequence space* by the formula

$$l_\Phi = \left\{ x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \ \text{ for some } c > 0 \right\},$$

where l^0 stands for the space of all real sequences. We will consider l_Φ equipped with the *Luxemburg norm*

$$\|x\| = \inf \left\{ \epsilon > 0 : I_\Phi\left(\frac{x}{\epsilon}\right) \leq 1 \right\}$$

or with the equivalent one

$$\|x\|_0 = \inf_{k>0} \frac{1}{k} \left(1 + I_\Phi(kx)\right),$$

called the *Orlicz norm* or the *Amemiya norm*.

By h_Φ we denote the subspace

$$h_\Phi = \left\{ x \in l_\Phi : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi(cx(i)) < \infty \text{ for any } c > 0 \right\}$$

To simplify notations, we put $l_\Phi = (l_\Phi, \|\cdot\|)$ and $l_\Phi^0 = (l_\Phi, \|\cdot\|_0)$.

Orlicz spaces l_Φ and l_Φ^0 are Banach lattices under the partial order $x \leq y$ iff $x(i) \leq y(i)$ for all $i \in \mathcal{N}$.

The set of all k's at which the infimum in the definition of $\|x\|_0$ for a fixed $x \in l_\Phi^0$ is attained, will be denoted by $K(x)$. In particular, the set $K(x)$ can be empty if the Orlicz sequence space l_Φ^0 is generated by an Orlicz function which does not satisfy condition (∞_1).

For every Orlicz function Φ the function $\Psi : \mathcal{R} \longrightarrow [0, \infty]$ *complementary to* Φ *in the sense of Young* is defined by the formula

$$\Psi(v) = \sup_{u>0} \{u|v| - \Phi(u)\}$$

for every $v \in \mathcal{R}$. It is well known that Ψ is also an Orlicz function.

We say an Orlicz function Φ *satisfies the* δ_2*-condition* ($\Phi \in \delta_2$ for short) if there exist constants $k \geq 2$ and $u_0 > 0$ such that $0 < \Phi(u_0) < \infty$ and

$$\Phi(2u) \leq k\Phi(u)$$

whenever $|u| \leq u_0$.

Let

$$\theta(x) = \sup\{c > 0 : I_\Phi(cx) < \infty\}$$

for any $x \in l_\Phi$. Obviously, $h_\Phi = \{x \in l_\Phi : \theta(x) = \infty\}$.

We say an Orlicz function Φ is *strictly convex on* $[a, b]$ if

$$\Phi\left(\frac{u+v}{2}\right) < \frac{1}{2}(\Phi(u) + \Phi(v))$$

whenever $u, v \in [a, b]$ and $u \neq v$. For more details on Orlicz functions and Orlicz spaces we refer to [1], [12], [13], [14] and [17].

Recall that in the previous papers on rotundity and local uniform rotundity in Orlicz sequence spaces l_Φ^0 equipped with the Orlicz norm there was assumed that the function Φ generating the space l_Φ^0 is an N-function (see [1]). We will omit this assumptions here. To do so it was necessary to use different techniques. We will also give criteria for properties **CLUR** and **H** in l_Φ^0.

2. Results

We begin with a general result.

Theorem 2.1. *If X is a reflexive Banach space, then both X and X^* are* **CLUR** *if and only if both X and X^* have property* **H**.

Proof. It is known that if X is **CLUR**, then X has property **H**. We only need to prove that X is **CLUR** if both X and X^* have property **H**. For every $x_0 \in S(X)$ and every sequence (x_n) in $S(X)$ with $\lim_{n\to\infty} \|x_n + x_0\| = 2$, take $(f_n) \subset S(X^*)$ such that $f_n(x_n + x_0) = \|x_n + x_0\|$ for every $n \in \mathcal{N}$. Then

$$f_n(x_0) = \|x_n + x_0\| - f_n(x_n)$$

for every $n \in \mathcal{N}$ and

$$\liminf_{n\to\infty} f_n(x_0) \geq \lim_{n\to\infty} \|x_n + x_0\| - \limsup_{n\to\infty} f_n(x_n),$$

whence $\lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty} f_n(x_n) = 1$.

By the reflexivity of X, there is a subsequence (f_{n_i}) of (f_n) and $f_0 \in X^*$ such that f_{n_i} tends to f_0 weakly. It is obvious that in virtue of $\lim_{n\to\infty} f_n(x_0) = 1$ this yields $f_0(x_0) = 1$, whence $\|f_0\| = 1$. By property **H** for X^*, we get that $f_{n_i} \to f_0$ in norm. Hence

$$f_0(x_{n_i}) = (f_0 - f_{n_i})(x_{n_i}) + f_{n_i}(x_{n_i}) \to 1 \quad \text{as} \quad i \to \infty.$$

Using now the reflexivity of X, we can find a subsequence (z_i) of (x_{n_i}) and $x^0 \in X$ such that z_i tends to x^0 weakly. Obviously, $f_0(x^0) = 1$, whence $\|x^0\| = 1$. By property **H** for X, z_i tends to x^0 strongly, i.e. the set $\{x_n : n \in \mathcal{N}\}$ is relatively compact in $S(X)$, which implies that X is **CLUR**. □

In the following we will consider some geometric properties of Orlicz sequence spaces.

Theorem 2.2. *Let $x \in l^0_\Phi$ $(x \neq 0)$. If $K(x) = \emptyset$, then*

$$\|x\|_0 = \lim_{k\to\theta(x)-} \frac{1}{k}\left(1 + I_\Phi(kx)\right).$$

Proof. Since the function $f(k) = \frac{1}{k}\left(1 + I_\Phi(kx)\right)$ is continuous on the interval $(0, \theta(x))$ and $\lim_{k\to 0+} f(k) = \infty$, the formula

$$\|x\|_0 = \lim_{k\to\theta(x)-} \frac{1}{k}(1 + I_\Phi(kx))$$

is true. □

Corollary 2.3. *If $x \in l^0_\Phi$ and $\theta(x) < \infty$, then $K(x) \neq \emptyset$.*

Proof. Assume for the contrary that $K(x) = \emptyset$. Then, by Theorem 2.2 and the Fatou Lemma, we have

$$\frac{1}{\theta(x)}\left(1 + I_\Phi(\theta(x)x)\right) \leq \lim_{k\to\theta(x)-} \frac{1}{k}\left(1 + I_\Phi(kx)\right) = \|x\|_0 < \infty,$$

whence $K(x) \neq \emptyset$. A contradiction. □

Corollary 2.4. *If $x \in l_\Phi^0$ and $K(x) = \emptyset$, then for any $n \in \mathcal{N}$ we have*

$$\left\| \sum_{i=1}^{n} x(i)e_i \right\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi \left(k \sum_{i=1}^{n} x(i)e_i \right)$$

and

$$\left\| \sum_{i=n+1}^{\infty} x(i)e_i \right\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi \left(k \sum_{i=n+1}^{\infty} x(i)e_i \right).$$

Proof. We first claim that the limit $\lim_{u\to\infty} \frac{\Phi(u)}{u}$ exists. This follows from the fact that for $0 < u_1 < u_2$, we have

$$\frac{\Phi(u_1)}{u_1} = \frac{1}{u_1}\Phi\left(\frac{u_1}{u_2}u_2\right) < \frac{u_1}{u_2} \cdot \frac{\Phi(u_2)}{u_1} = \frac{\Phi(u_2)}{u_2}.$$

