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On a Non-Standard Convex Regularization and the Relaxation of Unbounded Integral Functionals of the Calculus of Variations

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The analysis of the relationships between the functional $F^{(\infty)}(\Omega,\cdot)\colon u \in W^{1,\infty}(\Omega) \mapsto \inf\{\liminf_h \int_\Omega f(\nabla u_h)dx : \{u_h\} \subseteq W^{1,\infty}(\Omega),\ u_h \to u$ in weak*-$W^{1,\infty}(\Omega)\}$, and the sequential weak*-$W^{1,\infty}(\Omega)$-relaxed functional $\overline{F}^{(\infty)}(\Omega,\cdot)$ of the integral $u \in W^{1,\infty}(\Omega) \mapsto \int_\Omega f(\nabla u)dx$ is carried out, where $f\colon \mathbb{R}^n \to [0,+\infty]$, Ω is a bounded open subset of $\mathbb{R}^n$, and $u \in W^{1,\infty}(\Omega)$.
In [8] it has been proved the existence of $f^{(\infty)}\colon \mathbb{R}^n \to [0,+\infty]$ such that $F^{(\infty)}(\Omega,u) = \int_\Omega f^{(\infty)}(\nabla u)dx$ for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$ such that $F^{(\infty)}(\Omega,u) < +\infty$, and this result is exploited there to deduce that $\overline{F}^{(\infty)}(\Omega,u) = \int_\Omega f^{**}(\nabla u)dx$ for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$, where f^{**} is the bipolar of f.
In the present paper it is first proved that $f^{(\infty)}$ is the convex envelope of the lower semicontinuous envelope of f, and an example is produced showing that $f^{(\infty)}$ may be different from f^{**}. Conditions for their identity are then furnished.
Examples and conditions concerning the coincidence between $F^{(\infty)}(\Omega,u)$ and $\int_\Omega f^{(\infty)}(\nabla u)dx$ for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$ are also proposed.
By such results conditions for the identity between $F^{(\infty)}$ and $\overline{F}^{(\infty)}$ are deduced.

1. Introduction

Let $f\colon (x,s,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \mapsto f(x,s,z) \in [0,+\infty[$ be a function satisfying suitable growth and measurability conditions, and let Ω be a regular bounded open subset of $\mathbb{R}^n$, then a problem in the Calculus of Variations which has attracted much attention deals with the study of the sequential weak$^* - W^{1,\infty}(\Omega)$-relaxed functional $\overline{F}^{(\infty)}(\Omega,\cdot)$ of the integral $F(\Omega,\cdot)\colon u \in W^{1,\infty}(\Omega) \mapsto \int_\Omega f(x,u(x),\nabla u(x))dx$, i.e. of the greatest sequentially weak$^* - W^{1,\infty}(\Omega)$-lower semicontinuous functional less than or equal to $F(\Omega,\cdot)$ (cf. for example [1, 3, 4, 5, 8, 10, 12, 13]).

In many cases the analysis of the properties of $\overline{F}^{(\infty)}(\Omega,\cdot)$ is carried out by first studying

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the (simpler) functional $F^{(\infty)}(\Omega,\cdot)$ defined by

$$F^{(\infty)}(\Omega,\cdot)\colon u\in W^{1,\infty}(\Omega)\mapsto$$
$$\inf\left\{\liminf_h\int_\Omega f(x,u_h,\nabla u_h)dx : \{u_h\}\subseteq W^{1,\infty}(\Omega), u_h\to u \text{ in weak}^*-W^{1,\infty}(\Omega)\right\},$$

and then by proving some relationships between $\overline{F}^{(\infty)}(\Omega,\cdot)$ and $F^{(\infty)}(\Omega,\cdot)$ (cf. the above mentioned papers).

It is clear that $\overline{F}^{(\infty)}(\Omega,\cdot)$ is sequentially weakly* $W^{1,\infty}(\Omega)$-lower semicontinuous whilst in general, since the weak* $-W^{1,\infty}(\Omega)$ topology does not satisfy the first countability axiom, $F^{(\infty)}(\Omega,\cdot)$ is not.

The above outlined procedure applies also to some cases in which pointwise constraints on the gradient are taken into account, more precisely when for a.e. $x\in\mathbb{R}^n$ and every $s\in\mathbb{R}$, $f(x,s,\cdot)=+\infty$ outside a fixed ball of $\mathbb{R}^n$ not depending on x and s (cf. [13]).

In a more general setting, it has been carried out in [8] where it has been proved, among other things, that if $f\colon\mathbb{R}^n\to[0,+\infty]$ verifies suitable local boundedness and upper semicontinuity assumptions, if the set $\{z\in\mathbb{R}^n : f(z)<+\infty\}$ is convex, and if $f^{(\infty)}$ is defined by

$$f^{(\infty)}\colon z\in\mathbb{R}^n\mapsto\inf_{m\in\mathbb{N}}(f+I_{Q_m})^{**}(z),$$

then $f^{(\infty)}$ is convex, Borel and

$$F^{(\infty)}(\Omega,u)=\int_\Omega f^{(\infty)}(\nabla u)dx \tag{1.1}$$
$$\text{for every convex bounded open set } \Omega, u\in W^{1,\infty}(\Omega) \text{ such that } F^{(\infty)}(\Omega,u)<+\infty.$$

In addition, if $\{z\in\mathbb{R}^n : f(z)<+\infty\}^\circ\neq\emptyset$, (1.1) holds for every bounded open set Ω, $u\in W^{1,\infty}_{loc}(\mathbb{R}^n)$ for which $F^{(\infty)}(\Omega,u)$ is finite.

In the above formulas, for every $m\in\mathbb{N}$, $Q_m=]-m,m[^n$, I_{Q_m} is the indicator function of Q_m defined by $I_{Q_m}(z)=0$ if $z\in Q_m$, $I_{Q_m}(z)=+\infty$ if $z\in\mathbb{R}^n\setminus Q_m$, and $(f+I_{Q_m})^{**}$ is the bipolar of $f+I_{Q_m}$ defined as the greatest convex lower semicontinuous function less than or equal to $f+I_{Q_m}$.

By such result it is also deduced that

$$\overline{F}^{(\infty)}(\Omega,u)=\int_\Omega f^{**}(\nabla u)dx \tag{1.2}$$
$$\text{for every convex bounded open set } \Omega, u\in W^{1,\infty}(\Omega).$$

In the present paper, starting from the results of [8], we want to analyze more closely the functional $F^{(\infty)}$ and its relationships with $\overline{F}^{(\infty)}$ when the integrand f does not depend on x and s, is allowed to take also the value $+\infty$, and the set in which it is finite is not necessarily bounded.

Given $f : \mathbb{R}^n \to [0, +\infty]$, denoted by $\mathrm{sc}^- f$ the lower semicontinuous envelope of f, and by $\mathrm{co}\, f$ the convex hull of f (cf. Section 2), we establish, first of all, a finite representation formula for $f^{(\infty)}$ by means of these standard operations (cf. Proposition 3.3), i.e.

$$f^{(\infty)}(z) = \mathrm{co}(\mathrm{sc}^- f)(z) \quad \text{for every } z \in \mathbb{R}^n \tag{1.3}$$

and observe that formula (1.3) becomes of particular interest since it always true that (cf. (2.3))

$$f^{**}(z) = \mathrm{sc}^-(\mathrm{co}\, f)(z) \quad \text{for every } z \in \mathbb{R}^n.$$

Then we propose an example that points out an interesting phenomenon: contrary to what occurs when integrands taking only real values, or taking the value $+\infty$ outside a ball, are considered (cf. [13]), also in the case of upper semicontinuous integrands depending only on the z variable $f^{(\infty)}$ and f^{**} may be different (cf. Examples 3.4 and 3.5) and consequently $F^{(\infty)}$ may not agree with $\overline{F}^{(\infty)}$ (cf. Example 4.4).

Some sufficient conditions implying identity between $f^{(\infty)}$ and f^{**} are then proposed (cf. Section 4), for example we prove that $f^{(\infty)} = f^{**}$ if the convex envelope of $\{z \in \mathbb{R}^n : f(z) < +\infty\}$ is an affine set, or if it is a strictly convex set, or if $\lim_{z\to\infty} \frac{f(z)}{|z|} = +\infty$, or if $\{z \in \mathbb{R}^n : f(z) < +\infty\}$ is bounded. We also propose a geometric condition characterizing the convex subsets C of $\mathbb{R}^n$ such that for every f having C as the convex envelope of the finiteness set one has $f^{(\infty)} = f^{**}$ (cf. Theorem 3.13).

Obviously the identity $f^{(\infty)} = f^{**}$ trivially implies, by (1.1) and (1.2), that $F^{(\infty)}(\Omega, u) = \overline{F}^{(\infty)}(\Omega, u)$ for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$ such that $F^{(\infty)}(\Omega, u) < +\infty$.

In order to analyze the full identity between $F^{(\infty)}$ and $\overline{F}^{(\infty)}$, we give an example of a function f verifying the above quoted assumptions for which (1.1) holds, but such that for some convex bounded open set Ω, $u \in C^\infty(\mathbb{R}^n)$, $F^{(\infty)}(\Omega, u)$ is not finite, being $\int_\Omega f^{(\infty)}(\nabla u)dx$ finite (cf. Example 5.1). Then we are able to propose some sufficient conditions implying the validity of (1.1) on the whole $W^{1,\infty}(\Omega)$ (cf. Section 5). For example we prove that if f verifies the above mentioned assumptions yielding (1.1) and is bounded on the bounded subsets of $\{z \in \mathbb{R}^n : f(z) < +\infty\}$, then (1.1) holds for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$. We also prove the same result if $\{z \in \mathbb{R}^n : f(z) < +\infty\}$ is bounded, or if it is an affine set, or if $n = 1$.

Obviously the identities $f^{(\infty)} = f^{**}$ and, for a given convex bounded open set Ω, $F^{(\infty)}(\Omega, \cdot) = \int_\Omega f^{(\infty)}(\nabla \cdot)dx$ on the whole $W^{1,\infty}(\Omega)$ imply, by (1.2), that $F^{(\infty)}(\Omega, u) = \overline{F}^{(\infty)}(\Omega, u)$ for every $u \in W^{1,\infty}(\Omega)$. However in the last section, for sake of clearness, we explicitly state the results about this last identity.

Some of the results of the present paper have been announced in [6].

