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On the composition of quasiconvex functions and the transposition

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If $G : \mathbb{R}^{n\times m} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a convex, polyconvex or rank-one convex function, then the function $g : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}}$ defined as $g(A) = G(A^t)$ preserves convexity, polyconvexity, or rank-one convexity, respectively. The paper shows that this does not hold in general for quasiconvexity provided $n \geq 2$ and $m \geq 3$.

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1. Introduction

A function $\Psi : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is called quasiconvex at $A \in \mathbb{R}^{m\times n}$ if for any $\varphi \in W(\mathbb{R}^n;\mathbb{R}^m) := \{\theta \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m);\ \theta \text{ is } (0,1)^n\text{-periodic}\}$ (or equivalently any $\varphi \in W_0^{1,\infty}((0,1)^n;\mathbb{R}^m)$) (see e.g. [9, 10, 11, 16, 17, 18])

$$\Psi(A) \leq \int_{(0,1)^n} \Psi(A + \nabla\varphi(x))\,\mathrm{d}x \tag{1.1}$$

whenever the integral on the right hand side exists. We say that Ψ is quasiconvex if the previous inequality is valid for any $A \in \mathbb{R}^{m\times n}$. Quasiconvexity is the key property in the calculus of variations. Namely, if Ψ is only finite valued then quasiconvexity of Ψ is equivalent to sequential weak lower semicontinuity (omitting some growth conditions) of the functional

$$I(u) = \int_\Omega \Psi(\nabla u(x))\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $u : \Omega \to \mathbb{R}^m$ smooth enough; cf. e.g. [1, 5, 9]. If Ψ attains also the value $+\infty$ then it is known that quasiconvexity is the necessary condition for sequential weak lower semicontinuity of I; cf. [4, 5, 9, 16]. Unfortunately, quasiconvexity is very difficult to verify even in particular cases. On the other hand, there are known sufficient conditions and necessary conditions for quasiconvexity.

One sufficient condition is *polyconvexity*; cf. [2, 5]. Ψ given above is polyconvex if there is a convex function ϕ such that, for any $A \in \mathbb{R}^{m\times n}$, $\Psi(A) = \phi(T(A))$, where $T(A)$ is a vector of all subdeterminants of A, thus, $T : \mathbb{R}^{m\times n} \to \mathbb{R}^N$ where $N := \sum_{l=1}^{\min(m,n)} \binom{m}{l}\binom{n}{l}$.

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Dacorogna [5] showed that Ψ is polyconvex at $A \in \mathbb{R}^{m\times n}$ if and only if

$$\Psi(A) = \inf\left\{\sum_{i=1}^{N+1}\lambda_i\Psi(A_i);\ \sum_{i=1}^{N+1}\lambda_i T(A_i) = T(A),\ \sum_{i=1}^{N+1}\lambda_i = 1,\ \lambda_i \geq 0\ ,\ A_i \in \mathbb{R}^{m\times n}\right\}. \tag{1.2}$$

The necessary (if Ψ is real-valued) condition is *rank-one convexity.* The function Ψ (as above) is called rank-one convex if $\Psi(\lambda A + (1-\lambda)B) \leq \lambda\Psi(A) + (1-\lambda)\Psi(B)$ for any $0 \leq \lambda \leq 1$ and any $A, B \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(A-B) = 1$; cf. e.g. [5, 6, 13, 16, 19, 20].

If $\min(m,n) = 1$ then quasiconvexity, polyconvexity and rank-one convexity are equivalent to usual convexity. The question whether or not rank-one convexity implies quasiconvexity if $\min(m,n) > 1$ has been open for many years. In 1992 Šverák [18] found a counterexample showing that this is not the case when $m \geq 3$ and $n \geq 2$. In particular, he showed that for any $\varepsilon > 0$ there is $k = k(\varepsilon) > 0$ such that the function $f_k^\varepsilon : \mathbb{R}^{3\times 2} \to \mathbb{R}$

$$f_k^\varepsilon(A) = f(PA) + \varepsilon(|A|^2 + |A|^4) + k|A - PA|^2 \tag{1.3}$$

is rank-one convex but there is $\varepsilon > 0$ such that f_k^ε is not quasiconvex for any $k > 0$ at the point $A = 0$. Above, $P : \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 2}$ is an orthogonal projector given by

$$P\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \\ \frac{A_{31}+A_{32}}{2} & \frac{A_{31}+A_{32}}{2} \end{pmatrix}$$

and

$$f(PA) = -\frac{A_{11}A_{22}(A_{31}+A_{32})}{2},$$

where A_{ij}, $i = 1,2,3$, $j = 1,2$ mean the entries of A and $|\cdot|$ is the Euclidean norm.

