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Invariants of Pairs of Compact Convex Sets

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In a recent paper P. Diamond, P. Kloeden, A. Rubinov and A. Vladimirov [3] investigated comperative properties of three different metrics in the space of pairs of compact convex sets. These metrics describe invariant properties of the Rådström-Hörmander lattice [5] i.e. the space of equivalence classes of pairs of nonempty compact convex sets. In this paper we consider invariants of a class of equivalent pairs of nonempty compact convex sets. We show that the affine dimension of the minimal representant of an equivalence class is invariant and that each equivalence class has invariant convexificators.

Keywords: Pairs of convex sets, sublinear function, quasidifferential calculus

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1. Introduction

In this paper we consider invariants of inclusion minimal representants of elements of the Rådström-Hörmander [5] lattice. This lattice consists of equivalence classes of pairs of nonempty compact convex sets which have been investigated in a series of papers (see for instance [4], [6], [7], [8], [9], [12], [14]). The present paper extends a previous result on the characterization of flat inclusion minimal pairs (see Theorem 3.1 of [7]) to arbitrary minimal pairs and refines the characterization of C-minimal equivalence classes which was given in [14]. The major results of this work are the invariance of the affine dimension of inclusion minimal representants and the existence of an invariant convexificator for each equivalence class.

As in [6] we denote the set of all nonempty compact convex subsets for a real topological vector space X by $\mathcal{K}(X)$ and the set of all pairs of nonempty compact convex subsets by $\mathcal{K}^2(X)$, i.e. $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by: $(A,B) \sim (C,D)$ *if and only if* $A + D = B + C$ using the Minkowski-sum. A partial order on $\mathcal{K}^2(X)$ is given by the relation: $(A,B) \leq (C,D)$ *if and only if* $A \subseteq C$ *and* $B \subseteq D$. Pairs of compact convex sets arise in quasidifferential calculus as the sub- and superdifferentials of the directional derivative of a quasidifferentiable function and in formulas for the numerical evaluation of the Aumann-Integral (see [2] and [1]).

Let X be a real topological vector space, and X^* be the space of all continuous real valued

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linear functional. For two compact convex sets $A, B \in \mathcal{K}(X)$ we will use the notation

$$A \vee B := \operatorname{conv}(A \cup B),$$

where the operation "conv" denotes the convex hull. During the proofs, an easy identity for compact convex sets, which was first observed by A. Pinsker [11] will be used frequently, namely: For $A, B, C \in \mathcal{K}(X)$ we have:

$$(A + C) \vee (B + C) = C + (A \vee B).$$

We will use the abbreviation $A + B \vee C$ for $A + (B \vee C)$ and $C + d$ for $C + \{d\}$ for compact convex sets A, B, C and a point d. Moreover we will write $[a, b]$ instead of $\{a\} \vee \{b\}$.

By

$$H_f(K) := \{z \in K \mid f(z) = \max_{y \in K} f(y)\}$$

we denote the *face* of $K \in \mathcal{K}(X)$ with respect to $f \in X^*$. For the faces of two nonempty compact convex sets $A, B \subseteq X$ with respect to $f \in X^*$ the following identity holds:

$$H_f(A + B) = H_f(A) + H_f(B).$$

Finally we explicitly state the order cancellation law (see [5], [13]):

Let X be a real topological vector space and $A, B, C \subseteq X$ nonempty compact convex subsets.

Then the inclusion

$$A + B \subseteq A + C$$

implies

$$B \subseteq C.$$

For $A, B, S \in \mathcal{K}(X)$, we say that S *separates* the sets A and B if and only if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \emptyset$.

The following characterization of convex pairs of compact convex sets by a decomposition of the Minkowski sum, which was proved in [14], is of crucial importance for the remainder of this work.

Theorem 1.1. *Let X be a real topological vector space and $A, B \in \mathcal{K}(X)$.*

Then the following statements are equivalent:

(i) *the set $A \cup B$ is convex*
(ii) *the set $A \cap B$ separates the sets A and B*
(iii) *the set $A \vee B$ is a summand of the set $A + B$*
(iv) $A + B = A \vee B + A \cap B$ *and* $A \cap B \neq \emptyset$.

Note that property (iv) states the basic relationship between Minkowski-sum, convex hull and intersection.

In connection with the above theorem, we recall the following notations: a pair $(A, B) \in \mathcal{K}^2(X)$ is called *convex* (see [14]) if and only if $A \cup B$ is a convex set. Furthermore a pair

$(A,B) \in \mathcal{K}^2(X)$ is called *minimal* if and only if for every equivalent pair $(C,D) \in \mathcal{K}^2(X)$ the relation $(C,D) \leq (A,B)$ implies $C = A$ and $B = D$ and analogously we say that a convex pair $(A,B) \in \mathcal{K}^2(X)$ is *minimal convex* if and only if for every equivalent convex pair $(C,D) \in \mathcal{K}^2(X)$ the relation $(C,D) \leq (A,B)$ implies $C = A$ and $B = D$.