2. Notations and preliminary results of Convex Analysis

For every subset C of $\mathbb{R}^n$ we denote by $\mathrm{co}(C)$ the convex hull of C, i.e. the intersection of all the convex subsets of $\mathbb{R}^n$ containing C, and by $\mathrm{aff}(C)$ the affine hull of C, i.e. the intersection of all the affine subsets of $\mathbb{R}^n$ containing C. If C is also convex, we denote by $\mathrm{ri}(C)$ the relative interior of C, i.e. the set of the interior points of C, in the topology of

$\operatorname{aff}(C)$, once it is regarded as a subspace of $\operatorname{aff}(C)$, and by $\operatorname{rb}(C)$ the relative boundary of C, i.e. the set $\overline{C}\setminus\operatorname{ri}(C)$. When $\operatorname{aff}(C)=\mathbb{R}^n$ we write as usual $\operatorname{ri}(C)=C^\circ$ and $\operatorname{rb}(C)=\partial C$.

We recall that (cf. for example Theorem 6.2 and Theorem 6.1 in [15]), if C is a convex subset of $\mathbb{R}^n$, then $\overline{C}$ and $\operatorname{ri}(C)$ are convex subsets of $\mathbb{R}^n$ having the same affine hull and that, for every $z_0\in\overline{C}$, $z\in\operatorname{ri}(C)$, and $t\in[0,1[$, it results that $tz_0+(1-t)z\in\operatorname{ri}(C)$.

For every subset C of $\mathbb{R}^n$ we denote by I_C the indicator function of C defined by $I_C(z)=0$ if $z\in C$ and $I_C(z)=+\infty$ if $z\in\mathbb{R}^n\setminus C$.

For every $r>0$ and $x_0\in\mathbb{R}^n$, by $B_r(x_0)$ we denote the open ball of $\mathbb{R}^n$ centred in x_0 and with radius r, and by $Q_r(x_0)$ the open cube of $\mathbb{R}^n$ with faces parallel to the coordinate planes centred in x_0 and with sidelength r. We also set $Q_r=Q_r(0)$.

For every function $f\colon\mathbb{R}^n\to\,]-\infty,+\infty]$ we denote by $\operatorname{dom} f$ the effective domain of f, i.e. $\operatorname{dom} f=\{z\in\mathbb{R}^n : f(z)<+\infty\}$, by $\operatorname{sc}^- f$ the lower semicontinuous envelope of f, i.e.

$$\operatorname{sc}^- f\colon z\in\mathbb{R}^n\mapsto\liminf_{y\to z} f(y),$$

by $\operatorname{co} f$ the convex hull of f, i.e.

$$\operatorname{co} f\colon z\in\mathbb{R}^n\mapsto\sup\{\phi(z):\phi\colon\mathbb{R}^n\to]-\infty,+\infty]\text{ convex, }\phi\le f\text{ on }\mathbb{R}^n\},$$

and by f^{**} the bipolar of f, i.e. (cf. for example [12], Proposition 4.1 at page 18)

$$f^{**}\colon z\in\mathbb{R}^n\mapsto\sup\{\phi(z):\phi\colon\mathbb{R}^n\to\mathbb{R}\text{ affine, }\phi\le f\text{ on }\mathbb{R}^n\}.$$

Let $f\colon\mathbb{R}^n\to\,]-\infty,+\infty]$, then obviously $\operatorname{sc}^- f$ turns out to be lower semicontinuous, $\operatorname{co} f$ convex, f^{**} convex, lower semicontinuous and

$$f^{**}(z)\le\operatorname{co} f(z)\le f(z)\quad\text{for every } z\in\mathbb{R}^n. \tag{2.1}$$

Moreover we also have (cf. for example [12], Definition 3.2 and Proposition 3.1 in Chapter I)

$$f^{**}(z)=\sup\{\phi(z):\phi\colon\mathbb{R}^n\to\,]-\infty,+\infty]\text{ convex, lower semicontinuous, }\phi\le f\text{ on }\mathbb{R}^n\}\quad\text{for every } z\in\mathbb{R}^n \tag{2.2}$$

and (cf. for example Corollary 12.1.1 in [15])

$$f^{**}(z)=\operatorname{sc}^-(\operatorname{co} f)(z)\quad\text{for every } z\in\mathbb{R}^n. \tag{2.3}$$

The following representation result follows by Carathéodory Theorem (cf. Corollary 17.1.5 in [15]).

Theorem 2.1. *Let $f\colon\mathbb{R}^n\to\,]-\infty,+\infty]$, then*

$$\operatorname{co} f(z)=\inf\left\{\sum_{j=1}^{n+1}\lambda_j f(z_j):0\le\lambda_j\le 1\text{ for every } j\in\{1,\dots,n+1\},\ \sum_{j=1}^{n+1}\lambda_j=1,\sum_{j=1}^{n+1}\lambda_j z_j=z\right\}\quad\text{for every } z\in\mathbb{R}^n.$$

By Theorem 2.1 it follows that for every $f\colon \mathbb{R}^n \to]-\infty,+\infty]$ it results

$$\operatorname{dom}(\operatorname{co} f) = \operatorname{co}(\operatorname{dom} f). \tag{2.4}$$

The following theorem yields a representation result for f^{**}.

Theorem 2.2. *Let $f\colon \mathbb{R}^n \to]-\infty,+\infty]$ be lower semicontinuous, and assume that there exists a Borel function $\theta\colon [0,+\infty[\to [0,+\infty]$ with $\lim_{t\to+\infty}\theta(t)/t = +\infty$ such that $\theta(|z|) \leq f(z)$ for every $z \in \mathbb{R}^n$, then*

$$f^{**}(z) = \min\left\{\sum_{j=1}^{n+1}\lambda_j f(z_j) : 0 \leq \lambda_j \leq 1 \textit{ for every } j \in \{1,\dots,n+1\},\right.$$
$$\left.\sum_{j=1}^{n+1}\lambda_j = 1, \sum_{j=1}^{n+1}\lambda_j z_j = z\right\} \quad \textit{for every } z \in \mathbb{R}^n.$$

Proof. Similar to the one of Lemma 3.3 at page 280 of [12] in which $\theta\colon t \in [0,+\infty[\to t^p$ with $p > 1$. □

By Theorem 2.2 we deduce the following result.

Theorem 2.3. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$, and assume that $\lim_{z\to\infty}\frac{f(z)}{|z|} = +\infty$, then*

$$f^{**}(z) = \min\left\{\sum_{j=1}^{n+1}\lambda_j \operatorname{sc}^- f(z_j) : 0 \leq \lambda_j \leq 1 \textit{ for every } j \in \{1,\dots,n+1\},\right.$$
$$\left.\sum_{j=1}^{n+1}\lambda_j = 1, \sum_{j=1}^{n+1}\lambda_j z_j = z\right\} \quad \textit{for every } z \in \mathbb{R}^n.$$

Proof. Let us preliminarily observe that $f^{**} \leq \operatorname{sc}^- f \leq f$ from which, being clearly $(f^{**})^{**} = f^{**}$, it follows that

$$f^{**}(z) = (\operatorname{sc}^- f)^{**}(z) \quad \text{for every } z \in \mathbb{R}^n. \tag{2.5}$$

By the assumption on f it is easy to deduce the existence of a continuous function $\theta\colon [0,+\infty[\to [0,+\infty]$ with $\lim_{t\to+\infty}\theta(t)/t = +\infty$ such that $\theta(|z|) \leq f(z)$ for every $z \in \mathbb{R}^n$, from which we conclude that

$$\theta(|z|) \leq \operatorname{sc}^- f(z) \quad \text{for every } z \in \mathbb{R}^n. \tag{2.6}$$

At this point we observe that, by (2.6), the assumptions of Theorem 2.2 are fulfilled by $\operatorname{sc}^- f$, therefore by (2.5) and Theorem 2.2 we obtain that

$$f^{**}(z) = (\operatorname{sc}^- f)^{**}(z) = \min\left\{\sum_{j=1}^{n+1}\lambda_j \operatorname{sc}^- f(z_j) : 0 \leq \lambda_j \leq 1 \text{ for every } j \in \{1,\dots,n{+}1\},\right.$$
$$\left.\sum_{j=1}^{n+1}\lambda_j = 1, \sum_{j=1}^{n+1}\lambda_j z_j = z\right\} \quad \text{for every } z \in \mathbb{R}^n,$$

that is the thesis. □

Proposition 2.4. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$, then*

$$\begin{aligned} \mathrm{ri}(\mathrm{dom}\, f^{**}) &= \mathrm{ri}(\mathrm{dom}(\mathrm{co}\, f)) = \mathrm{ri}(\mathrm{co}(\mathrm{dom}\, f)), \\ \mathrm{rb}(\mathrm{dom}\, f^{**}) &= \mathrm{rb}(\mathrm{dom}(\mathrm{co}\, f)) = \mathrm{ri}(\mathrm{co}(\mathrm{dom}\, f)) \end{aligned} \tag{2.7}$$

and

$$f^{**}(z) = \mathrm{co}\, f(z) \quad \text{for every } z \in \mathbb{R}^n \setminus \mathrm{rb}(\mathrm{co}(\mathrm{dom}\, f)). \tag{2.8}$$

Proof. Equalities in (2.7) follow by (2.3), Corollary 7.4.1 in [15], and (2.4), whilst (2.8) by Theorem 7.4 in [15], and (2.4). □

In particular, given $f\colon \mathbb{R}^n \to [0,+\infty]$, by Proposition 2.4 we deduce that

$$\mathrm{dom}\, f^{**} \subseteq \overline{\mathrm{co}(\mathrm{dom}\, f)}. \tag{2.9}$$

Let C be a convex subset of $\mathbb{R}^n$. A supporting half-space to C is a closed half-space containing C and having a point of C in its boundary. A non-trivial supporting hyperplane to C is a hyperplane not containing C which is the boundary of a supporting half-space to C.

The following result is well known (cf. for example Theorem 11.6 in [15]).

Theorem 2.5. *Let C be a convex subset of $\mathbb{R}^n$, and let $z \in C$, then there exists a non-trivial supporting hyperplane to C containing z if and only if $z \notin \mathrm{ri}(C)$.*

3. Some new results of Convex Analysis: the function $f^{(\infty)}$

In [8], for every $f\colon \mathbb{R}^n \to [0,+\infty]$, the function $f^{(\infty)}$ given by

$$f^{(\infty)}\colon z \in \mathbb{R}^n \mapsto \inf_{m\in\mathbb{N}} (f + I_{Q_m})^{**}(z) \tag{3.1}$$

has been introduced.

In the present section, given $f\colon \mathbb{R}^n \to [0,+\infty]$, we want to carry out the study of $f^{(\infty)}$ and, in particular, of its relationships with f^{**}. We first compare $f^{(\infty)}$ with f^{**} and $\mathrm{co}\, f$, then prove that, similarly to f^{**} (cf. (2.3)), also $f^{(\infty)}$ can be deduced by f by means of convexification and relaxation operations, and observe that, in general, they are different. Finally we propose some conditions in order to have identity.