Since $K(x) = \emptyset$, $\lim_{u\to\infty} \frac{\Phi(u)}{u}$ is finite, and consequently

$$\begin{aligned} \|x\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi(kx) &= \lim_{k\to\infty} \frac{1}{k} \left(I_\Phi \left(k \sum_{i=1}^{n} x(i)e_i \right) + I_\Phi \left(k \sum_{i=n+1}^{\infty} x(i)e_i \right) \right) \\ &= \lim_{k\to\infty} \frac{1}{k} I_\Phi \left(k \sum_{i=1}^{n} x(i)e_i \right) + \lim_{k\to\infty} \frac{1}{k} I_\Phi \left(k \sum_{i=n+1}^{\infty} x(i)e_i \right) \\ &\geq \left\| \sum_{i=1}^{n} x(i)e_i \right\|_0 + \left\| \sum_{i=n+1}^{\infty} x(i)e_i \right\|_0 \end{aligned}$$

for every $n \in \mathcal{N}$. On the other hand, by the triangle inequality, we have

$$\|x\|_0 \leq \left\| \sum_{i=1}^{n} x(i)e_i \right\|_0 + \left\| \sum_{i=n+1}^{\infty} x(i)e_i \right\|_0$$

for every $n \in \mathcal{N}$. So the corollary is proved. □

Corollary 2.5. *If $x \in l_\Phi^0$ and $K(x) = \emptyset$, then*

$$\|x\|_0 = A \sum_{i=1}^{\infty} |x(i)|,$$

where $A = \lim_{u\to\infty} \frac{\Phi(u)}{u}$.

Proof. Since we can assume without loss of generality that $x(i) \neq 0$ for any $i \in \mathcal{N}$, by Corollary 2.4, we have

$$\left\| \sum_{i=1}^{n} x(i)e_i \right\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi \left(k \sum_{i=1}^{n} x(i)e_i \right)$$

$$= \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{n} \Phi(kx(i)) = \lim_{k\to\infty} \sum_{i=1}^{n} |x(i)| \frac{\Phi(kx(i))}{k\,|x(i)|} = A \sum_{i=1}^{n} |x(i)|$$

for every $n \in \mathcal{N}$. Since $\|x\|_0 = \lim_{n\to\infty} \left\| \sum_{i=1}^{n} x(i)e_i \right\|_0$, we obtain

$$\|x\|_0 = A \sum_{i=1}^{\infty} |x(i)| .$$

□

Corollary 2.6. *Assume that Φ is an Orlicz function with*

$$\lim_{u\to\infty} (\Phi(u)/u) = A < \infty$$

and $a > 0$ is the number satisfying $\Phi(a) = 1$. If there exists $b \in (0,1)$ such that $Au - b \leq \Phi(u)$ for all $u \geq \frac{a}{2}$, then $K(x) = \emptyset$ for some $x \in S(l^0_\Phi)$.

Proof. Define $y = (a, 0, 0, ...)$. Then $I_\Phi(y) = 1$, whence $\|y\| = 1$ and $\|y\|_0 \leq 2\,\|y\| = 2$. So, for $x = \frac{1}{2}y$ there holds $\|x\|_0 \leq 1$. Assuming for the contrary that $K(x) \neq \emptyset$, we conclude that there is $k \geq 1$ such that

$$\|x\|_0 = \frac{1}{k}\left(1 + \Phi\left(\frac{ak}{2}\right)\right)$$

$$\geq \frac{1}{k}\left(1 + \frac{Aak}{2} - b\right) = \frac{Aa}{2} + \frac{1-b}{k} > \frac{Aa}{2}.$$

On the other hand

$$\|x\|_0 \leq \lim_{k\to\infty} \frac{1}{k}\Phi\left(\frac{ak}{2}\right)$$

$$= \frac{a}{2} \lim_{k\to\infty} \frac{2}{ak}\Phi\left(\frac{ak}{2}\right) = \frac{Aa}{2},$$

a contradiction, which shows that $K(x) = \emptyset$. □

Remark 2.7.
(i) If Φ is an N'-function, then $K(x) \neq \emptyset$ for any $x \in l^0_\Phi \backslash \{0\}$.
(ii) Let Φ be an Orlicz function satisfying condition (0_1). If $x \in l^0_\Phi \backslash \{0\}$ and $K(x) = \emptyset$, then $\operatorname{supp} x$ is a finite set.

Proof. The assertion (i) has been proved in [6]. To prove (ii) assume for the contrary that $\operatorname{supp} x$ is infinite. Denote by μ the counting measure. Then

$$\mu\left\{i \in \mathcal{N} : |x(i)| > \frac{1}{n}\right\} \to \infty \quad \text{as } n \to \infty.$$

By the assumption (0_1), we conclude that both functions Ψ and p vanish only at zero. Fix $a > 0$ and denote $b = p(a)$, where p is the right hand derivative of Φ on $\mathcal{R}_+$. Let $m \in \mathcal{N}$ satisfy $m\Psi(b) \geq 1$. Next, let $n_0 \in \mathcal{N}$ be such that

$$\mu\left\{i \in \mathcal{N} : |x(i)| > \frac{1}{n}\right\} \geq m$$

for $n \geq n_0$. Let $k = n_0 a$ and

$$A = \left\{i \in \mathcal{N} : |x(i)| > \frac{1}{n_0}\right\}.$$

Then $k\,|x(i)| > a$ and consequently $p\,(k\,|x(i)|) \geq b$ for any $i \in A$. Therefore,

$$I_\Psi(p \circ k\,|x|) \geq \mu(A)\Psi(b) \geq m\Psi(b) \geq 1.$$

This yields

$$k_x^* := \inf\{k > 0 : I_\Psi(p \circ k\,|x|) \geq 1\} < \infty,$$

whence $K(x) \neq \emptyset$ (see [1]). □

Lemma 2.8. *Let $x_0 \in S(l_\Phi^0)$ be such that $K(x_0)$ is nonempty and bounded. If $(x_n) \subset S(l_\Phi^0)$ is coordinatewise convergent to x_0, then there exists $n_0 \in \mathcal{N}$ such that $K(x_n) \neq \emptyset$ for all $n \geq n_0$ and $\sup_{n \geq n_0}\{k_n\} < \infty$ for any sequence (k_n) with $k_n \in K(x_n)$ $(n \in \mathcal{N})$. If additionally $\Phi \in \delta_2$, then there exists a subsequence (x_{n_k}) of (x_n) such that all elements of (x_{n_k}) have equi-absolutely continuous norms, i.e. for any $\epsilon > 0$ there is $i_\epsilon \in \mathcal{N}$ such that*

$$\left\|\sum_{i=i_\epsilon}^{\infty} x_{n_k}(i)e_i\right\|_0 < \epsilon$$

for all $k \in \mathcal{N}$.