The definition of *convex minimality* can be considered as a definition of minimality under a constraint. The following more general definition of minimality under constraints was investigated in [10].

Let X be a real topological vector space and $C \in \mathcal{K}(X)$. Then the pair $(A,B) \in \mathcal{K}^2(X)$ is called *C-minimal* if and only if the pair $(A+C, B+C)$ is convex, and if for every $C' \in \mathcal{K}(X)$ with $C' \subseteq C$ and $(A+C', B+C')$ is a convex pair follows that $C' = C$.

It is shown in section 3 of [10], that the above definition of *C-minimality* can be extended to the whole equivalence class $[A,B]$. In the proof of this statement the following definition of *C-convexity* plays a central role.

Let $A,B,C \in \mathcal{K}(X)$ be given, then the class $[A,B] = \{(A',B') \in \mathcal{K}^2(X) \mid (A',B') \sim (A,B)\}$ is called *C-convex* if and only if for every representant $(A',B') \in [A,B]$ the pair $(A'+C, B'+C)$ is convex. The set $C \in \mathcal{K}(X)$ is called a *convexificator* of $[A,B]$. Below the above mentioned results of section 3 in [10] are refined by showing that for every equivalence class $[A,B]$ the set $C = A \vee B$ is an invariant convexificator.

2. Pairs of Polar Polyhedra are Minimal

In the case of a locally convex vector space X we proved in [7], Theorem 2.1, a sufficient criterion for minimality of a pair $(A,B) \in \mathcal{K}(X)$ of compact convex sets for which the sets A and B are in general position relative to another. This intuitive meaning of *general position* relative to another was described in term of a *shape* for the set A, that is a set of continuous linear functionals $\mathcal{S} \subseteq X^* \setminus \{0\}$ with

$$\overline{\operatorname{conv}\Big(\bigcup_{f \in \mathcal{S}} H_f(A)\Big)} \;=\; A.$$

In the case where one of the compact convex sets is a polytope, this criterion can be strenghtend as follows:

Theorem 2.1. *Let (X,τ) be a topological vector space and $A \subset X$ a polytope and $B \in \mathcal{K}(X)$. Furthermore let us assume that A has k faces $S_1 = H_{f_1}(A), ..., S_k = H_{f_k}(A)$ of maximal dimension and that for every $i \in \{1,...,k\}$ we have $H_{f_i}(B) = \{b_i\}$.*

Then the pair $(A,B) \in \mathcal{K}^2(X)$ is minimal.

Proof. Let us assume that there exists a pair $(A',B') \in \mathcal{K}^2(X)$ with $(A',B') \sim (A,B)$ and $A' \subset_{\neq} A$ and $B' \subset B$. By virtue of the formula on the addition of faces we deduce from $A + B' = B + A'$, that for all $i \in \{1,...,k\}$

$$S_i + H_{f_i}(B') \;=\; b_i + H_{f_i}(A')$$

holds. Now let us choose elements $b_i^{'} \in H_{f_i}(B'), \;\; i \in \{1,...,k\}$. Then for each $i \in \{1,...,k\}$ holds $S_i + b_i^{'} \subset b_i + H_{f_i}(A')$. By defining $x_i = b_i^{'} - b_i$ the inclusion

$$S_i + x_i \subset H_{f_i}(A') \subset A' \subset_{\neq} A. \tag{2.1}$$

is satisfied for each $i \in \{1, ..., k\}$.

Since the polytope $A \in \mathcal{K}(X)$ has k faces $S_1 = H_{f_1}(A), ..., S_k = H_{f_k}(A)$ of maximal dimension it can be described in terms of inequalities as

$$A = \{x \in X \mid f_i(x) \leq \alpha_i \ , \ i \in \{1, ..., k\} \ \}$$

for some $\alpha_1, ..., \alpha_k \in \mathbf{R}$. Hence from equation (2.1) follows, that there exist numbers $\beta_1, ..., \beta_k \in \mathbf{R}$ such that

$$\hat{A} = \{x \in X \mid f_i(x) \leq \beta_i \ , \ i \in \{1, ..., k\} \ \} \subset_{\neq} A$$

and that for each $i \in \{1, ..., k\}$ we have

$$S_i + x_i = H_{f_i}(A) + x_i \subset H_{f_i}(\hat{A}). \tag{2.2}$$

Since $\hat{A} \subset A$, the inequality $\beta_i \leq \alpha_i$ holds for each $i \in \{1, ..., k\}$ holds. Combining equation (2.2) with the fact that each functional f_i, $i \in \{1, ..., k\}$ determines a face of maximal dimension of A results in the validity of the inequality $\alpha_i \leq \beta_i$ for each $i \in \{1, ..., k\}$ which in turn means that $\hat{A} = A$. Hence the pair $(A, B) \in \mathcal{K}^2(X)$ is minimal. □

Now let $X = \mathbf{R}^n$ and $P \in \mathcal{K}(X)$ be a polytope. Then the polar polytope is defined by:

$$P^o = \{u \in \mathbf{R}^n \mid \sup_{x \in P} \langle u, x \rangle \leq 1\},$$

where $\langle ., . \rangle$ denotes the inner product of $\mathbf{R}^n$.