Let us first recall the following elementary properties of $f^{(\infty)}$ already established in [8] (cf. Proposition 6.3 in [8]).

Proposition 3.1. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$, then $f^{(\infty)}$ is convex, Borel and*

$$f^{**}(z) \le f^{(\infty)}(z) \le \mathrm{co}\, f(z) \quad \text{for every } z \in \mathbb{R}^n, \tag{3.2}$$

$$\begin{aligned} \mathrm{ri}(\mathrm{dom}\, f^{**}) &= \mathrm{ri}(\mathrm{dom}\, f^{(\infty)}) = \mathrm{ri}(\mathrm{dom}(\mathrm{co}\, f)), \\ \mathrm{rb}(\mathrm{dom}\, f^{**}) &= \mathrm{rb}(\mathrm{dom}\, f^{(\infty)}) = \mathrm{rb}(\mathrm{dom}(\mathrm{co}\, f)), \end{aligned} \tag{3.3}$$

$$f^{**}(z) = f^{(\infty)}(z) = \mathrm{co}\, f(z) \quad \text{for every } z \in \mathbb{R}^n \setminus \mathrm{rb}(\mathrm{dom}\, f^{(\infty)}). \tag{3.4}$$

Remark 3.2. We observe explicitly that, given $f\colon \mathbb{R}^n \to [0,+\infty]$, the definition of $f^{(\infty)}$ does not depend on the choice of the sets involved, in fact it is easy to see that

$$\inf_{m\in\mathbb{N}}(f+I_{Q_m})^{**}=\inf_{m\in\mathbb{N}}(f+I_{z_0+m(A-z_0)})^{**}\quad\text{whenever } A \text{ is a bounded open set, } z_0\in A.$$

Proposition 3.3. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$, then*

$$f^{(\infty)}(z)=\mathrm{co}(\mathrm{sc}^- f)(z)\quad \textit{for every } z\in\mathbb{R}^n.$$

Proof. It is clear that

$$\begin{gathered} f^{(\infty)}(z)\le (f+I_{Q_m})^{**}(z)\le \mathrm{sc}^-(f+I_{Q_m})(z)=\mathrm{sc}^- f(z)\\ \text{for every } z\in\mathbb{R}^n, m\in\mathbb{N} \text{ with } z\in Q_m, \end{gathered}\tag{3.5}$$

therefore, being $f^{(\infty)}$ convex, by (3.5) we deduce that

$$f^{(\infty)}(z)\le \mathrm{co}(\mathrm{sc}^- f)(z)\quad \text{for every } z\in\mathbb{R}^n.\tag{3.6}$$

In order to prove the reverse inequality to (3.6), we fix $z\in\mathbb{R}^n$ and $m\in\mathbb{N}$, then by Theorem 2.3 and Theorem 2.1 we get $z_1^m,\dots,z_{n+1}^m\in\mathbb{R}^n$, $\lambda_1^m,\dots,\lambda_{n+1}^m\in[0,1]$, with $\sum_{j=1}^{n+1}\lambda_j^m=1$, $\sum_{j=1}^{n+1}\lambda_j^m z_j^m=z$, such that

$$\begin{gathered} (f+I_{Q_m})^{**}(z)=\sum_{j=1}^{n+1}\lambda_j^m\,\mathrm{sc}^-(f+I_{Q_m})(z_j^m)\ge\sum_{j=1}^{n+1}\lambda_j^m\,\mathrm{sc}^- f(z_j^m)\ge \mathrm{co}(\mathrm{sc}^- f)(z)\\ \text{for every } m\in\mathbb{N}, \end{gathered}\tag{3.7}$$

therefore by (3.1) and (3.7) we conclude as m diverges that

$$f^{(\infty)}(z)\ge \mathrm{co}(\mathrm{sc}^- f)(z)\quad \text{for every } z\in\mathbb{R}^n.\tag{3.8}$$

By (3.6) and (3.8) the thesis follows. □

Example 3.4. Let $n=2$, and let f be defined by

$$f\colon (z_1,z_2)\in\mathbb{R}^2\mapsto\begin{cases} z_2-z_1e^{z_2} & \text{if } z_2\ge 0 \text{ and } 0<z_1\le z_2e^{-z_2}\\ 0 & \text{if } z_2\ge 0 \text{ and } z_2e^{-z_2}<z_1\\ +\infty & \text{otherwise,}\end{cases}$$

then $\mathrm{dom}\, f$ is convex, f is upper semicontinuous in $\mathbb{R}^2$ and locally Lipschitz in $\mathrm{dom}\, f$. Moreover, by (2.3), it is clear that

$$f^{**}(z_1,z_2)=\mathrm{sc}^-(\mathrm{co}\, f)(z_1,z_2)=\begin{cases} 0 & \text{if } z_1\ge 0 \text{ and } z_2\ge 0\\ +\infty & \text{otherwise}\end{cases}\quad \text{for every } (z_1,z_2)\in\mathbb{R}^2,$$

whilst, by Proposition 3.3, we it is easy to see that

$$f^{(\infty)}(z_1,z_2)=\mathrm{co}(\mathrm{sc}^- f)(z_1,z_2)=\begin{cases} z_2 & \text{if } z_2\ge 0 \text{ and } z_1=0\\ 0 & \text{if } z_2\ge 0 \text{ and } z_1>0\\ +\infty & \text{otherwise}\end{cases}\quad \text{for every } (z_1,z_2)\in\mathbb{R}^2.$$

Note that in this case $f^{(\infty)}$ is convex but not lower semicontinuous.

In the example below we observe that $f^{(\infty)}$ and f^{**} can be different also when f is bounded in $\operatorname{dom} f$, and $\operatorname{dom} f$ is very regular.

Example 3.5. Let $n = 2$, and let f be defined by

$$f\colon (z_1, z_2) \in \mathbb{R}^2 \mapsto \begin{cases} +\infty & \text{if } z_1 \le 0 \\ 1 - z_1 e^{z_2^2} & \text{if } 0 < z_1 \le e^{-z_2^2} \\ 0 & \text{if } z_1 > e^{-z_2^2}, \end{cases}$$

then $\operatorname{dom} f$ is convex, f is bounded and upper semicontinuous in $\mathbb{R}^2$, locally Lipschitz in $\operatorname{dom} f$. Moreover it is clear that

$$f^{**}(z_1, z_2) = \begin{cases} +\infty & \text{if } z_1 < 0 \\ 0 & \text{if } z_1 \ge 0 \end{cases} \quad \text{for every } (z_1, z_2) \in \mathbb{R}^2,$$

whilst $f^{(\infty)}$ is given by

$$f^{(\infty)}(z_1, z_2) = \begin{cases} +\infty & \text{if } z_1 < 0 \\ 1 & \text{if } z_1 = 0 \\ 0 & \text{if } z_1 > 0 \end{cases} \quad \text{for every } (z_1, z_2) \in \mathbb{R}^2.$$

Also in this case $f^{(\infty)}$ is convex but not lower semicontinuous.

We now propose some conditions in order to have identity between $f^{(\infty)}$ and f^{**}.

Remark 3.6. Let $f\colon \mathbb{R}^n \to [0, +\infty]$, then by using the convexity of $f^{(\infty)}$ and (2.2) we deduce that the following conditions are equivalent

$$f^{(\infty)}(z) = f^{**}(z) \quad \text{for every } z \in \mathbb{R}^n,$$

$$f^{(\infty)} \text{ is lower semicontinuous.}$$

Proposition 3.7. *Let* $f\colon \mathbb{R}^n \to [0, +\infty]$*, and assume that* $\operatorname{co}(\operatorname{dom} f)$ *is an affine set, then* $f^{(\infty)} = f^{**}$*.*

In particular $f^{(\infty)} = f^{**}$ *if* $\operatorname{co}(\operatorname{dom} f) = \mathbb{R}^n$*, or if* $\operatorname{dom} f = \mathbb{R}^n$*.*

Proof. By (2.4), $\operatorname{co} f$ turns out to be convex and finite in $\operatorname{co}(\operatorname{dom} f)$. Therefore, being by our assumptions $\operatorname{co}(\operatorname{dom} f) = \operatorname{ri}(\operatorname{co}(\operatorname{dom} f))$, it turns out to be continuous in $\operatorname{co}(\operatorname{dom} f)$.

On the other side our assumptions imply also that $\operatorname{co}(\operatorname{dom} f)$ is closed. This, together with the continuity of $\operatorname{co} f$ in $\operatorname{co}(\operatorname{dom} f)$, yields the lower semicontinuity of $\operatorname{co} f$ on the whole $\mathbb{R}^n$, and hence that $\operatorname{co} f(z) \le f^{**}(z)$ for every $z \in \mathbb{R}^n$.

By virtue of this and (2.1) the first part of the thesis follows.

The second part follows by the first one, being by (2.1) $\operatorname{dom} f \subseteq \operatorname{co}(\operatorname{dom} f)$. $\square$

Proposition 3.8. *Let* $f\colon \mathbb{R}^n \to [0, +\infty]$*, and assume that* $\lim_{z \to \infty} \frac{f(z)}{|z|} = +\infty$*, then* $f^{(\infty)} = f^{**}$*.*

Proof. By Theorem 2.3, Theorem 2.1 and Proposition 3.3 we obtain that

$$f^{**}(z) = \min\left\{ \sum_{j=1}^{n+1} \lambda_j \operatorname{sc}^- f(z_j) : 0 \leq \lambda_j \leq 1 \text{ for every } j \in \{1, \dots, n+1\}, \right.$$
$$\left. \sum_{j=1}^{n+1} \lambda_j = 1, \sum_{j=1}^{n+1} \lambda_j z_j = z \right\} = \operatorname{co}(\operatorname{sc}^- f)(z) = f^{(\infty)}(z) \quad \text{for every } z \in \mathbb{R}^n,$$

that is the thesis. □

Remark 3.9. Let $f\colon \mathbb{R}^n \to [0, +\infty]$, then by using Proposition 3.1 and (2.4) we deduce that

$$f^{**}(z) = f^{(\infty)}(z) = \operatorname{co} f(z) \quad \text{for every } z \in \mathbb{R}^n \setminus \operatorname{rb}(\operatorname{co}(\operatorname{dom} f)),$$

therefore, to prove identity between $f^{(\infty)}$ and f^{**}, we have to prove only their coincidence in $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$.