Proof. Suppose that $K(x_0)$ is nonempty and bounded. First, we will prove that there is $n_1 \in \mathcal{N}$ such that $K(x_n) \neq \emptyset$ for $n \geq n_1$. Otherwise, we may assume without loss of generality that $K(x_n) = \emptyset$ $(n = 1, 2, ...)$. This implies that $\lim_{u\to\infty}(\Phi(u)/u) = A < \infty$ because otherwise $K(x) \neq \emptyset$ for any $x \in l_\Phi^0$ (see [6]). Since $K(x_0) \neq \emptyset$ and it is bounded, there is $\epsilon_0 > 0$ such that

$$\|x_0\|_0 + 2\epsilon_0 < \lim_{k\to\infty} \frac{1}{k} I_\Phi(kx_0) = A \sum_{i=1}^{\infty} |x_0(i)|.$$

Next, there exists $i_1 \in \mathcal{N}$ such that

$$\|x_0\|_0 + \epsilon_0 < A \sum_{i=1}^{i_1} |x_0(i)|.$$

Since $x_n \to x_0$ coordinatewise, there is $n_2 \in \mathcal{N}$ such that

$$\sum_{i=1}^{i_1} |x_n(i)| > \sum_{i=1}^{i_1} |x_0(i)| - \frac{\epsilon_0}{2A}$$

for $n \geq n_2$. Hence

$$1 = \|x_n\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi(kx_n)$$

$$= A\sum_{i=1}^{\infty} |x_n(i)| = A\sum_{i=1}^{i_1} |x_n(i)| + A\sum_{i=i_1+1}^{\infty} |x_n(i)|$$

$$> A\left(\sum_{i=1}^{i_1} |x_0(i)| - \frac{\epsilon_0}{2A}\right) + A\sum_{i=i_1+1}^{\infty} |x_n(i)| = A\sum_{i=1}^{i_1} |x_0(i)| - \frac{\epsilon_0}{2} + A\sum_{i=i_1+1}^{\infty} |x_n(i)|$$

$$> \|x_0\|_0 + \epsilon_0 - \frac{\epsilon_0}{2} + A\sum_{i=i_1+1}^{\infty} |x_n(i)| \geq 1 + \frac{\epsilon_0}{2}$$

for $n \geq n_2$. This contradiction shows that there is $n_1 \in \mathcal{N}$ such that $K(x_n) \neq \emptyset$ for $n \geq n_1$. Without loss of generality, we may assume that $K(x_n) \neq \emptyset$ for all $n \in \mathcal{N}$. Therefore, there are $k_n \geq 1$ such that

$$\|x_n\|_0 = \frac{1}{k_n}\left(1 + I_\Phi(k_n x_n)\right), \quad n = 0, 1, 2, \ldots .$$

We will prove that $\sup\{k_n : n = 0, 1, 2, \ldots\} < \infty$. If not, we can assume that $k_n \uparrow \infty$. Since

$$\|x_0\|_0 + 2\epsilon_0 < \lim_{k\to\infty} \frac{1}{k} I_\Phi(kx_0) = A\sum_{i=1}^{\infty} |x_0(i)|,$$

there exists $i_2 \in \mathcal{N}$ such that

$$\|x_0\|_0 + \epsilon_0 < A\sum_{i=1}^{i_2} |x_0(i)|.$$

Hence

$$1 = \|x_n\|_0 = \frac{1}{k_n}\left(1 + I_\Phi(k_n x_n)\right)$$

$$= \frac{1}{k_n}\left(1 + \sum_{i=1}^{\infty} \Phi(k_n x_n(i))\right) = \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_2} \Phi(k_n x_n(i)) + \sum_{i=i_2+1}^{\infty} \Phi(k_n x_n(i))\right)$$

$$\geq \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_2} \Phi(k_n x_n(i))\right) \to A\sum_{i=1}^{i_2} |x_0(i)| > 1 + \epsilon_0,$$

a contradiction which shows that $\sup\{k_n : n = 0, 1, 2, \ldots\} < \infty$ whenever $K(x_n) \neq \emptyset$ for all $n \in \mathcal{N}$. Taking n_0 large enough and using the fact that $K(x_n) \neq \emptyset$ for all $n \geq n_1$, we get the first part of the lemma.

Now, suppose that $\Phi \in \delta_2$. Passing to a subsequence if necessary, we may assume without loss of generality that there is $k \in [1, \infty)$ such that

$$\lim_{n\to\infty} k_n = k.$$

Now, we will prove that the elements of (x_n) have equi-absolutely continuous norms. Since $\Phi \in \delta_2$, we only need to prove that for every $\epsilon > 0$ there is $i_\epsilon \in \mathcal{N}$ such that

$$I_\Phi\left(\sum_{i=i_\epsilon}^{\infty} x_n(i)e_i\right) < \epsilon$$

for all $n \in \mathcal{N}$. Fix $\epsilon > 0$. Then there is $i_3 \in \mathcal{N}$ such that

$$\left\|\sum_{i=1}^{i_3} x_0(i)e_i\right\|_0 > 1 - \frac{\epsilon}{2}.$$

Since $x_n \to x_0$ coordinatewise, there is $n_3 \in \mathcal{N}$ such that

$$\frac{1}{k_n}\left(1 + I_\Phi\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right)\right) \geq \frac{1}{k}\left(1 + I_\Phi\left(\sum_{i=1}^{i_3} k x_0(i)e_i\right)\right) - \frac{\epsilon}{2}$$

for $n \geq n_3$. Hence

$$\begin{aligned}
1 = \frac{1}{k_n}(1 + I_\Phi(k_n x_n)) &= \frac{1}{k_n}\left(1 + I_\Phi\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right) + I_\Phi\left(\sum_{i=i_3+1}^{\infty} k_n x_n(i)e_i\right)\right) \\
&\geq \frac{1}{k_n}\left(1 + I_\Phi\left(\sum_{i=1}^{i_3} k_n x_n(i)e_i\right)\right) + I_\Phi\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right) \\
&\geq \frac{1}{k}\left(1 + I_\Phi\left(\sum_{i=1}^{i_3} k x_0(i)e_i\right)\right) - \frac{\epsilon}{2} + I_\Phi\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right) \\
\geq \left\|\sum_{i=1}^{i_3} x_0(i)e_i\right\|_0 - \frac{\epsilon}{2} &+ I_\Phi\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right) > 1 - \epsilon + I_\Phi\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right)
\end{aligned}$$

for $n \geq n_3$. So we have

$$I_\Phi\left(\sum_{i=i_3+1}^{\infty} x_n(i)e_i\right) < \epsilon$$

for $n \geq n_3$. This finishes the proof of the lemma. □

Theorem 2.9. *If $\Phi \in \delta_2$, then for each sequence (x_n) in $S(l_\Phi^0)$ and $x_0 \in S(l_\Phi^0)$ such that $x_n(i) \to x_0(i)$ as $n \to \infty$ for $i = 1, 2, \ldots$, we have that $x_n \to x_0$ in norm.*

Proof. Note that $\Phi \in \delta_2$ yields that Φ vanishes only at zero. In order to prove the theorem, we will consider two cases.