The aim of this paper is to discuss properties of the function $F_k^\varepsilon : \mathbb{R}^{2\times 3} \to \mathbb{R}$ defined as

$$F_k^\varepsilon(A) = f_k^\varepsilon(A^t) \tag{1.4}$$

and its limit for $k \to \infty$. The superscript "t" denotes the transposition (i.e. $A^t_{ij} = A_{ji}$). Namely, we show that $\lim_{k\to\infty} F_k^\varepsilon$ is quasiconvex although $\lim_{k\to\infty} f_k^\varepsilon$ is not.

In other words, quasiconvexity is not invariant under composition with linear mappings. This is a completely different situation in comparison with rank-one convexity (for mappings, which map rank-one matrices into rank-one matrices), polyconvexity, or convexity.

We will start with an easy lemma.

Lemma 1.1. *Let $G : \mathbb{R}^{n\times m} \to \bar{\mathbb{R}}$ be rank-one convex (polyconvex). Then $g : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}}$, $g(A) = G(A^t)$ is also rank-one convex (polyconvex).*

Proof. Take $A, B \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(A-B) = 1$ and $0 \leq \lambda \leq 1$. Then $\mathrm{rank}(A^t - B^t) = 1$ and

$$g(\lambda A + (1-\lambda)B) = G(\lambda A^t + (1-\lambda)B^t) \leq \lambda G(A^t) + (1-\lambda)G(B^t) = \lambda g(A) + (1-\lambda)g(B)\ ,$$

which shows rank-one convexity of g.

Take $\tilde{T} : \mathbb{R}^{n\times m} \to \mathbb{R}^N$ that $\tilde{T}(A)$ is the vector of all subdeterminants of $A \in \mathbb{R}^{n\times m}$. Let $T : \mathbb{R}^{m\times n} \to \mathbb{R}^N$ be such that $T(B)$ is the vector of all subdeterminants of $B \in \mathbb{R}^{m\times n}$. Then $T(A^t) = L\tilde{T}(A)$ where $L : \mathbb{R}^N \to \mathbb{R}^N$ is a suitable permutation (reordering) of elements of the vector $\tilde{T}(A)$. We have a convex function ϕ such that

$$g(A) = G(A^t) = \phi(T(A^t)) = \phi(L\tilde{T}(A)) = (\phi \circ L)(\tilde{T}(A)).$$

As L is linear and ϕ convex, $\phi \circ L$ is also convex and the proof is finished. □

2. Properties of F_k^ε

First, let us show some obvious properties of F_k^ε. We will suppose that k, ε are such that f_k^ε is rank-one convex.

Proposition 2.1. *If f_k^ε is rank-one convex, then F_k^ε is rank-one convex but not polyconvex.*

Proof. We need only to show that f_k^ε is not polyconvex. But it cannot be polyconvex because it is not quasiconvex at the origin. Thus due to (1.2) there are $A_i \in \mathbb{R}^{3\times 2}$ and $\lambda_i \geq 0$, for $i = 1, \dots, 10$, such that $\sum_{i=1}^{10} \lambda_i = 1$, $\sum_{i=1}^{10} \lambda_i T(A_i) = T(0) = 0$ and $0 = f_k^\varepsilon(0) > \sum_{i=1}^{10} \lambda_i f_k^\varepsilon(A_i)$. In fact, it follows from [18, Proof of Theorem] and [5, Ch. 5] that there is $\delta > 0$ independent of k that

$$0 = f_k^\varepsilon(0) > -\delta > \sum_{i=1}^{10} \lambda_i f_k^\varepsilon(A_i). \tag{2.1}$$

The assertion of the proposition is implied by Lemma 1.1. □

Set $\mathcal{L} = \{\varphi \in W(\mathbb{R}^3; \mathbb{R}^2); P\nabla^t\varphi = \nabla^t\varphi\}$ (we abbreviated $\nabla^t\varphi := (\nabla\varphi)^t$). We have the following lemma.