In the following results, given $f\colon \mathbb{R}^n \to [0, +\infty]$, we prove that coincidence of $f^{(\infty)}$ with f^{**} depends, in some cases, only on some geometric properties of $\operatorname{dom} f$. We characterize the convex subsets of $\mathbb{R}^n$ that are convex envelopes of effective domains of functions for which such coincidence holds.

We start with some results of local nature.

Proposition 3.10. *Let $f\colon \mathbb{R}^n \to [0, +\infty]$, $z_0 \in \operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$, and assume that there exists a non-trivial supporting hyperplane to $\operatorname{co}(\operatorname{dom} f)$ having bounded intersection with $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ and containing z_0, then*

$$f^{(\infty)}(z_0) = f^{**}(z_0). \tag{3.9}$$

Proof. Let H be the non-trivial supporting hyperplane to $\operatorname{co}(\operatorname{dom} f)$ having bounded intersection with $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ and containing z_0, Σ be the closed half-space containing $\operatorname{co}(\operatorname{dom} f)$ whose boundary is H and $r > 0$ be such that

$$H \cap \operatorname{rb}(\operatorname{co}(\operatorname{dom} f)) \subseteq B_r(z_0). \tag{3.10}$$

Let $m \in \mathbb{N}$ be such that $B_{2r}(z_0) \subseteq Q_m$, α be an affine function with $\alpha(z) \leq (f + I_{Q_m})(z)$ for every $z \in \mathbb{R}^n$, $\eta \in \mathbb{R}$ with $\eta < \min\{\alpha(z_0), 0\}$, and, for every $\tau > 0$, let α_τ be an affine function verifying

$$\begin{aligned}
&\alpha_{\tau_2}(z) < \alpha_{\tau_1}(z) < \alpha(z) \text{ for every } \tau_1, \tau_2 \in]0, +\infty[\text{ with } \tau_1 < \tau_2, z \in \Sigma^\circ, \\
&\lim_{\tau \to +\infty} \alpha_\tau(z) = -\infty \text{ for every z } \in \Sigma^\circ, \\
&\alpha_\tau(z) = \alpha(z) \text{ for every } \tau > 0, z \in H.
\end{aligned} \tag{3.11}$$

Moreover, for every $\tau > 0$, let us set $P_\tau = \{z \in \mathbb{R}^n : \alpha_\tau(z) = \eta\}$, and denote by Σ_τ the closed half-space containing z_0 whose boundary is P_τ.

Let us prove that

$$\text{there exists } \tau_0 > 0 \text{ such that } \Sigma \cap \Sigma_{\tau_0} \cap \overline{\operatorname{co}(\operatorname{dom} f)} \subseteq B_{2r}(z_0). \tag{3.12}$$

To this aim we argue by contradiction. We assume that for every $h \in \mathbb{N}$ there exists $z_h \in \Sigma \cap \Sigma_h \cap \overline{\mathrm{co}(\mathrm{dom}\, f)}$ with $|z_h - z_0| \geq 2r$, then by the convexity of $\overline{\mathrm{co}(\mathrm{dom}\, f)}$ we get that

$$\xi_h = z_0 + 2r\frac{z_h - z_0}{|z_h - z_0|} \in \overline{\mathrm{co}(\mathrm{dom}\, f)} \quad \text{for every } h \in \mathbb{N}. \tag{3.13}$$

It is clear that $|\xi_h - z_0| = 2r$, that by (3.11) $\lim_h \mathrm{dist}(\xi_h, H) = 0$, and that there exist $\{\xi_{h_k}\} \subseteq \{\xi_h\}$ and $\xi \in \mathbb{R}^n$ such that $\lim_k \xi_{h_k} = \xi$. Then, once observed that $\overline{\mathrm{co}(\mathrm{dom}\, f)} \cap H = \mathrm{rb}(\mathrm{co}(\mathrm{dom}\, f)) \cap H$, by (3.13) it follows that $\xi \in \mathrm{rb}(\mathrm{co}(\mathrm{dom}\, f)) \cap H$ and $|\xi - z_0| = 2r$, contrary to (3.10).

Let τ_0 be given by (3.12), then, being $f(z) = +\infty$ for every $z \in \mathbb{R}^n \setminus \Sigma$ and $f(z) \geq 0 > \eta$ for every $z \in \Sigma$, it turns out that

$$\alpha_{\tau_0}(z) \leq f(z) \quad \text{for every } z \in (\mathbb{R}^n \setminus \Sigma) \cup (\mathbb{R}^n \setminus \Sigma_{\tau_0}). \tag{3.14}$$

Moreover, since $B_{2r}(z_0) \subseteq Q_m$, by (3.12) we get that $f(z) = +\infty$ for every $z \in (\Sigma \cap \Sigma_{\tau_0}) \setminus Q_m$ and hence, taking into account also (3.11), that

$$\alpha_{\tau_0}(z) \leq f(z) \quad \text{for every } z \in \Sigma \cup \Sigma_{\tau_0}. \tag{3.15}$$

In conclusion, by (3.14) and (3.15), we have that $\alpha_{\tau_0}(z) \leq f(z)$ for every $z \in \mathbb{R}^n$ from which, together with (3.11), we infer that

$$\alpha(z_0) = a_{\tau_0}(z_0) \leq f^{**}(z_0). \tag{3.16}$$

By (3.16), being α a generic affine function with $\alpha \leq f + I_{Q_m}$ on $\mathbb{R}^n$, we conclude that $(f + I_{Q_m})^{**}(z_0) \leq f^{**}(z_0)$ and, by (3.2) of Proposition 3.1, that

$$f^{**}(z_0) \leq f^{(\infty)}(z_0) \leq (f + I_{Q_m})^{**}(z_0) \leq f^{**}(z_0),$$

that is (3.9). □

In order to invert Proposition 3.10 let us first prove the following result.

Lemma 3.11. *Let C be a convex subset of $\mathbb{R}^n$, and H be a non-trivial supporting hyperplane to C, then $H \cap \mathrm{rb}(C)$ is unbounded if and only if $H \cap \mathrm{rb}(C)$ contains an half-line.*

Proof. It is clear that, if $H \cap \mathrm{rb}(C)$ contains an half-line, then $H \cap \mathrm{rb}(C)$ is unbounded.

Conversely let us assume that $H \cap \mathrm{rb}(C)$ is unbounded, let $z_0 \in H \cap \mathrm{rb}(C)$ and observe that it is not restrictive to assume that $z_0 = 0$.

For every $h \in \mathbb{N}$ there exists $z_h \in H \cap \mathrm{rb}(C)$ with $|z_h| > h$ and set $\xi_h = z_h/|z_h|$, then, being $0 \in H \cap \overline{C}$, by the convexity of $H \cap \overline{C}$ we deduce that $\xi_h \in H \cap \overline{C}$ for every $h \in \mathbb{N}$. Let $\xi_0 \in \mathbb{R}^n$ be such that $|\xi_0| = 1$ and, up to subsequences, $\xi_h \to \xi_0$, then, being $H \cap \overline{C}$ closed, we get also that $\xi_0 \in H \cap \overline{C}$.

Let us prove that the half-line $\{t\xi_0 : t \geq 0\}$ is contained in $H \cap \overline{C}$, this will conclude the proof since $H \cap \overline{C} = H \cap \mathrm{rb}(C)$.

Let $t > 0$, then it is clear that $t\xi_0 \in H$, so we only have to prove that $t\xi_0 \in \overline{C}$.

Let $r > 0$, and take $h \in \mathbb{N}$ be such that $|z_h| > t$ and $\xi_h \in B_{r/(2t)}(\xi_0)$, then, being $0 \in \overline{C}$, by the convexity of $\overline{C}$ we conclude that $t\xi_h = \frac{t}{|z_h|} z_h \in \overline{C}$ and that $t\xi_h \in B_{r/2}(t\xi_0)$. By virtue of this we infer that $B_r(t\xi_0) \cap \mathrm{ri}(C) \neq \emptyset$ for every $r > 0$, i.e. $t\xi_0 \in \overline{C}$. □

Proposition 3.12. *Let C be a convex subset of $\mathbb{R}^n$, H be a non-trivial supporting hyperplane to C, and assume that $f^{(\infty)}(z) = f^{**}(z)$ for every $f\colon \mathbb{R}^n \to [0, +\infty]$ with $\mathrm{co}(\mathrm{dom}\, f) = C$ and every $z \in H \cap \mathrm{rb}(C)$, then $H \cap \mathrm{rb}(C)$ is bounded.*

Proof. If $n = 1$ the thesis is certainly true since $\mathrm{rb}(C)$ is empty or bounded.

If $n > 1$ let us prove that if $H \cap \mathrm{rb}(C)$ is unbounded, then

$$\begin{gathered} \text{there exist } f\colon \mathbb{R}^n \to [0,+\infty] \text{ with } \mathrm{co}(\mathrm{dom}\, f) = C \text{ and } \overline{z} \in H \cap \mathrm{rb}(C) \\ \text{such that } f^{(\infty)}(\overline{z}) \neq f^{**}(\overline{z}). \end{gathered} \tag{3.17}$$

To this aim let l be the half-line with $l \subseteq H \cap \mathrm{rb}(C)$ given by Lemma 3.11, and assume for the moment that $H = \{z \in \mathbb{R}^n : z_1 = 0\}$, $\{z \in \mathbb{R}^n : z_1 = z_2 = \ldots = z_{n-1} = 0, z_n \geq -1\} \subseteq l$ and that $C \subseteq \{z \in \mathbb{R}^n : z_1 \geq 0\}$.

As in Example 3.4, let f_0 be given by

$$f_0\colon (y_1, y_2) \in \mathbb{R}^2 \mapsto \begin{cases} y_2 - y_1 e^{y_2} & \text{if } y_2 > 0 \text{ and } 0 \leq y_1 < y_2 e^{-y_2} \\ 0 & \text{if } y_1 \geq \max\{y_2 e^{-y_2}, 0\} \\ +\infty & \text{if } y_1 < 0, \end{cases}$$

and set

$$f\colon (z_1, \ldots, z_n) \in \mathbb{R}^n \mapsto f_0(z_1, z_n) + I_C(z_1, \ldots, z_n),$$

then $\mathrm{co}(\mathrm{dom}\, f) = \mathrm{dom}\, f = C$.