Case I. $K(x_0) = \emptyset$ or $K(x_0)$ is nonempty and unbounded. Assume first that there is a subsequence (z_n) of (x_n) such that $K(z_n) = \emptyset$ for all $n \in \mathcal{N}$. Then, by Corollary 2.5, $\|z_n\|_0 = A\sum_{i=1}^{\infty}|z_n(i)|$ for each $n \in \mathcal{N}$ and $\|x_0\|_0 = A\sum_{i=1}^{\infty}|x_0(i)|$. Since l_1 has the Schur property, we get

$$A\sum_{i=1}^{\infty}|z_n(i) - x_0(i)| \to 0$$

as $n \to \infty$. Since

$$\|z_n - x_0\|_0 \le A \sum_{i=1}^{\infty} |z_n(i) - x_0(i)|$$

for all $n \in \mathcal{N}$, we get

$$\lim_{n\to\infty} \|z_n - x_0\|_0 = 0.$$

Assume now that there is a subsequence (y_n) of (x_n) such that there are $k_n \ge 1$ satisfying

$$\|y_n\|_0 = \frac{1}{k_n}\left(1 + I_\Phi(k_n y_n)\right)$$

for $n = 1, 2, \ldots$. In virtue of $\Phi \in \delta_2$, for any $\epsilon > 0$ there is $\delta > 0$ such that $I_\Phi(x) < \delta$ implies $\|x\|_0 < \epsilon$. Moreover, there is $i_0 \in \mathcal{N}$ such that

$$\left\|\sum_{i=1}^{i_0} x_0(i) e_i\right\|_0 > 1 - \delta$$

and

$$\left\|\sum_{i=i_0+1}^{\infty} x_0(i) e_i\right\|_0 < \epsilon.$$

Since $y_n(i) \to x_0(i)$ as $n \to \infty$ for $i = 1, 2, \ldots$, there exists $n_0 \in \mathcal{N}$ such that

$$\left\|\sum_{i=1}^{i_0} y_n(i) e_i\right\|_0 > 1 - \delta$$

for $n \ge n_0$. So,

$$1 = \frac{1}{k_n}\left(1 + I_\Phi(k_n y_n)\right) = \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_0} \Phi(k_n y_n(i)) + \sum_{i=i_0+1}^{\infty} \Phi(k_n y_n(i))\right)$$

$$\ge \frac{1}{k_n}\left(1 + \sum_{i=1}^{i_0} \Phi(k_n y_n(i))\right) + \sum_{i=i_0+1}^{\infty} \Phi(y_n(i))$$

$$\ge \left\|\sum_{i=1}^{i_0} y_n(i) e_i\right\|_0 + \sum_{i=i_0+1}^{\infty} \Phi(y_n(i)) > 1 - \delta + \sum_{i=i_0+1}^{\infty} \Phi(y_n(i))$$

for $n \ge n_0$. This means that

$$\sum_{i=i_0+1}^{\infty} \Phi(y_n(i)) < \delta$$

for $n \ge n_0$. Hence

$$\left\|\sum_{i=i_0+1}^{\infty} y_n(i) e_i\right\|_0 < \epsilon$$

for $n \geq n_0$. Since $y_n \to x_0$ coordinatewise, there is $n_1 \geq n_0$ such that

$$\left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i)) e_i \right\|_0 < \epsilon$$

for $n \geq n_1$. Thus

$$\|y_n - x_0\|_0 = \left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i))\, e_i + \sum_{i=i_0+1}^{\infty} y_n(i) e_i - \sum_{i=i_0+1}^{\infty} x_0(i) e_i \right\|_0$$

$$\leq \left\| \sum_{i=1}^{i_0} (y_n(i) - x_0(i))\, e_i \right\|_0 + \left\| \sum_{i=i_0+1}^{\infty} y_n(i) e_i \right\|_0 + \left\| \sum_{i=i_0+1}^{\infty} x_0(i) e_i \right\|_0 < 3\epsilon$$

for $n \geq n_1$. Note that we have proved that we always can find a subsequence (y_n) of (x_n) such that $\|y_n - x_0\| \to 0$. So, by the double extract subsequence theorem, there holds $\|x_n - x_0\| \to 0$.

Case II. $K(x_0)$ is nonempty and bounded. By Lemma 2.8, we can assume without loss of generality that $K(x_n) \neq \emptyset$ for each $n \in \mathcal{N}$. Now, repeating the procedure from case I, we get

$$\lim_{n \to \infty} \|x_n - x_0\|_0 = 0.$$

We can also get the same conclusion by applying equi-absolute continuity of the norm of (x_n) and the fact that $x_n \to x_0$ coordinatewise. So the proof of the theorem is finished. $\square$

Remark 2.10. An analogue of Theorem 2.9 for the Luxemburg norm has been proved in [10].

Remark 2.11. Criteria for **H**-property of Orlicz sequence spaces equipped with the Luxemburg norm and the Orlicz norm, but generated by N-functions, were first given in [19] and [3]. Next the problem of a characterization of **H**-points in Orlicz sequence spaces was considered in [7], [16] and [10]. Criteria for **H**-property of Orlicz sequence spaces l_Φ equipped with the Luxemburg norm in the case of convex Orlicz functions (without the assumption that it is an N-function) were given in [10]. The next theorem solves an analogous problem for the Orlicz norm for arbitrary Orlicz function. Although the criterion is the same as for N-functions, the proof is much more complicated and it is based on Theorem 2.9, the proof of which use some new techniques in comparison with the ones used in [19] and [3].

Theorem 2.12. *The space l_Φ^0 has property* **H** *if and only if $\Phi \in \delta_2$.*

Proof. *Sufficiency.* Assume for a sequence (x_n) in $S(l_\Phi^0)$ that $x_n \to x_0$ weakly. This implies that $x_n \to x_0$ coordinatewise. By Theorem 2.9, in view of $\Phi \in \delta_2$, $x_n \to x_0$ in norm.

Necessity. Assume first that Φ vanishes only at zero. If $\Phi \notin \delta_2$, there is $x_0 \in S(l_\Phi)$ such that

$$I_\Phi(x_0) \leq 1 \text{ and } I_\Phi(\lambda x_0) = \infty$$

for any $\lambda > 1$ (see [1]). Take an increasing sequence (i_n) of natural numbers such that $i_n \to \infty$ as $n \to \infty$ and

$$\left\| \sum_{i=i_n+1}^{i_{n+1}} x_0(i)e_i \right\|_0 \geq \frac{1}{2}$$

for every $n \in \mathcal{N}$. Put

$$x_n = (x_0(1), ..., x_0(i_n), 0, ...0, x_0(i_{n+1}+1), x_0(i_{n+1}+2), ...)$$

and $z_n = x_n / \|x_0\|_0$ for $n = 0, 1, 2, ...$.

We will show that $z_n \to z_0$ weakly. Any $f \in (l^0_\Phi)^*$ is uniquely represented in the form $f = y + s$, where $y \in l_\Psi$ and $s \in (h^0_\Phi)^\perp$. Since $y \in l_\Psi$, there is $\lambda > 0$ such that

$$\sum_{i=1}^{\infty} \Psi(\lambda y(i)) < \infty.$$

Moreover, since $x_n - x_0 \in h^0_\Phi$, we have $\langle x_n - x_0, s\rangle = 0$ for each $n \in \mathcal{N}$. Hence

$$|\langle x_n - x_0, f\rangle| = |\langle x_n - x_0, y\rangle| = \left| \sum_{i=i_n+1}^{i_{n+1}} x_0(i)y(i) \right|$$

$$\leq \frac{1}{\lambda}\left(\sum_{i=i_n+1}^{i_{n+1}} \Phi(x_0(i)) + \Psi(\lambda y(i)) \right) \to 0$$

as $n \to \infty$. So, $z_n \to z_0$ weakly. Hence, by property **H** of l^0_Φ, $\|z_n - z_0\|_0 \to 0$.