Lemma 2.2. *Let $\varphi \in \mathcal{L}$. Then there exist $\alpha, \beta, \gamma \in W(\mathbb{R}; \mathbb{R})$ such that*

$$\varphi_1(x) = \alpha(x_1) + \gamma(x_3)\ ,\ \ \varphi_2(x) = \beta(x_2) + \gamma(x_3).$$

Proof. By the definition of P we have

$$\frac{\partial\varphi_1}{\partial x_2} = \frac{\partial\varphi_2}{\partial x_1} = \frac{\partial(\varphi_2 - \varphi_1)}{\partial x_3} = 0 \quad \text{for a.a. } x \in (0,1)^3,$$

whence $\varphi_1(x) = a(x_1, x_3)$ and $\varphi_2(x) = b(x_2, x_3)$, and also

$$\frac{\partial a}{\partial x_3}(x_1, x_3) = \frac{\partial b}{\partial x_3}(x_2, x_3);$$

the left hand side must be independent of x_1 and the right hand side independent of x_2, therefore both are functions of x_3 alone. Thus $a(x_1, x_3) = \alpha(x_1) + \gamma(x_3)$ and $b(x_2, x_3) = \beta(x_2) + \gamma(x_3)$; the summability and periodicity properties of the three functions are straightforward. □

An important role in Šverák's counterexample is played by the fact that the test function which "breaks" (1.1) fulfills $\nabla\varphi = P\nabla\varphi$, $\varphi \in W(\mathbb{R}^2;\mathbb{R}^3)$. The next proposition shows that this does not work in the case of F_k^ε.

Proposition 2.3. *Let $\varphi \in \mathcal{L}$. Then for any $A \in \mathbb{R}^{2\times 3}$*

$$F_k^\varepsilon(A) \le \int_{(0,1)^3} F_k^\varepsilon(A + \nabla\varphi(x))\, dx. \tag{2.2}$$

Proof. Let $A \in \mathbb{R}^{2\times 3}$ and $\varphi \in \mathcal{L}$ be arbitrary. We have

$$\begin{aligned}
\int_{(0,1)^3} F_k^\varepsilon(A + \nabla\varphi(x))\,\mathrm{d}x &= \int_{(0,1)^3} f_k^\varepsilon(A^t + \nabla^t\varphi(x))\,\mathrm{d}x \\
&= \int_{(0,1)^3} f(P(A^t + \nabla^t\varphi(x))\,\mathrm{d}x \\
&+ \int_{(0,1)^3} \varepsilon(|A^t + \nabla^t\varphi(x)|^2 + |A^t + \nabla^t\varphi(x)|^4)\,\mathrm{d}x \\
&+ k|A^t - PA^t|^2.
\end{aligned}$$

The only term we have to deal with is $\int_{(0,1)^3} f(P(A + \nabla\varphi(x))^t)\,\mathrm{d}x$ because all of other terms are convex functions invariant on transposition.

By Lemma 2.2 we obtain

$$\begin{aligned}
&\int_{(0,1)^3} f(P(A^t + \nabla^t\varphi(x)))\,\mathrm{d}x \\
= &-\frac{1}{2}\int_{(0,1)^3} (A_{11} + \alpha'(x_1))\,(A_{22} + \beta'(x_2))\,(A_{13} + 2\gamma'(x_3))\;\mathrm{d}x \\
= &-\frac{1}{2}\int_0^1 A_{11}\,\mathrm{d}x_1 \int_0^1 A_{22}\,\mathrm{d}x_2 \int_0^1 A_{13}\,\mathrm{d}x_3 \\
= &\; f(PA^t).
\end{aligned}$$

As all of other terms in the definition of F_k^ε are convex functions we end up with

$$F_k^\varepsilon(A) \le \int_{(0,1)^3} F_k^\varepsilon(A + \nabla\varphi(x))\,\mathrm{d}x.$$

The proposition is proved. □

3. The functions F_∞^ε and f_∞^ε

Let us define $F_\infty^\varepsilon : \mathbb{R}^{2\times 3} \to \bar{\mathbb{R}}$, by $F_\infty^\varepsilon := \lim_{k\to\infty} F_k^\varepsilon$. We have

$$F_\infty^\varepsilon(A) = \begin{cases} f(PA^t) + \varepsilon(|A^t|^2 + |A^t|^4) & \text{if } A^t = PA^t \\ +\infty & \text{otherwise .} \end{cases} \tag{3.1}$$

Proposition 3.1. *F^ε_∞ is quasiconvex but not polyconvex.*

Proof. We must verify (1.1) with F^ε_∞ instead of Ψ. Take $A \in \mathbb{R}^{2\times 3}$ and $\varphi \in W(\mathbb{R}^3;\mathbb{R}^2)$ arbitrary. We will divide the proof into four steps.