Let $\tilde{z} \in \mathrm{ri}(C)$ with $\tilde{z}_n = 0$, and set $S = \{t\tilde{z} + (1-t)z : z \in l \text{ with } z_n \geq 0, t \in]0, 1]\}$, then it is clear that $S \subseteq \mathrm{ri}(C)$ and hence that

$$\begin{gathered} \text{for every } z \in S \text{ there exist } \xi_1, \xi_2 \in S, \tau \in [0,1] \\ \text{such that } z = (1-\tau)\xi_1 + \tau\xi_2 \text{ and } f(\xi_1) = f(\xi_2) = 0, \end{gathered} \tag{3.18}$$

therefore by the convexity of $\mathrm{co}\, f$, (2.1) and (3.18) and we conclude that

$$\begin{gathered} \mathrm{co}\, f(z) \leq (1-\tau)\,\mathrm{co}\, f(\xi_1) + \tau\,\mathrm{co}\, f(\xi_2) \leq (1-\tau) f(\xi_1) + \tau f(\xi_2) = 0 \\ \text{for every } z \in S \end{gathered} \tag{3.19}$$

and, by (2.3) and (3.19), that

$$f^{**}(0, 0, \ldots, 0, z_n) = 0 \quad \text{for every } z_n > 0. \tag{3.20}$$

Let now $m \in \mathbb{N}$, and observe that the affine function $\alpha_m\colon (z_1, \ldots, z_n) \in \mathbb{R}^n \mapsto z_n - e^{m/2} z_1$ is such that $\alpha_m \leq f_0 + I_{Q_m} \leq f + I_{Q_m}$ on $\mathbb{R}^n$, and that this yields

$$\begin{gathered} \overline{z}_n = \alpha_m(\overline{z}) \leq (f + I_{Q_m})^{**}(\overline{z}) \\ \text{for every } m \in \mathbb{N}, \overline{z} \in \{z \in \mathbb{R}^n : z_1 = z_2 = \ldots = z_{n-1} = 0, 0 \leq z_n \leq m/2\}. \end{gathered} \tag{3.21}$$

In conclusion by (3.20), (3.21) and (3.1) we obtain that

$$f^{**}(\overline{z}) < f^{(\infty)}(\overline{z}) \quad \text{for every } \overline{z} \in \{z \in \mathbb{R}^n : z_1 = z_2 = \ldots = z_{n-1} = 0, z_n > 0\}$$

provided that $H = \{z \in \mathbb{R}^n : z_1 = 0\}$, $\{z \in \mathbb{R}^n : z_1 = z_2 = \ldots = z_{n-1} = 0, z_n \geq -1\} \subseteq l$, and that $C \subseteq \{z \in \mathbb{R}^n : z_1 \geq 0\}$.

In order to prove (3.17) in the general case let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a one to one affine mapping such that $A(H) = \{\zeta \in \mathbb{R}^n : \zeta_1 = 0\}$, $A(l) \supseteq \{\zeta \in \mathbb{R}^n : \zeta_1 = \zeta_2 = \ldots = \zeta_{n-1} = 0, \zeta_n \geq -1\}$, and $A(C) \subseteq \{\zeta \in \mathbb{R}^n : \zeta_1 \geq 0\}$, then by (3.17) in the just considered particular case we deduce the existence of a function $g\colon \mathbb{R}^n \to [0, +\infty]$ with $\mathrm{co}(\mathrm{dom}\, g) = A(C)$ such that

$$g^{(\infty)}(\overline{\zeta}) > g^{**}(\overline{\zeta}) \quad \text{for some } \overline{\zeta} \in A(H) \cap A(C), \tag{3.22}$$

and set $f = g(A(\cdot))$.

By Theorem 2.1 it is not difficult to verify that $\mathrm{co}\, f(z) = \mathrm{co}\, g(A(z))$ for every $z \in \mathbb{R}^n$, from which, together with (2.3), we conclude that

$$f^{**}(z) = g^{**}(A(z)) \quad \text{for every } z \in \mathbb{R}^n. \tag{3.23}$$

Analogously, for every $m \in \mathbb{N}$, we have that $f + I_{Q_m} = g(A(\cdot)) + I_{A(Q_m)}(A(\cdot))$, and therefore that $(f + I_{Q_m})^{**} = (g + I_{A(Q_m)})^{**}(A(\cdot))$. Therefore by (3.1) and Remark 3.2, we infer that

$$\begin{aligned} f^{(\infty)}(z) &= \inf_{m\in\mathbb{N}} (f + I_{Q_m})^{**}(z) = \inf_{m\in\mathbb{N}} (g + I_{A(Q_m)})^{**}(A(z)) = \inf_{m\in\mathbb{N}} (g + I_{Q_m})^{**}(A(z)) \\ &= g^{(\infty)}(A(z)) \quad \text{for every } z \in \mathbb{R}^n. \end{aligned} \tag{3.24}$$

By (3.24), (3.22) and (3.23) we obtain that

$$f^{(\infty)}(\overline{z}) > f^{**}(\overline{z}) \quad \text{for some } \overline{z} \in H \cap C,$$

from which (3.17) and the thesis follow. □

By the previous results we deduce the following characterization of global nature.

Theorem 3.13. *Let C be a convex subset of $\mathbb{R}^n$, then the following conditions are equivalent:*

(i) *for every $z_0 \in \mathrm{rb}(C)$ there exists a non-trivial supporting hyperplane H to C containing z_0 such that $H \cap \mathrm{rb}(C)$ is bounded,*
(ii) *$f^{(\infty)} = f^{**}$ for every $f\colon \mathbb{R}^n \to [0, +\infty]$ with $\mathrm{co}(\mathrm{dom}\, f) = C$,*
(iii) *for every non-trivial supporting hyperplane H to C, $H \cap \mathrm{rb}(C)$ is bounded.*

Proof. Let us prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

It is clear that (i), together with Remark 3.9 and Proposition 3.10, implies (ii), and that, by Proposition 3.12, (iii) follows by (ii).

Finally let $z_0 \in \mathrm{rb}(C)$, and let H be the non-trivial supporting hyperplane to C containing z_0 given by Theorem 2.5, then (iii) yields (i). □

By Theorem 3.13 we deduce the following corollary.

Corollary 3.14. *Let* $f\colon \mathbb{R}^n \to [0,+\infty]$*, and assume that* $\operatorname{dom} f$ *is bounded, then* $f^{(\infty)} = f^{**}$.

Proof. Follows by Theorem 3.13 once observed that if $\operatorname{dom} f$ is bounded, so is also $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$. □

Let C be a convex set, we recall that C is said to be strictly convex if for every $z_1, z_2 \in \operatorname{rb}(C)$ with $z_1 \neq z_2$ and $t \in]0,1[$, it results $tz_1 + (1-t)z_2 \in \operatorname{ri}(C)$ or, equivalently, if every point of $\operatorname{rb}(C)$ is an extremal point of $\overline{C}$.

Corollary 3.15. *Let* $f\colon \mathbb{R}^n \to [0,+\infty]$*, and assume that* $\operatorname{co}(\operatorname{dom} f)$ *is strictly convex, then* $f^{(\infty)} = f^{**}$.

Proof. Follows by Theorem 3.13 once observed that if $\operatorname{co}(\operatorname{dom} f)$ is strictly convex, then for every non-trivial supporting hyperplane H to $\operatorname{co}(\operatorname{dom} f)$, $H \cap \operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ consists of only one point. □

Corollary 3.16. *Let* $f\colon \mathbb{R} \to [0,+\infty]$*, then* $f^{(\infty)} = f^{**}$.

Proof. Let us observe that in one dimension $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ can be empty or made up by one or two points.

If it is empty then $\operatorname{co}(\operatorname{dom} f) = \mathbb{R}$ and the thesis follows by Proposition 3.7, otherwise $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ is bounded and Theorem 3.13 applies. □

We conclude this section with a result that will be useful in the sequel.

Proposition 3.17. *Let* $f\colon \mathbb{R}^n \to [0,+\infty]$ *be bounded on the bounded subsets of* $\operatorname{dom} f$*, and such that* $\operatorname{dom} f$ *is convex, then for every open set* A *it results that*

$$\operatorname{dom} f^{(\infty)} \cap A \subseteq \operatorname{dom}(f + I_A)^{**}.$$

Proof. Let us preliminarly prove that the boundedness of f on the bounded subsets of $\operatorname{dom} f$ implies that

$$\overline{\operatorname{dom} f} \cap A \subseteq \operatorname{dom}(f + I_A)^{**}. \tag{3.25}$$

To do this, we observe that if $z \in \overline{\operatorname{dom} f} \cap A$ and $\{z_h\} \subseteq \operatorname{dom} f \cap A$ is such that $z_h \to z$, then by the lower semicontinuity of $(f + I_A)^{**}$, (2.1), and the boundedness of f on the bounded subsets of $\operatorname{dom} f$, we infer that

$$(f + I_A)^{**}(z) \leq \liminf_h (f + I_A)^{**}(z_h) \leq \liminf_h (f + I_A)(z_h) = \liminf_h f(z_h) < +\infty$$
$$\text{for every } z \in \overline{\operatorname{dom} f} \cap A,$$

from which inclusion in (3.25) follows.

At this point by (3.2) of Proposition 3.1, (2.9), the convexity of $\operatorname{dom} f$, and (3.25) we conclude that

$$\operatorname{dom} f^{(\infty)} \cap A \subseteq \operatorname{dom} f^{**} \cap A \subseteq \overline{\operatorname{dom} f} \cap A \subseteq \operatorname{dom}(f + I_A)^{**},$$

that is the thesis. □

4. Recalls and preliminary results of Calculus of Variations

For every $z \in \mathbb{R}^n$ we denote by u_z the function defined by $u_z\colon x \in \mathbb{R}^n \mapsto z \cdot x$.

Let Ω be an open set. Given $\{u_h\} \subseteq W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega)$, we say that $\{u_h\}$ converges to u in $w^* - W^{1,\infty}(\Omega)$, and write $u_h \to u$ in $w^* - W^{1,\infty}(\Omega)$, if $\{u_h\}$ converges to u in $L^\infty(\Omega)$ and $\{\nabla u_h\}$ converges to ∇u weakly* in $(L^\infty(\Omega))^n$.

We recall the following lower semicontinuity result (cf. for example Corollary 3.4.2 in [3]).

Proposition 4.1. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be convex, lower semicontinuous, and Ω be an open set, then the functional $u \in W^{1,\infty}(\Omega) \mapsto \int_\Omega f(\nabla u)dx$ is sequentially $w^* - W^{1,\infty}(\Omega)$-lower semicontinuous.*

Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function.