On the other hand,

$$\|z_n - z_0\|_0 = \frac{1}{\|x_0\|_0} \left\| \sum_{i=i_n+1}^{i_{n+1}} x_0(i)e_i \right\|_0 \geq \frac{1}{2\|x_0\|_0} > 0$$

for every $n \in \mathcal{N}$. This contradiction shows that $\Phi \in \delta_2$ is necessary for property **H** of l^0_Φ if Φ vanishes only at zero.

Assume now that Φ vanishes outside zero and define

$$a = a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}.$$

Then $a > 0$. Let $x = (a, a, ...)$. We have $1 + I_\Phi(x) = 1$ and

$$\frac{1}{k}(1 + I_\Phi(kx)) > 1$$

for each $k > 0, k \neq 1$. Therefore $\|x\|_0 = 1$. Define

$$x_n = (\underbrace{a, ..., a,}_{n} 0, a, a, ...)$$

for every $n \in \mathcal{N}$. Then we can prove in the same way as for x that $\|x_n\|_0 = 1$ for each $n \in \mathcal{N}$. Since $x - x_n = ae_{n+1} \in h_\Phi$, $x^*(x - x_n) = 0$ for each singular functional x^* over l_Φ. Take now any $y = (y_1, y_2, ...) \in l_\Psi^0$. There is $\lambda > 0$ such that $I_\Psi(\lambda y) < \infty$, whence it follows that $\Psi(\lambda y_n) \to 0$ as $n \to \infty$. Therefore, since Ψ vanishes only at zero, we get $y_n \to 0$ as $n \to \infty$ and consequently

$$\langle x - x_n, y\rangle = ay_{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $x - x_n \to 0$ weakly. However

$$\|x - x_n\| = a\,\|e_{n+1}\| = a\,\|e_1\| > 0$$

for each $n \in \mathcal{N}$. So, l_Φ^0 fails to have property **H** if Φ vanishes outside zero, which finishes the proof of the theorem. □

Theorem 2.13. *The space l_Φ^0 is* **CLUR** *if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.*

Proof. *Sufficiency.* If $\Phi \in \delta_2$ and $\Psi \in \delta_2$, then l_Φ^0 is reflexive. Consequently, by the previous theorem and Theorem 6 in [10], both l_Φ^0 and l_Ψ have property **H**. In virtue of Theorem 2.1, we obtain that l_Φ^0 is **CLUR**.

Necessity. Since property **CLUR** implies property **H**, in view of Theorem 2.12, we get $\Phi \in \delta_2$. To prove the necessity of $\Psi \in \delta_2$ assume first that Ψ vanishes only at zero. If $\Psi \notin \delta_2$, there is a sequence (u_n) of positive numbers such that $u_n \downarrow 0$ and

$$\Psi\left(\left(1 + \frac{1}{n}\right)u_n\right) \geq 2^n\Psi(u_n) \text{ and } \Psi(u_n) \leq \frac{1}{2^{n-1}} \tag{2.1}$$

for each $n \in \mathcal{N}$. Take positive integers N_m such that

$$\frac{1}{2^m} < N_m\Psi(u_m) \leq \frac{1}{2^{m-1}} \tag{2.2}$$

for $m = 1, 2, ...$ (we can pass to a subsequence of (u_n) if necessary). Define $k_i = \sum_{m=1}^{i} N_m$ $(i = 1, 2, ...)$,

$$b(\Psi) = \sup\{u \geq 0 : \Psi(u) \leq 1\},$$

and

$$z_0 = (b(\Psi), 0, 0, ...),$$

$$z_1 = (\overbrace{u_1, u_1, ..., u_1}^{N_1}, 0, 0, ...),$$

$$z_2 = (\overbrace{0, 0, ..., 0}^{N_1}, \overbrace{u_2, u_2, ..., u_2}^{N_2}, 0, 0, ...),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$z_m = (\overbrace{0, 0, ...0}^{k_{m-1}}, \overbrace{u_m, u_m, ..., u_m}^{N_m}, 0, 0, ...),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

Then, denoting by $\|\cdot\|$ the Luxemburg norm in l_Ψ, we easily get from inequalities (2.1) and (2.2) that

(i) $\frac{m}{m+1} \leq \|z_m\| \leq 1,\ m = 1, 2, \dots$.

Moreover,

(ii) there is a sequence (x_m) in $S(l^0_\Phi)$ such that $x_m(i) = 0$ for $1 \leq i \leq k_{m-1}$ and $i \geq k_m+1$, and x_m generate support functionals at z_m, i.e.

$$\|z_m\| = \langle z_m, x_m\rangle = u_m \sum_{i=k_{m-1}+1}^{k_m} x_m(i).$$

This follows by the Hahn-Banach theorem and the fact that $(l_\Psi)^* = l^0_\Phi$. Moreover for

$$x_0 = \left(\frac{1}{b(\Psi)}, 0, 0, \dots\right)$$

there holds $\|x_0\|_0 = 1$. Put

$$g_m = \left(1 - \frac{1}{2^{m-1}}\right)(\overbrace{b(\Psi), 0, 0, \dots 0}^{k_m}, z_m(k_{m-1}+1), z_m(k_{m-1}+2), \dots, z_m(k_m), 0, 0, \dots)$$

for $m = 1, 2, \dots$. Then

$$I_\Psi(g_m) \leq \left(1 - \frac{1}{2^{m-1}}\right)\left(1 + \frac{1}{2^{m-1}}\right) = 1 - \frac{1}{4^{m-1}} \leq 1,$$

whence $\|g_m\| \leq 1$. Moreover, in virtue of (ii), we have

$$\|x_m + x_0\|_0 \geq \langle x_m + x_0, g_m\rangle = \sum_{i=1}^{\infty}(x_m(i) + x_0(i))g_m(i)$$

$$= \left(1 - \frac{1}{2^{m-1}}\right)\left(1 + u_m \sum_{i=k_{m-1}+1}^{k_m} x_m(i)\right) \to 2$$

as $m \to \infty$. But, by the orthogonality of x_m and x_n for $n \neq m$, there holds

$$\|x_m - x_n\|_0 \geq \|x_n\|_0 = 1 \text{ for } n \neq m,$$

which means that l^0_Φ is not **CLUR**. So, in the case when Ψ vanishes only at zero, $\Psi \in \delta_2$ is necessary for property **CLUR** of l^0_Φ.

Assume now that

$$c(\Psi) = \sup\{u \geq 0 : \Psi(u) = 0\} > 0$$

and define

$$u_n = \left(1 - \frac{1}{n^2}\right)c(\Psi) \text{ and } \lambda_n = \frac{n}{n-1}$$

for $n = 2, 3, \ldots$. Then

$$\lambda_n u_n = \frac{n^2 - 1}{n^2 - n} c(\Psi) > c(\Psi)$$

for each $n \in \mathcal{N} \setminus \{1\}$, but $\lambda_n u_n \to c(\Psi)$ as $n \to \infty$. Let $(N_n)_{n=1}^{\infty}$ be a sequence of natural numbers such that

$$N_{n-1} \Psi(\lambda_n u_n) \geq 1$$

for $n = 2, 3, \ldots$. Define

$$z_1 = \sum_{i=2}^{N_1+1} u_2 e_i$$

and

$$z_n = \sum_{i=k_{n-1}+1}^{k_n} u_{n+1} e_i,$$

where $k_n = 1 + \sum_{i=1}^{n-1} N_i$ for $n = 2, 3, \ldots$. Define also $z_0 = (b(\Psi), 0, 0, \ldots)$, where $b(\Psi)$ is the number defined above. Then $\|z_0\| = 1$ and