STEP 1. Let $A^t = PA^t$ and let $\nabla^t\varphi = P\nabla^t\varphi$. Then (1.1) follows from Proposition 2.3.

STEP 2. Let $A^t = PA^t$ and $\nabla^t\varphi \neq P\nabla^t\varphi$. Then $P(A^t + \nabla^t\varphi) \neq A^t + \nabla^t\varphi$ on a subset of $(0,1)^3$ with a positive Lebesgue measure, so that the right side in (1.1) equals to $+\infty$.

STEP 3. Let $A^t \neq PA^t$ and $\nabla^t\varphi = P\nabla^t\varphi$. Then both of the sides in (1.1) are $+\infty$. This is clear for the left side. To show it for the right one we argue as in Step 2.

STEP 4. Let $A^t \neq PA^t$ and $\nabla^t\varphi \neq P\nabla^t\varphi$. To show that both sides of (1.1) are $+\infty$ in this case, we need only to exclude that $\partial\varphi_1/\partial x_2 = -A_{12} \neq 0$, $\partial\varphi_2/\partial x_1 = -A_{21} \neq 0$ and that $\partial(\varphi_1 - \varphi_2)/\partial x_3 = A_{23} - A_{13} \neq 0$. But this contradicts the fact that φ is $(0,1)^3$-periodic, hence its first or second components (or their difference) cannot be affine in the directions x_2, x_1 or x_3, respectively. Note that $A_{12} = 0$, $A_{21} = 0$ and $A_{13} = A_{23}$ would contradict $A^t \neq PA^t$.

Altogether we verified that F^ε_∞ is quasiconvex.

We showed in the proof of Proposition 2.1 that f^ε_k is not polyconvex at 0. The limit passage for $k \to \infty$ in (2.1) shows that $f^\varepsilon_\infty(0)$ does not fulfill (1.2) and therefore neither f^ε_∞ nor F^ε_∞ are polyconvex. $\square$

As for infinity-valued functions quasiconvexity does not imply rank-one convexity we have to show explicitly that F^ε_∞ is rank-one convex. But this is easy because it is the point limit of rank-one convex functions $\{F^\varepsilon_k\}_{k\geq k_0}$ for some $k_0 > 0$.

Proposition 3.2. *Let $f^\varepsilon_\infty := \lim_{k\to\infty} f^\varepsilon_k$. Then $f^\varepsilon_\infty(A) = F^\varepsilon_\infty(A^t)$ is not quasiconvex but it is rank-one convex.*

Proof. It is shown in [18] that

$$\int_{(0,1)^2} f^\varepsilon_k(\nabla\psi(x))\,\mathrm{d}x = \int_{(0,1)^2} f^\varepsilon_\infty(\nabla\psi(x))\,\mathrm{d}x < f^\varepsilon_k(0) = f^\varepsilon_\infty(0) = 0, \tag{3.2}$$

where $\psi(x) = 1/(2\pi)(\sin 2\pi x_1, \sin 2\pi x_2, \sin 2\pi(x_1 + x_2))$. Note that $\nabla\psi = P\nabla\psi$.

Finally, rank-one convexity of f^ε_∞ follows from the same pointwise convergence argument as above. $\square$

Šverák ([18, Corollary]) constructed a mapping $M : \mathbb{R}^{m\times n} \to \mathbb{R}^{3\times 2}$, $n \geq 2$, $m \geq 3$,

$$\forall B \in \mathbb{R}^{m\times n} \quad MB = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix}$$

and defined $\tilde f^\varepsilon_k : \mathbb{R}^{m\times n} \to \mathbb{R}$ by $\tilde f^\varepsilon_k(B) = f^\varepsilon_k(MB)$. He showed that for suitable values of $k > 0$ and $\varepsilon > 0$ $\tilde f^\varepsilon_k$ is rank-one convex but not quasiconvex. Similarly as above we can get that $\tilde f^\varepsilon_\infty := \lim_{k\to\infty} \tilde f^\varepsilon_k$ is not quasiconvex but $\tilde F^\varepsilon_\infty : \mathbb{R}^{n\times m} \to \bar{\mathbb{R}}$ given by $\tilde F^\varepsilon_\infty(B) := \tilde f^\varepsilon_\infty(B^t)$ is quasiconvex. Namely, following the proof of Proposition 2.3 we get for $\varphi \in W(\mathbb{R}^m;\mathbb{R}^n)$