For every bounded open set Ω let us denote by $\overline{F}^{(\infty)}(\Omega,\cdot)$ the greatest sequentially weak$^*-$ $W^{1,\infty}(\Omega)$-lower semicontinuous functional less than or equal to the integral $F(\Omega,\cdot)\colon u \in W^{1,\infty}(\Omega) \mapsto \int_\Omega f(\nabla u)dx$, and by $F^{(\infty)}(\Omega,\cdot)$ the one defined by

$$F^{(\infty)}(\Omega,\cdot)\colon u \in W^{1,\infty}(\Omega) \mapsto \\ \inf\left\{ \liminf_h \int_\Omega f(\nabla u_h)dx : \{u_h\} \subseteq W^{1,\infty}_{loc}(\mathbb{R}^n), u_h \to u \text{ in } w^* - W^{1,\infty}(\Omega)\right\}. \tag{4.1}$$

When f may take he value $+\infty$ the study of $F^{(\infty)}$ is carried out in [8] where the following assumptions are introduced

$$\operatorname{dom} f \text{ is convex,} \tag{4.2}$$

$$f \text{ is locally bounded in } \operatorname{ri}(\operatorname{dom} f), \tag{4.3}$$

i.e. for every compact subset K of $\operatorname{ri}(\operatorname{dom} f)$ there exists $M_K > 0$ such that $f(z) \leq M_K$ for every $z \in K$,

$$\begin{gathered}\text{for every bounded subset } L \text{ of } \operatorname{dom} f \text{ there exists } z_L \in \operatorname{ri}(\operatorname{dom} f) \\ \text{such that the function } t \in [0,1] \mapsto f((1-t)z_L + tz) \text{ is} \\ \text{upper semicontinuous at } t=1 \text{ uniformly as } z \text{ varies in } L,\end{gathered} \tag{4.4}$$

i.e. for every $\varepsilon > 0$ there exists $t_\varepsilon < 1$ such that $f((1-t)z_L + tz) \leq f(z) + \varepsilon$ for every $t \in]t_\varepsilon, 1]$ and $z \in L$.

Remark 4.2. Assumption (4.4) looks like a sort of uniform radial upper semicontinuity on bounded subsets of $\operatorname{dom} f$, nevertheless it does not imply in general (4.3) (think for example to the case in which $n = 2$, $f(z_1,z_2) = |z_2|/|z_1|$ if $|z_1|^2 + |z_2|^2 \leq 1$ and $z_1 z_2 \neq 0$, $f(z_1,z_2) = 0$ if $|z_1|^2 + |z_2|^2 \leq 1$ and $z_1 z_2 = 0$, $f(z_1,z_2) = +\infty$ otherwise in $\mathbb{R}^2$, and $z_L = (0,0)$ independently on L). It is fulfilled if f is finite and continuous in $\mathbb{R}^n$ or if there exists $z_0 \in \operatorname{ri}(\operatorname{dom} f)$ such that the function $t \in [0,1] \mapsto f((1-t)z_0 + tz)$ is increasing for every $z \in \operatorname{dom} f$.

In [8] the following representation result is proved (cf. Theorem 6.1, Remark 6.2 and Proposition 6.3 in [8]).

Theorem 4.3. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function verifying* (4.2)÷(4.4)*,* $f^{(\infty)}$ *be given by* (3.1)*, and* $F^{(\infty)}$ *by* (4.1)*, then*

$$F^{(\infty)}(\Omega,u)=\lim_m \int_\Omega (f+I_{Q_m})^{**}(\nabla u)dx \geq \int_\Omega f^{(\infty)}(\nabla u)dx \quad \text{for every convex bounded open set } \Omega, u\in W^{1,\infty}(\Omega), \tag{4.5}$$

$$F^{(\infty)}(\Omega,u)=\lim_m \int_\Omega (f+I_{Q_m})^{**}(\nabla u)dx = \int_\Omega f^{(\infty)}(\nabla u)dx \quad \text{for every convex bounded open set } \Omega, u\in W^{1,\infty}(\Omega) \text{ such that } F^{(\infty)}(\Omega,u)<+\infty. \tag{4.6}$$

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$*, then*

$$F^{(\infty)}(\Omega,u)=\lim_m \int_\Omega (f+I_{Q_m})^{**}(\nabla u)dx \geq \int_\Omega f^{(\infty)}(\nabla u)dx \quad \text{for every bounded open set } \Omega, u\in W^{1,\infty}_{loc}(\mathbb{R}^n), \tag{4.7}$$

$$F^{(\infty)}(\Omega,u)=\lim_m \int_\Omega (f+I_{Q_m})^{**}(\nabla u)dx = \int_\Omega f^{(\infty)}(\nabla u)dx \quad \text{for every bounded open set } \Omega, u\in W^{1,\infty}_{loc}(\mathbb{R}^n) \text{ such that } F^{(\infty)}(\Omega,u)<+\infty. \tag{4.8}$$

Example 4.4. Let f be given by Example 3.4, then f fulfils (4.2)÷(4.4).

Let us prove that, given a bounded open set Ω, $F^{(\infty)}(\Omega,\cdot)$ is not even strongly $W^{1,\infty}(\Omega)$-lower semicontinuous.

To see this take $\overline{z}=(0,\overline{b})$ with $\overline{b}>0$, and $\{z_h\}\subseteq]0,+\infty[^2$ such that $z_h\to\overline{z}$, then $u_{z_h}\to u_{\overline{z}}$ in $W^{1,\infty}(\Omega)$, and by (4.7) of Theorem 4.3 we get that

$$F^{(\infty)}(\Omega,u_{\overline{z}}) \geq f^{(\infty)}(\overline{z})\operatorname{meas}(\Omega) > \liminf_h f^{(\infty)}(z_h)\operatorname{meas}(\Omega). \tag{4.9}$$

On the other side, since we have that

$$F^{(\infty)}(\Omega,u_{z_h}) \leq f(z_h)\operatorname{meas}(\Omega) < +\infty \quad \text{for every } h\in\mathbb{N},$$

by (4.9) and (4.8) of Theorem 4.3 we conclude that

$$F^{(\infty)}(\Omega,u_{\overline{z}}) > \liminf_h F^{(\infty)}(\Omega,u_{z_h}).$$

Note that in this case $F^{(\infty)}(\Omega,\cdot)$ cannot agree with $\overline{F}^{(\infty)}(\Omega,\cdot)$.

Proposition 4.5. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function verifying* (4.2)÷(4.4)*,* $F^{(\infty)}$ *be given by* (4.1)*. Assume that for every $z\in\mathbb{R}^n$,* $F^{(\infty)}(Q_1,\cdot)$ *is* $W^{1,\infty}(Q_1)$*-lower semicontinuous in* u_z*, then for every convex bounded open set* Ω*,* $F^{(\infty)}(\Omega,\cdot)$ *is sequentially* $w^*-W^{1,\infty}(\Omega)$*-lower semicontinuous.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$*, then for every bounded open set* Ω*,* $F^{(\infty)}(\Omega,\cdot)$ *is sequentially* $w^*-W^{1,\infty}(\Omega)$*-lower semicontinuous.*

Proof. Let us first prove that $f^{(\infty)}$ is lower semicontinuous.

To this aim let $\{z_h\} \subseteq \mathbb{R}^n$, $z \in \mathbb{R}^n$ with $z_h \to z$ and $\liminf_h f^{(\infty)}(z_h) < +\infty$. Moreover let, by (4.2), $z_0 \in \mathrm{ri}(\mathrm{dom}\, f)$ and $\{t_h\} \subseteq [0,1[$ with $t_h \to 1$, then by the convexity of $f^{(\infty)}$ we obtain that

$$\begin{aligned}\liminf_h f^{(\infty)}(t_h z_h + (1-t_h)z_0) &\leq \liminf_h \{t_h f^{(\infty)}(z_h) + (1-t_h) f^{(\infty)}(z_0)\} \\ &= \liminf_h f^{(\infty)}(z_h) < +\infty.\end{aligned} \tag{4.10}$$

For every $h \in \mathbb{N}$ we infer, by (4.2), that $t_h z_h + (1-t_h)z_0 \in \mathrm{ri}(\mathrm{dom}\, f)$. This implies that $F^{(\infty)}(Q_1, u_{t_h z_h + (1-t_h)z_0}) \leq f(t_h z_h + (1-t_h)z_0) < +\infty$, and hence, by Theorem 4.3, that

$$F^{(\infty)}(Q_1, u_{t_h z_h + (1-t_h)z_0}) = f^{(\infty)}(t_h z_h + (1-t_h)z_0) \quad \text{for every } h \in \mathbb{N}. \tag{4.11}$$

By the $W^{1,\infty}(Q_1)$-lower semicontinuity of $F^{(\infty)}(Q_1, \cdot)$ in u_z, (4.11) and (4.10) we obtain that

$$\begin{aligned}F^{(\infty)}(Q_1, u_z) &\leq \liminf_h F^{(\infty)}(Q_1, u_{t_h z_h + (1-t_h)z_0}) = \liminf_h f^{(\infty)}(t_h z_h + (1-t_h)z_0) \\ &\leq \liminf_h f^{(\infty)}(z_h) < +\infty\end{aligned}$$

from which we deduce that $F^{(\infty)}(Q_1, u_z)$ too is finite and, again by Theorem 4.3, that

$$f^{(\infty)}(z) = F^{(\infty)}(Q_1, u_z) \leq \liminf_h f^{(\infty)}(z_h),$$

i.e. the lower semicontinuity of $f^{(\infty)}$.

Finally by the convexity and the lower semicontinuity of $f^{(\infty)}$, Proposition 4.1 and Theorem 4.3, the thesis follows. □

5. Integral representation of $F^{(\infty)}$ on the whole space of Lipschitz functions

Let $f \colon \mathbb{R}^n \to [0, +\infty]$, $f^{(\infty)}$ be defined in (3.1), and $F^{(\infty)}$ be given by (4.1).

In this section we first prove that in some cases, even being the assumptions of Theorem 4.3 fulfilled, one can have $+\infty = F^{(\infty)}(\Omega, u) > \int_\Omega f^{(\infty)}(\nabla u)dx$ for some regular bounded open set Ω, u in $C^\infty(\mathbb{R}^n)$, and then propose some conditions in order to deduce identity between $F^{(\infty)}(\Omega, \cdot)$ and $\int_\Omega f^{(\infty)}(\nabla \cdot)dx$ on the whole $W^{1,\infty}(\Omega)$ for every convex bounded open set Ω, or on the whole $W^{1,\infty}_{loc}(\mathbb{R}^n)$ for every bounded open set Ω if $(\mathrm{dom}\, f)^\circ \neq \emptyset$.