$$I_\Psi(z_n) = 0, \quad I_\Psi(\lambda_n z_n) \geq 1$$

for each $n \in \mathcal{N}$. Hence

$$\lambda_n^{-1} \leq \|z_n\| \leq 1$$

for each $n \in \mathcal{N}$. Since $z_n \in h_\Psi$ for every $n \in \mathcal{N} \cup \{0\}$, there is a sequence $(x_n)_{n=0}^{\infty}$ with $\operatorname{supp} x_n = \operatorname{supp} z_n$ such that

$$\|x_n\|_0 = 1 \quad \text{and} \quad \langle z_n, x_n \rangle = \|z_n\|$$

for each $n \in \mathcal{N} \cup \{0\}$. Define $g_n = z_n + z_0$ for each $n \in \mathcal{N}$. Then

$$I_\Psi(g_n) = I_\Psi(z_n) + I_\Psi(z_0) = I_\Psi(z_0) \leq 1,$$

whence $\|g_n\| \leq 1$. Moreover, $\|g_n\| \geq \|z_0\| = 1$ and consequently $\|g_n\| = 1$ for each $n \in \mathcal{N}$. Hence

$$2 \geq \|x_n + x_0\|_0 \geq \langle x_n + x_0, g_n \rangle = \|x_n\|_0 + \|x_0\|_0 \to 2,$$

whence

$$\lim_{n\to\infty} \|x_n + x_0\|_0 = 2.$$

Since

$$\|x_n - x_0\|_0 \geq \|x_n\|_0 = 1$$

for every $n \neq m$, the sequence (x_n) contains no convergent sequence, i.e. l_Φ^0 is not **CLUR**. This finishes the proof of the theorem. □

Denote by p the right derivative of Φ and define

$$\pi_\Phi(1) = \sup\{t > 0 : \Psi(p(t)) \leq 1\}.$$

It is known that if Φ vanishes only at zero and $\Phi \notin \delta_2$, then l_Φ^0 does not contain an order isometric copy of l_∞ since it is then strictly monotone (see [11]) although it contains an order isomorphic (even an order almost isometric) copy of l_∞ because it is not order continuous by $\Phi \notin \delta_2$.

Note that $\Phi \notin \delta_2$ whenever Φ vanishes outside zero. We will show in the following lemma that in this special case of $\Phi \notin \delta_2, l_\Phi^0$ contains an order isometric copy of l_∞.

Lemma 2.14. *If Φ is an Orlicz function which vanishes outside zero, then l_Φ^0 contains an order isometric copy of l_∞.*

Proof. To build an isometry P of l_∞ onto a closed subspace of l_Φ^0 preserving the order it is enough to find a sequence $(x_n) \subset S(l_\Phi^0)$ with pairwise disjoint supports and such that $\|\sum_{n=1}^\infty x_n\| = 1$. Then the operator $P : l_\infty \to l_\Phi^0$ defined by

$$Py = \sum_{n=1}^{\infty} y_n x_n$$

for each $y = (y_n) \in l_\infty$ is the desired isometry.

Assume that

$$a = a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\} > 0.$$

Divide the set $\mathcal{N}$ of natural numbers into a sequence $(\mathcal{N}_n)_{n=1}^\infty$ of pairwise disjoint and infinite sets. Define $x_n = (x_n(i))_{i=1}^\infty$, where

$$x_n(i) = \begin{cases} a & \text{if } i \in \mathcal{N}_n \\ 0 & \text{otherwise} \end{cases}$$

for any $n \in \mathcal{N}$. Then $1 + I_\Phi(x_n) = 1$ for each $n \in \mathcal{N}$. Moreover,

$$\frac{1}{k}\left(1 + I_\Phi(kx_n)\right) = \frac{1}{k} > 1$$

for each $k \in (0, 1)$ and

$$\frac{1}{k}\left(1 + I_\Phi(kx_n)\right) = \infty$$

for each $k > 1$, because the sets $\mathcal{N}_n$ are infinite and $\Phi(ka) > 0$ for $k > 1$. This shows that

$$\|x_n\|_0 = \inf_{k>0} \frac{1}{k}\left(1 + I_\Phi(kx_n)\right) = 1 + I_\Phi(x_n) = 1$$

for each $n \in \mathcal{N}$. In the same way it can be proved that $\|\sum_{n=1}^\infty x_n\| = 1$. This finishes the proof. $\square$

Corollary 2.15. *Let Φ be a finitely Orlicz function satisfying the condition (∞_1). Then l_Φ^0 is smooth if and only if*

(i) $\Phi \in \delta_2$,
(ii) Φ *is smooth on the interval* $[0, \pi_\Phi(1/2))$,
(iii) $\Psi\left(p^{-1}(\pi_\Phi(1/2))\right) = \frac{1}{2}$.

Proof. In the class of Orlicz functions which vanish only at zero, the proof is given in [2]. However, by Lemma 2.14, the fact that l_∞ is not a smooth space and by the Hahn-Banach theorem, we easily get that l_Φ^0 is not smooth if Φ vanishes outside zero. □

Theorem 2.16. *The space l_Φ^0 is* **R** *if and only if the following conditions are satisfied:*

(i) Φ *vanishes only at zero,*
(ii) *there is $u > 0$ such that $\Psi(p(u)) \geq \frac{1}{2}$,*
(iii) Φ *is strictly convex on the interval* $[0, \pi_\Phi(1)]$.

Proof. *Sufficiency.* First we prove that if $z \in l_\Phi^0 \setminus \{0\}$ and z has at least two coordinates different from zero, then $K(z) \neq \emptyset$. Assume without loss of generality that $z(1) > 0$ and $z(2) > 0$. There is $k > 0$ such that

$$\Psi\left(p(kz(1))\right) \geq \frac{1}{2} \text{ and } \Psi\left(p(kz(2))\right) \geq \frac{1}{2},$$

whence $I_\Psi\left(p \circ k\,|z|\right) \geq 1$. This yields that

$$k^*(z) := \inf\left\{k > 0 : I_\Psi\left(p \circ k\,|z|\right) \geq 1\right\} < \infty$$

and consequently $K(z) \neq \emptyset$.

Let $x, y \in S(l_\Phi^0)$ and $x \neq y$. It is obvious by the definition of the number $\pi_\Phi(1)$ that if $k \in K(x) \neq \emptyset$ and $l \in K(y) \neq \emptyset$, then

$$k\,|x(i)| \in [0, \pi_\Phi(1)] \quad \text{and} \quad l\,|y(i)| \in [0, \pi_\Phi(1)]$$

for all $i \in \mathcal{N}$. Then we can repeat the proof of the sufficiency of Theorem 2.9 from [1] to obtain

$$\left\|\frac{1}{2}(x+y)\right\|_0 < 1.$$

Consider now the case $K(x) = \emptyset$ or $K(y) = \emptyset$. Then

$$\lim_{u\to\infty}\left(\Phi(u)/u\right) = A < \infty.$$

Assume first that $K(x) = K(y) = \emptyset$. Then it must be

$$\operatorname{supp} x = \operatorname{supp} y = \{i\}.$$

Really, it is obvious by the above considerations that if μ is the counting measure in $2^{\mathcal{N}}$,

$$\mu(\operatorname{supp} x) = \mu(\operatorname{supp} y) = 1.$$

Assume for the contrary that $\operatorname{supp} x = \{i\}$ and $\operatorname{supp} y = \{j\}$, where $i \neq j$. Then $\operatorname{supp}(x+y) = \{i, j\}$ and we get $K(x+y) \neq \emptyset$. The assumption $K(x) = K(y) = \emptyset$ yields

$$\|x\|_0 = A\,|x(i)| \quad \text{and} \quad \|y\|_0 = A\,|y(j)|.$$

Assume first additionally that the set $K(x+y)$ is bounded. Then assuming for the contrary that $\|x+y\|_0 = 2$, we get

$$2 = \|x+y\|_0 < A\left(|x(i)| + |y(j)|\right) = \|x\|_0 + \|y\|_0 = 2,$$

a contradiction. So $\|x+y\|_0 < 2$, which is the desired inequality.