such that $M\nabla^t\varphi = PM\nabla^t\varphi$ that $\partial\varphi_1/\partial x_1$ does not depend on x_2 and x_3, $\partial\varphi_2/\partial x_2$ does not depend on x_1 and x_3 and, finally, that $\partial(\varphi_1+\varphi_2)/\partial x_3$ does not depend on x_1 and x_2. Therefore, we get for any $A \in \mathbb{R}^{n\times m}$ and any φ as above

$$\tilde{F}^{\varepsilon}_{\infty}(A) \le \int_{(0,1)^m} \tilde{F}^{\varepsilon}_{\infty}(A+\nabla\varphi(x))\,\mathrm{d}x.$$

The proof of quasiconvexity of $\tilde{F}^{\varepsilon}_{\infty}$ is then the same as the proof of Proposition 3.1.

Eventually, we have the following assertion.

Proposition 3.3. *Let $n \ge 2$ and $m \ge 3$. Then there is a quasiconvex function $G : \mathbb{R}^{n\times m} \to \bar{\mathbb{R}}$ such that $g : \mathbb{R}^{m\times n} \to \bar{\mathbb{R}}$, $g(A) = G(A^t)$ is not quasiconvex.*

4. Concluding remarks

- Let $m, n \ge 3$ and let "C" be such a condition on $B_1, B_2 \in \mathbb{R}^{n\times m}$ that if B_1, B_2 satisfy "C" then also B_1^t, B_2^t satisfy "C". Then quasiconvexity of $h : \mathbb{R}^{n\times m} \to \bar{\mathbb{R}}$ cannot be expressed e.g as Jensen's inequality $h(\lambda B_1 + (1-\lambda)B_2) \le \lambda h(B_1) + (1-\lambda)h(B_2)$, $0<\lambda<1$, provided B_1, B_2 satisfy "C". In particular, our result partly negatively answers the question by Pedregal [14, 15] whether quasiconvexity can be described by means of discrete inequalities invariant on transposition. See [8] for more results in this direction.
- Recently, I learned from [12] that the infiniteness of F^{ε}_{∞} is not essential for the example above. We conjecture that F^{ε}_k becomes quasiconvex once f^{ε}_k (and also F^{ε}_k) starts to be rank-one convex. This conjecture is based on many numerical experiments checking quasiconvexity of F^{ε}_k. We used an element-wise affine approximation of $\varphi \in W^{1,\infty}_0((0,1)^3;\mathbb{R}^2)$ on tetrahedral elements. Here we rely on results by Dacorogna and Haeberly [7] who numerically computed that, for $\varepsilon = 0.01$, f^{ε}_k is rank-one convex if $k \ge 1/0.12978763$.
- Assume for a while that rank-one convexity would be equivalent to quasiconvexity for functions $\mathbb{R}^{2\times 2} \to \mathbb{R}$ and let $g : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be quasiconvex. Then also $G : \mathbb{R}^{2\times 2} \to \mathbb{R}$, $G(A) = g(A^t)$, have to be quasiconvex. Therefore, we have for any $\varphi \in W(\mathbb{R}^2;\mathbb{R}^2)$ and any $A \in \mathbb{R}^{2\times 2}$

$$\int_{(0,1)^2} G(A^t+\nabla\varphi(x))\,\mathrm{d}x = \int_{(0,1)^2} g(A+\nabla^t\varphi(x))\,\mathrm{d}x \ge G(A^t) = g(A).$$

Eventually, we would get that any quasiconvex function $g : \mathbb{R}^{2\times 2} \to \mathbb{R}$ satisfies both

$$\int_{(0,1)^2} g(A+\nabla\varphi(x))\,\mathrm{d}x \ge g(A)$$

and

$$\int_{(0,1)^2} g(A+\nabla^t\varphi(x))\,\mathrm{d}x \ge g(A)$$

for any $A \in \mathbb{R}^{2\times 2}$ and any $\varphi \in W(\mathbb{R}^2;\mathbb{R}^2)$.

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