Example 5.1. Let $n = 2$, and let f be defined by

$$f \colon (z_1, z_2) \in \mathbb{R}^2 \mapsto \begin{cases} +\infty & \text{if } z_1 \leq 0 \\ \frac{1}{z_1} - e^{z_2^2} & \text{if } 0 < z_1 \leq e^{-z_2^2} \\ 0 & \text{if } z_1 > e^{-z_2^2}, \end{cases}$$

then f is continuous and verifies (4.2)÷(4.4). Moreover, by Proposition 3.3, it is clear that

$$f^{(\infty)}(z_1, z_2) = \begin{cases} +\infty & \text{if } z_1 \leq 0 \\ 0 & \text{if } z_1 > 0 \end{cases} \quad \text{for every } (z_1, z_2) \in \mathbb{R}^2. \tag{5.1}$$

In addition let us also observe that

$$(f + I_{Q_m})^{**}(z_1, z_2) = \begin{cases} +\infty & \text{if } z_1 \leq 0 \text{ or } z_1 > m \text{ or } |z_2| > m \\ \frac{1}{z_1} - e^{m^2} & \text{if } 0 < z_1 \leq e^{-m^2} \text{ and } -m \leq z_2 \leq m \\ 0 & \text{if } e^{-m^2} < z_1 \leq m \text{ and } -m \leq z_2 \leq m \end{cases} \tag{5.2}$$
$$\text{for every } m \in \mathbb{N}, (z_1, z_2) \in \mathbb{R}^2.$$

Let $\Omega =]0,1[\times]-1,1[$, and $u\colon (x_1, x_2) \in \mathbb{R}^2 \mapsto x_1^2/2$, then by Theorem 4.3 and (5.2) it follows that $F^{(\infty)}(\Omega, u) = +\infty$ whilst, by (5.1), it results $\int_\Omega f^{(\infty)}(\nabla u)dx = 0$.

We now propose some sufficient conditions ensuring the validity of (4.6) and (4.8) of Theorem 4.3 without any finiteness restriction. More precisely that

$$F^{(\infty)}(\Omega, u) = \int_\Omega f^{(\infty)}(\nabla u)dx$$
$$\text{for every convex bounded open set } \Omega, u \in W^{1,\infty}(\Omega), \tag{5.3}$$

or, if $(\operatorname{dom} f)^\circ \neq \emptyset$, that

$$F^{(\infty)}(\Omega, u) = \int_\Omega f^{(\infty)}(\nabla u)dx \quad \text{for every bounded open set } \Omega, u \in W^{1,\infty}_{loc}(\mathbb{R}^n). \tag{5.4}$$

Proposition 5.2. *Let $f\colon \mathbb{R}^n \to [0, +\infty]$ be a Borel function verifying* (4.2)÷(4.4)*, and $F^{(\infty)}$ be given by* (4.1)*. Let Ω be a convex bounded open set, or simply a bounded open set if $(\operatorname{dom} f)^\circ \neq \emptyset$, let $u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$, and assume that one of the following conditions is fulfilled:*

(i) $\int_\Omega f(\nabla u)dx < +\infty$,
(ii) *there exists a compact set $K \subseteq \operatorname{ri}(\operatorname{dom} f)$ such that $\nabla u(x) \in K$ for a.e. $x \in K$,*
(iii) $\int_\Omega f^{(\infty)}(\nabla u)dx = +\infty$,

then

$$F^{(\infty)}(\Omega, u) = \int_\Omega f^{(\infty)}(\nabla u)dx.$$

Proof. If (i) holds, by (4.1) it results $F^{(\infty)}(\Omega, u) < +\infty$, and the thesis follows by Theorem 4.3.

If (ii) holds, by (4.3) condition (i), and hence the thesis, follow.

If (iii) holds, the thesis follows by Theorem 4.3. □

Proposition 5.3. *Let $f\colon \mathbb{R}^n \to [0, +\infty]$ be a Borel function verifying* (4.2)÷(4.4)*, and $F^{(\infty)}$ be given by* (4.1)*. Assume that* $\operatorname{dom} f$ *is bounded, then* (5.3) *holds.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$*, then* (5.4) *too holds.*

Proof. Let us first observe that, if $\operatorname{dom} f$ is bounded, then $f + I_{Q_m} = f$ for every $m \in \mathbb{N}$ sufficiently large, and therefore, by Theorem 4.3, that $F^{(\infty)}(\Omega, u) = \int_\Omega f^{**}(\nabla u)dx$ for every convex bounded open set Ω, $u \in W^{1,\infty}(\Omega)$ or, if $(\operatorname{dom} f)^\circ \neq \emptyset$, for every bounded open set Ω, $u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$.

By virtue of this and by Corollary 3.14 the thesis follows. □

Theorem 5.4. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function verifying* (4.2), (4.4), *and* $F^{(\infty)}$ *be given by* (4.1). *Assume that f is bounded on the bounded subsets of* $\operatorname{dom} f$, *then* (5.3) *holds.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$, *then* (5.4) *too holds.*

Proof. Let us prove (5.3), the proof of (5.4) being similar.

It is clear that, by our assumptions on f, condition (4.3) too follows.

Let Ω be a convex bounded open set, $u \in W^{1,\infty}(\Omega)$, then it is clear that, by Theorem 4.3, we have to treat only the case in which $F^{(\infty)}(\Omega, u) = +\infty$. If this is the case let $m_0 > \|\nabla u\|_{(L^\infty(\Omega))^n}$, then by Theorem 4.3 we get that $\int_\Omega (f + I_{Q_{m_0}})^{**}(\nabla u)dx = +\infty$ from which, taking into account the boundedness of f on the bounded subsets of $\operatorname{dom} f$, we conclude that

$$\begin{gathered}\text{there exists a measurable set } E \subseteq \Omega \text{ with positive measure}\\ \text{such that } \nabla u(x) \notin \operatorname{dom}(f + I_{Q_{m_0}})^{**} \text{ for a.e. } x \in E.\end{gathered} \tag{5.5}$$

By (5.5) and Proposition 3.17, applied with $A = Q_{m_0}$, we deduce that $\nabla u(x) \notin \operatorname{dom} f^{(\infty)}$ for a.e. $x \in E$, and hence that $\int_\Omega f^{(\infty)}(\nabla u)dx = +\infty$, from which (5.3) follows. □

By Theorem 5.4 we deduce the following corollaries.

Corollary 5.5. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function verifying* (4.2), (4.4), *and* $F^{(\infty)}$ *be given by* (4.1). *Assume that* $\operatorname{dom} f$ *is closed, and that f is upper semicontinuous, then* (5.3) *holds.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$, *then* (5.4) *too holds.*

Proof. Follows by Theorem 5.4. □

Corollary 5.6. *Let $g\colon \mathbb{R}^n \to [0,+\infty[$ be continuous, C be a convex subset of* $\mathbb{R}^n$, $f = g + I_C$, *and let* $F^{(\infty)}$ *be given by* (4.1), *then* (5.3) *holds.*

If in addition $C^\circ \neq \emptyset$, *then* (5.4) *too holds.*

Proof. Follows by Theorem 5.4 once observed that f verifies (4.2)÷(4.4). □

Corollary 5.7. *Let $f\colon \mathbb{R}^n \to [0,+\infty]$ be a Borel function verifying* (4.2)÷(4.4), *and* $F^{(\infty)}$ *be given by* (4.1). *Assume that* $\operatorname{dom} f$ *is an affine set, then* (5.3) *holds.*

If $\operatorname{dom} f = \mathbb{R}^n$, *then* (5.4) *holds.*

Proof. Follows by (4.3) and Theorem 5.4. □

The following result shows that Example 5.1 needs to be settled at least in dimension two.

Proposition 5.8. *Let $f\colon \mathbb{R} \to [0,+\infty]$ be a Borel function verifying* (4.2)÷(4.4), *and* $F^{(\infty)}$ *be given by* (4.1), *then*

$$F^{(\infty)}(\Omega, u) = \int_\Omega f^{(\infty)}(u')dx \quad \text{for every bounded open set } \Omega, u \in W^{1,\infty}_{loc}(\mathbb{R}). \tag{5.6}$$

Proof. It is clear that we can assume that $(\operatorname{dom} f)^\circ \neq \emptyset$, so that $\operatorname{dom} f$ turns out to be an interval.

If $\operatorname{dom} f$ is a bounded interval, the thesis follows by Proposition 5.3.

If $\operatorname{dom} f = \mathbb{R}$, the thesis follows by Corollary 5.7, therefore we have to treat only the case in which $\operatorname{dom} f$ is an unbounded interval with one real endpoint, say for example $\operatorname{dom} f =]a, +\infty[$ or $\operatorname{dom} f = [a, +\infty[$ for some $a \in \mathbb{R}$.

Let us prove that

$$\begin{gathered}(f + I_{Q_m})^{**}(z) \leq f^{(\infty)}(z) + f(z_0) + 1 \\ \text{for every } z_0 > a, m > |a| + |z_0| + 1, z \in]a, z_0[.\end{gathered} \tag{5.7}$$

To do this let $z_0 > a$, $m > |a| + |z_0| + 1$, $z \in]a, z_0[$, then by Theorem 2.1 there exist z_1, $z_2 \in \operatorname{dom} f$ with $z_1 \leq z$, $\lambda \in [0, 1]$ such that $z = \lambda z_1 + (1 - \lambda) z_2$, and

$$\lambda f(z_1) + (1 - \lambda) f(z_2) < \operatorname{co} f(z) + 1. \tag{5.8}$$

Since a, $z_0 \in Q_m$ and $z_1 \in [a, z]$, it is clear that $z_1 \in Q_m$, and we treat separately the cases in which $z_2 \in Q_m$ and $z_2 \notin Q_m$.

If $z_2 \in Q_m$, by (2.1), Theorem 2.1, (5.8), and Proposition 3.1 we have that

$$\begin{aligned}(f + I_{Q_m})^{**}(z) &\leq \operatorname{co}(f + I_{Q_m})(z) \\ &\leq \lambda (f + I_{Q_m})(z_1) + (1 - \lambda)(f + I_{Q_m})(z_2) = \lambda f(z_1) + (1 - \lambda) f(z_2) \\ &< \operatorname{co} f(z) + 1 = f^{(\infty)}(z) + 1,\end{aligned}$$

from which (5.7) follows.

If $z_2 \notin Q_m$, let $\mu \in [0, 1]$ be such that $z = \mu z_1 + (1 - \mu) z_0$, and let us consider the two cases in which $\mu f(z_1) + (1 - \mu) f(z_0) \leq \lambda f(z_1) + (1 - \lambda) f(z_2)$ and $\mu f(z_1) + (1 - \mu) f(z_0) > \lambda f(z_1) + (1 - \lambda) f(z_2)$.