Assume now that $K(x+y)$ is unbounded. Then

$$\|x+y\|_0 = \lim_{k\to\infty} \frac{1}{k} I_\Phi\left(k(x+y)\right),$$

whence

$$\left\|\frac{x+y}{2}\right\|_0 = \lim_{k\to\infty} \frac{1}{2k} I_\Phi\left(2k\,\frac{x+y}{2}\right) = \lim_{k\to\infty} \frac{1}{k} I_\Phi\left(k\,\frac{x+y}{2}\right).$$

Denote $x+y=z$. By unboundedness of $K(x+y)$ there is an increasing sequence (l_n) in $K(x+y)$ such that $l_n \nearrow \infty$ as $n\to\infty$. Then

$$\|z\|_0 \le \frac{k+l_n}{2kl_n}\left(1+I_\Phi\left(\frac{2kl_n}{k+l_n}\,z\right)\right)$$

$$= \frac{k+l_n}{2kl_n}\left(1+I_\Phi\left(\frac{l_n}{k+l_n}\,kz+\frac{k}{k+l_n}\,l_n z\right)\right)$$

$$\le \frac{k+l_n}{2kl_n}\left(1+\frac{l_n}{k+l_n}\,I_\Phi(kz)+\frac{k}{k+l_n} I_\Phi(\,l_n z)\right) = \|z\|_0$$

for any $n\in\mathcal{N}$ and $k\in K(z)$. This shows that

$$\|z\|_0 = \frac{k+l_n}{2kl_n}\left(1+I_\Phi\left(\frac{2kl_n}{k+l_n}\,z\right)\right) \tag{2.3}$$

for each $n\in\mathcal{N}$ and $k\in K(z)$. Consequently, by the left continuity of I_Φ and the fact that $\frac{2kl_n}{k+l_n}\nearrow 2k$ as $n\to\infty$, we get

$$\|z\|_0 = \frac{1}{2k}\left(1+I_\Phi\left(2kz\right)\right).$$

Therefore, $2k\in K(z)$. Hence, putting $l=2k$ and repeating the same argumentation as in the proof of (2.3), we get

$$\|z\|_0 = \frac{k+l}{2kl}\left(1+I_\Phi\left(\frac{2kl}{k+l}\,z\right)\right).$$

Since

$$\frac{2kl}{k+l}\,z(m) = \frac{k}{k+l}\,l\,z(m)+\frac{l}{k+l}\,k\,z(m),$$

we conclude that $\frac{2kl}{k+l}\,z(m)$, $kz(m)$ and $lz(m)$ belong to the same interval of affinity of Φ for $m=i$ and $m=j$. But $k\neq l$, whence $kz(i)\neq lz(i)$ and $kz(i),\ lz(i)\in[0,\pi_\Phi(1)]$, which contradicts the assumption that Φ is strictly convex on the interval $[0,\pi_\Phi(1)]$. This contradiction shows that $K(x+y)$ is bounded whenever $K(x)=\emptyset, K(y)=\emptyset, \operatorname{supp} x=\{i\}, \operatorname{supp} y=\{j\}$ and $i\neq j$. Therefore $\left\|\frac{x+y}{2}\right\|_0<1$, as it was already shown.

Consider now the case when $K(x) = \emptyset$ and $K(y) \neq \emptyset$. Denote $x_0 = \frac{1}{2}(x+y)$ and assume that $0 < k_0 \in K(y)$. If $\|x_0\|_0 < 1$, then the proof of the sufficiency is finished. So consider the case $\|x_0\|_0 = 1$. Then for all $n \in \mathcal{N}$ we get

$$2 = \|2x_0\|_0 \leq \frac{n+k_0}{nk_0}\left(1 + I_\Phi\left(\frac{nk_0}{n+k_0}2x_0\right)\right)$$

$$= \frac{n+k_0}{nk_0}\left(1 + I_\Phi\left(\frac{n}{n+k_0}k_0y + \frac{k_0}{n+k_0}nx\right)\right)$$

$$\leq \frac{1}{k_0}\left(1 + I_\Phi\left(k_0y\right)\right) + \frac{1}{n}\left(1 + I_\Phi\left(nx\right)\right) \underset{n\to\infty}{\longrightarrow} 2,$$

whence we get as above that $2k_0 \in K(x_0)$. If for some $\alpha \in (0,1)$ there holds

$$\|\alpha x + (1-\alpha)y\|_0 < 1,$$

then, by the convexity of the norm $\|\cdot\|_0$, we get $\|x_0\|_0 < 1$, a contradiction. So assume that

$$\|\alpha x + (1-\alpha)y\|_0 = 1 \tag{2.4}$$

for all $\alpha \in (0,1)$. This will also give a contradiction, finishing the proof of the sufficiency. Indeed, defining

$$x_n = \frac{x_{n-1}+y}{2}$$

for $n = 1, 2, \ldots$, we will get in the same way as for x_0 that $2^{n+1}k_0 \in K(x_n)$. In order to prove that $K(\alpha x + (1-\alpha)y) \neq \emptyset$ for each $\alpha \in (0,1)$ it is enough to prove that if $w, v \in S\left(l_\Phi^0\right)$ are such that $K(w) \neq \emptyset$ and $K(v) \neq \emptyset$, then $K(\alpha w + (1-\alpha)v) \neq \emptyset$. Let $0 < k \in K(w)$ and $0 < l \in K(v)$. If we can prove that for

$$c = \frac{\alpha k(1-\alpha)l}{\alpha k + (1-\alpha)l}$$

there holds

$$\|w+v\|_0 = \frac{1}{c}\left(1 + I_\Phi(c(w+v))\right)$$

whenever $\|w+v\|_0 = \|w\|_0 + \|v\|_0$, then, by the obvious equalities

$$\|\alpha w\|_0 = \frac{\alpha}{k}\left(1 + I_\Phi\left(\frac{k}{\alpha}(\alpha w)\right)\right)$$

and

$$\|(1-\alpha)v\|_0 = \frac{1-\alpha}{l}\left(1 + I_\Phi\left(\frac{l}{1-\alpha}(1-\alpha)v\right)\right),$$

we get

$$\frac{(1-\alpha)k + \alpha l}{kl} \in K\left(\alpha w + (1-\alpha)v\right).$$

But we easily get

$$\|w\|_0 + \|v\|_0 = \|w+v\|_0 \le \frac{k+l}{kl}\left(1 + I_\Phi\left(\frac{kl}{k+l}(w+v)\right)\right)$$

$$= \frac{k+l}{kl}\left(1 + I_\Phi\left(\frac{l}{k+l}kw + \frac{k}{k+l}lv\right)\right)$$

$$\le \frac{k+l}{kl}\left(1 + \frac{l}{k+l}I_\Phi(kw) + \frac{k}{k+l}I_\Phi(lv)\right)$$

$$= \frac{k+l}{kl} + \frac{1}{k}I_\Phi(kw) + \frac{1}{l}I_\Phi(lv)$$

$$= \frac{1}{k}\left(1 + I_\Phi(kw)\right) + \frac{1}{l}\left(1 + I_\Phi(lv)\right) = \|w\|_0 + \|v\|_0,$$

whence

$$\|w+v\|_0 = \frac{k+l}{kl}\left(1 + I_\Phi\left(\frac{kl}{k+l}(w+v)\right)\right).$$

Note that $x_0 = \frac{1}{2}(w+v)$, where $w = \frac{3}{4}x + \frac{1}{4}y$ and $v = \frac{3}{4}y + \frac{1}{4}x$. Since $w, v \in S\left(l_\Phi^0\right)$ and $K(w) \ne \emptyset$, $K(v) \ne \emptyset$, we obtain by strict convexity of Φ on the interval $[0, \pi_\Phi(1)]$ that $\|x_0\|_0 < 1$, which contradicts the assumption (2.4). This ends the proof of the sufficiency.

Necessity. The necessity of the property that Φ vanishes only at zero follows by Lemma 2.14. Let us prove the necessity of the condition $\Psi(p(u)) \ge \frac{1}{2}$ for some $u > 0$. First we will show that l_Φ^0 is not rotund if there is $x \in S(l_\Phi^0)$ such that $x(1) > 0$, $x(2) > 0$ and $K(x) = \emptyset$. Note that the last assumption implies that

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = A < \infty.$$

Take $\epsilon \in (0, \min\{x(1), x(2)\})$. Define

$$x_1 = (x(1) + \epsilon, x(2) - \epsilon, x(3), x(4), \ldots)$$

and

$$x_2 = (x(1) - \epsilon, x(2) + \epsilon, x(3), x(4), \ldots).$$

Then

$$\|x_1\|_0 \le A\left(x(1) + \epsilon + x(2) - \epsilon + \sum_{i=3}^{\infty}|x(i)|\right) = A\sum_{i=1}^{\infty}|x(i)| = \|x\|_0 = 1$$

and similarly $\|x_2\|_0 \le 1$. Hence

$$1 = \|x\|_0 = \left\|\frac{x_1 + x_2}{2}\right\|_0 \le \frac{1}{2}\|x_1\|_0 + \frac{1}{2}\|x_2\|_0 \le 1,$$

and consequently $\frac{x_1+x_2}{2}$, x_1, $x_2 \in S(l_\Phi^0)$. But $x_1 \ne x_2$, so l_Φ^0 is not rotund. Therefore, to show the necessity of $\Psi(p(u)) \ge \frac{1}{2}$ for some $u > 0$ it is enough to show that if $\Psi(p(u)) < \frac{1}{2}$

for any $u > 0$, then there is $x \in S(l^0_\Phi)$ with $x(1) > 0$, $x(2) > 0$ and $K(x) = \emptyset$. Let $a > 0$ be such that

$$x = (a, a, 0, 0, ...) \in S(l^0_\Phi).$$

For any $k > 0$ there holds $I_\Psi(p \circ kx) < 1$, which shows that $k^*(x) = k^{**}(x) = \infty$, where

$$k^{**}(x) = \sup\{k > 0 : I_\Psi(p \circ kx) \leq 1\}$$

and

$$k^*(x) = \inf\{k > 0 : I_\Psi(p \circ kx) \geq 1\}.$$

Therefore $K(x) = \emptyset$ (see [1], pp. 18-19).

It remains to prove the necessity of strict convexity of Φ on the interval $[0, \pi_\Phi(1)]$. Assume that Φ is affine on some interval $[a-\epsilon, a+\epsilon] \subset [0, \pi_\Phi(1)]$, where $\epsilon > 0$. Then $\Psi(p(a)) < 1$. Suppose first that $\Psi(p(a)) > 0$. Let $m \in \mathcal{N}$ be the biggest natural number such that $m\Psi(p(a)) < 1$ and define

$$b = \sup\{u > 0 : m\Psi(p(a)) + \Psi(p(u)) \leq 1\}.$$

Note that $0 < b < a$ by the definitions of b and m. If $\Phi(u)/u \to 0$ as $u \to 0$, then $p(0) = 0$, and the inequality $b > 0$ follows by the right continuity of p. If $\Phi(u)/u \to A < \infty$ as $u \to 0$, then $p(0) = A$ and Ψ vanishes on the interval $[0, A]$, whence $b > 0$ again. Define

$$k = 1 + m\Phi(a) + \Phi(b)$$

and

$$x = k^{-1}(\underbrace{a, ..., a}_{m \text{ times}}, b, 0, 0, ...).$$

Then $k \in K(x)$ and so

$$\|x\|_0 = \frac{1}{k}(1 + I_\Phi(kx)) = \frac{1 + m\Phi(a) + \Phi(b)}{1 + m\Phi(a) + \Phi(b)} = 1.$$

Since $\mu(\operatorname{supp} x) = 2$ and $kx(i) = a$ is a midpoint of the interval of the affinity of Φ for $i = 1, 2, ..., m$, x is not an extreme point. We can prove this in an analogous way to the proof of Theorem 2.8, pp. 54-55 in [1]. The only changing that is needed is taking

$$b(\Psi) = \sup\{u > 0 : \Psi(u) \leq 1\}$$

instead of $\Psi^{-1}(1)$, since we did not exclude the situation when Ψ vanishes outside zero or attains infinite values.

Assume now that $\Psi(p(a)) = 0$. Then define

$$c = \sup\{u > 0 : \Psi(p(a)) + 2\Psi(p(u)) \leq 1\}.$$

By the previous part of the necessity, we may assume that $\Psi(p(u)) \geq \frac{1}{2}$ for some $u > 0$, whence it follows that $0 < c < \infty$. Define

$$k = 1 + \Phi(a) + 2\Phi(c)$$

and

$$x = k^{-1}(a, c, c, 0, 0, ...).$$

Then it follows by the definition of c that $k \in K(x)$ and so

$$\|x\|_0 = \frac{1}{k}(1 + \Phi(a) + 2\Phi(c)) = 1.$$

Repeating the same argumentation as above, we conclude that x is not an extreme point, i.e. l^0_Φ is not rotund, which finishes the proof of the theorem. □

Remark 2.17. Note that Theorem 2.16 can be formulated equivalently that l^0_Φ is rotund if and only if Φ is strictly convex on the interval $[0, \pi_\Phi(1)]$ and $K(x) \neq \emptyset$ for any $x \in S(l^0_\Phi)$ with $\mu(\operatorname{supp} x) \geq 2$.

Corollary 2.18. *Let Φ be an arbitrary Orlicz function. Then l^0_Φ is* **LUR** *if and only if $\Phi \in \delta_2$, $\Psi \in \delta_2$, Φ is strictly convex on the interval $[0, \pi_\Phi(1)]$ and $\Psi(p(u)) \geq \frac{1}{2}$ for some $u > 0$.*

Proof. Since a Banach space X is **LUR** if and only if it is **CLUR** and **R** (see [15]), the corollary follows immediately from Theorems 2.12 and 2.13. □

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