If $\mu f(z_1) + (1 - \mu) f(z_0) \leq \lambda f(z_1) + (1 - \lambda) f(z_2)$, by (2.1), Theorem 2.1, (5.8), and Proposition 3.1 we have that

$$\begin{aligned}(f + I_{Q_m})^{**}(z) &\leq \operatorname{co}(f + I_{Q_m})(z) \\ &\leq \mu (f + I_{Q_m})(z_1) + (1 - \mu)(f + I_{Q_m})(z_0) = \mu f(z_1) + (1 - \mu) f(z_0) \\ &\leq \lambda f(z_1) + (1 - \lambda) f(z_2) < \operatorname{co} f(z) + 1 = f^{(\infty)}(z) + 1,\end{aligned}$$

from which (5.7) follows.

If $\mu f(z_1) + (1 - \mu) f(z_0) > \lambda f(z_1) + (1 - \lambda) f(z_2)$, by (2.1), Theorem 2.1, and (5.8) we have that

$$\begin{aligned}(f + I_{Q_m})^{**}(z) &\leq \operatorname{co}(f + I_{Q_m})(z) \\ &\leq \mu (f + I_{Q_m})(z_1) + (1 - \mu)(f + I_{Q_m})(z_0) \\ &= \mu f(z_1) + (1 - \mu) f(z_0) \\ &= \lambda f(z_1) + (1 - \lambda) f(z_2) + \mu f(z_1) + (1 - \mu) f(z_0) - \\ &\quad - (\lambda f(z_1) + (1 - \lambda) f(z_2)) \\ &< \operatorname{co} f(z) + 1 + \mu f(z_1) + (1 - \mu) f(z_0) - (\lambda f(z_1) + (1 - \lambda) f(z_2)).\end{aligned} \tag{5.9}$$

We now observe that $\lambda f(z_1) + (1-\lambda) f(z_2)$ is the value at z of the affine function α verifying $\alpha(z_1) = f(z_1)$ and $\alpha(z_2) = f(z_2)$, whilst $\mu f(z_1) + (1-\mu) f(z_0)$ is the one at z of the affine function β verifying $\beta(z_1) = f(z_1)$ and $\beta(z_0) = f(z_0)$, therefore, once observed that $\beta(z_1) = \alpha(z_1)$, and that $a(z_0) \geq 0$, we obtain that

$$\begin{aligned} &\mu f(z_1) + (1-\mu) f(z_0) - (\lambda f(z_1) + (1-\lambda) f(z_2)) \\ &= \beta(z) - \alpha(z) \leq \beta(z_0) - \alpha(z_0) \leq f(z_0). \end{aligned} \tag{5.10}$$

By (5.9), (5.10), and Proposition 3.1 we conclude that

$$(f + I_{Q_m})^{**}(z) \leq \operatorname{co} f(z) + 1 + f(z_0) = f^{(\infty)}(z) + 1 + f(z_0),$$

from which (5.7) follows also in this case.

Let us observe now that, if $z_0 > a$, $m > |a| + |z_0| + 1$, by the lower semicontinuity of $(f + I_{Q_m})^{**}$, (5.7), the convexity of $f^{(\infty)}$, and (3.2) of Proposition 3.1 it results that

$$\begin{aligned} (f + I_{Q_m})^{**}(a) &\leq \liminf_{t \to 1-} (f + I_{Q_m})^{**}(ta + (1-t) z_0) \\ &\leq \limsup_{t \to 1-} \{ t f^{(\infty)}(a) + (1-t) f(z_0) \} + f(z_0) + 1 \\ &\leq f^{(\infty)}(a) + f(z_0) + 1, \end{aligned} \tag{5.11}$$

whilst by Proposition 3.1, and (2.4) it clearly follows that

$$(f + I_{Q_m})^{**}(z) \leq f^{(\infty)}(z) + f(z_0) + 1 \quad \text{for every } z < a. \tag{5.12}$$

Hence by (5.7), (5.11), and (5.12) we conclude that

$$\begin{gathered} (f + I_{Q_m})^{**}(z) \leq f^{(\infty)}(z) + f(z_0) + 1 \\ \text{for every } z_0 > a, m > |a| + |z_0| + 1, z \in]-\infty, z_0[. \end{gathered} \tag{5.13}$$

In conclusion, if Ω is a bounded open set, $u \in W^{1,\infty}_{loc}(\mathbb{R})$ with $F^{(\infty)}(\Omega, u) = +\infty$, and $z_0 > \|u'\|_{L^\infty(\Omega)}$, we deduce by Theorem 4.3, and by the monotonicity properties of $\left\{ \int_\Omega (f + I_{Q_m})^{**}(u') dx \right\}$, that $\int_\Omega (f + I_{Q_m})^{**}(u') dx = +\infty$ for every $m \in \mathbb{N}$ and, by (5.13), that $\int_\Omega f^{(\infty)}(u') dx = +\infty$. By virtue of this, and again Theorem 4.3, (5.6) follows. □

6. Applications to the identity between $\overline{F}^{(\infty)}$ and $F^{(\infty)}$

Let $f \colon \mathbb{R}^n \to [0, +\infty]$, $f^{(\infty)}$ be defined in (3.1), and $\overline{F}^{(\infty)}$, $F^{(\infty)}$ in Section 4.

In the present section we apply the previously obtained results to prove identity between $F^{(\infty)}$ and $\overline{F}^{(\infty)}$. More precisely that

$$\begin{gathered} F^{(\infty)}(\Omega, u) = \overline{F}^{(\infty)}(\Omega, u) = \int_\Omega f^{**}(\nabla u) dx \\ \text{for every convex bounded open set } \Omega, u \in W^{1,\infty}(\Omega), \end{gathered} \tag{6.1}$$

and

$$F^{(\infty)}(\Omega, u) = \overline{F}^{(\infty)}(\Omega, u) = \int_\Omega f^{**}(\nabla u)dx \qquad (6.2)$$
$$\text{for every bounded open set } \Omega, u \in W^{1,\infty}_{loc}(\mathbb{R}^n).$$

Let us preliminarily observe that, by using also Proposition 4.1, it follows that

$$\int_\Omega f^{**}(\nabla u)dx \leq \overline{F}^{(\infty)}(\Omega, u) \leq F^{(\infty)}(\Omega, u) \qquad (6.3)$$
$$\text{for every bounded open set } \Omega, u \in W^{1,\infty}(\Omega).$$

Theorem 6.1. *Let* $f\colon \mathbb{R}^n \to [0, +\infty]$ *be a Borel function verifying* (4.2)÷(4.4), *and assume that* $\operatorname{dom} f$ *is an affine set, then* (6.1) *holds.*

If $\operatorname{dom} f = \mathbb{R}^n$, *then* (6.2) *too holds.*

Proof. Follows by (6.3), Corollary 5.7, and Proposition 3.7. □

Theorem 6.2. *Let* $f\colon \mathbb{R}^n \to [0, +\infty]$ *be a Borel function verifying* (4.2)÷(4.4), *and assume that* $\operatorname{dom} f$ *is bounded, then* (6.1) *holds.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$, *then* (6.2) *too holds.*

Proof. Follows by (6.3), Proposition 5.3, and Corollary 3.14. □

Theorem 6.3. *Let* $f\colon \mathbb{R}^n \to [0, +\infty]$ *be a Borel function verifying* (4.2), (4.4). *Assume that* f *is bounded on the bounded subsets of* $\operatorname{dom} f$, *and that one of the following conditions is fulfilled*

(i) $\lim_{z\to\infty} \frac{f(z)}{|z|} = +\infty$,

(ii) *for every* $z_0 \in \operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$ *there exists a non-trivial supporting hyperplane to* $\operatorname{co}(\operatorname{dom} f)$ *containing* z_0 *having a bounded intersection with* $\operatorname{rb}(\operatorname{co}(\operatorname{dom} f))$,

then (6.1) *holds.*

If in addition $(\operatorname{dom} f)^\circ \neq \emptyset$, *then* (6.2) *too holds.*

Proof. Follows by (6.3), Theorem 5.4, and Proposition 3.8 or Theorem 3.13. □

Theorem 6.4. *Let* $f\colon \mathbb{R} \to [0, +\infty]$ *be a Borel function verifying* (4.2)÷(4.4), *then*

$$F^{(\infty)}(\Omega, u) = \overline{F}^{(\infty)}(\Omega, u) = \int_\Omega f^{**}(u')dx \quad \text{for every bounded open set } \Omega, u \in W^{1,\infty}_{loc}(\mathbb{R}).$$

Proof. Follows by (6.3), Proposition 5.8, and Corollary 3.16. □

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References

[1] E. Acerbi, N. Fusco: Semicontinuity problems in the calculus of variations, Arch. Rational Mech. Anal. 86 (1984) 125–145.

[2] G. Bouchitté, G. Dal Maso: Integral representation and relaxation of convex local functionals on $BV(\Omega)$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 20 (1993) 483–533.

[3] G. Buttazzo: Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations, Pitman Res. Notes Math. Ser. 207, Longman Scientific & Technical, 1989.

[4] G. Buttazzo, G. Dal Maso: Integral representation and relaxation of local functionals, Nonlinear Anal. 9 (1985) 515–532.

[5] G. Buttazzo, G. Dal Maso: A characterization of nonlinear functionals on Sobolev spaces which admit an integral representation with a Carathéodory integrand, J. Math. Pures Appl. (9) 64 (1985) 337–361.

[6] L. Carbone, R. De Arcangelis: On integral representation, relaxation and homogenization for unbounded functionals, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 8 (1997) 129–135.

[7] L. Carbone, R. De Arcangelis: Integral representation for some classes of unbounded functionals, Atti Sem. Mat. Fis. Univ. Modena 46 - Suppl. (1998) 533–567.

[8] L. Carbone, R. De Arcangelis: On the relaxation of some classes of unbounded integral functionals, Matematiche 51 (1996) 221–256.

[9] A. Corbo Esposito, R. De Arcangelis: A characterization of families of function sets described by constraints on the gradient, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994) 553–609.

[10] G. Dal Maso: An Introduction to Γ-convergence, Progr. Nonlinear Differential Equations Appl. 8, Birkhäuser-Verlag, 1993.

[11] E. De Giorgi, T. Franzoni: Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 58 (1975) 842–850.

[12] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, Stud. Math. Appl. 1, North-Holland American Elsevier, 1976.

[13] P. Marcellini, C. Sbordone: Semicontinuity problems in the calculus of variations, Nonlinear Anal. 4 (1980) 241–257.

[14] C. B. Morrey: Multiple Integrals in the Calculus of Variations, Grundlehren Math. Wiss. 130, Springer-Verlag, 1966.

[15] R. T. Rockafellar: Convex Analysis, Princeton Math. Ser. 28, Princeton University Press, 1972.

[16] J. Serrin: On the definition and properties of certain variational integrals, Trans. Amer. Math. Soc. 101 (1961) 139–167.