From the well known correspondence between extremal points of P and faces of maximal dimension of P^o follows:

Corollary 2.2. *Let $X = \mathbf{R}^n$ be a finite dimensional topological vector space and $P \in \mathcal{K}(X)$ be a polytope.*

Then the pair

$$(P, P^o) \in \mathcal{K}^2(\mathbf{R}^n)$$

is minimal.

Remark 2.3. A similar minimality criterium has been stated in [7], Theorem 2.1. To explain the difference between the above Theorem 2.1 and the corresponding result of [7] we present the following example: Let us consider in $X = \mathbf{R}^2$ three polyhedra $A, B, C \in \mathcal{K}(X)$, as indicated in the following figure.

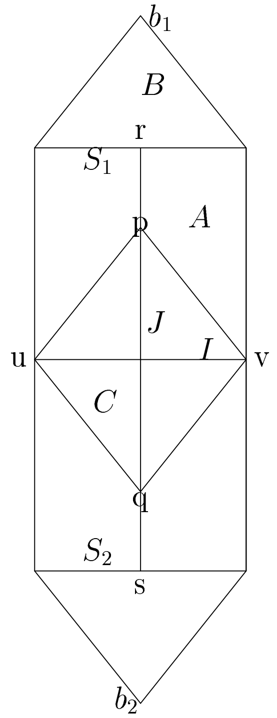

Figure 1

These sets are contructed as follows: Let $I = [u, v]$ and $J = [r, s]$ be two orthogonally intersecting intervals such that $[p, q]$ is properly contained in J as indicated in Fig. 1. Now put $A = I + J$, $\quad C = I \vee [p, q]$ and $B = C + J$. Then the rectangle A has two faces S_1 and S_2 which are parallel to I. Now we choose a shape $\mathcal{S} = \{f_1, f_2\}$ for A with $H_{f_i}(A) = S_i, \ i \in \{1, 2\}$. Then we have:

$$A \quad = \quad S_1 \vee S_2 \qquad \text{and} \qquad \begin{array}{ccc} H_{f_1}(A) = S_1 & , & H_{f_2}(A) \ = \ S_2 \\ \\ H_{f_1}(B) = \{b_1\} & , & H_{f_2}(B) \ = \ \{b_2\} \end{array}$$

On the other side we have:

$$\left.\begin{array}{ccccc} A & = & I & + & J \\ \\ B & = & C & + & J \end{array}\right. \quad \Longrightarrow \quad A + C = B + I$$

which implies that the pair (A, B) is not minimal since $(A, B) \sim (I, C)$.

3. The Dimension-Invariance of Minimal Pairs

In this section we will show, that the affine dimension and codimension of the union a minimal pair of compact convex sets is invariant. We begin with the following proposition:

Proposition 3.1. *Let (X, τ) be a topological vector space and $(A, B), (C, D) \in \mathcal{K}^2(X)$ be equivalent pairs. Furthermore let us assume, that the pair $(C, D) \in \mathcal{K}^2(X)$ is minimal.*

If $A \cup B \subset X_0 \subset X$, where X_0 is a closed subspace of X, then there exists a point $x_0 \in X$ such that $C \cup D \subset X_0 + x_0$.

Proof. By assumption the pairs $(A,B),(C,D) \in \mathcal{K}^2(X)$ are equivalent, i.e.

$$A + D \;=\; B + C \tag{3.1}$$

holds. Hence for every $a_0 \in A$ and $d_0 \in D$ there exist points $b_0 \in B$ and $c_0 \in C$ with $a_0 + d_0 \;=\; b_0 + c_0$. Therefore we have:

$$(A - a_0) \;+\; (D - d_0) \;=\; (B - b_0) \;+\; (C - c_0).$$

If we put $A_0 = A - a_0,\; B_0 = B - b_0,\; C_0 = C - c_0$ and $D_0 = D - d_0$, then we can rewrite equation (3.1) as

$$A_0 + D_0 \;=\; B_0 + C_0 \quad \text{with} \quad 0 \in A_0 \cap B_0 \cap C_0 \cap D_0. \tag{3.1'}$$

Now we consider the sets $C' = C_0 \cap X_0$ and $D' = D_0 \cap X_0$. By equation (3.1') there exist for every $x \in A_0$ and $y \in D'$ elements $b \in B$ and $c \in C$ such that the equation

$$x + y \;=\; (b - b_0) + (c - c_0)$$

holds. Since

$$c - c_0 \;=\; (x + y) \;-\; (b - b_0) \in X_0 + X_0 + X_0 + X_0 \;=\; X_0,$$

we obtain $c - c_0 \in C_0 \cap X_0 = C'$, and hence that

$$A_0 + D' \;\subset\; B_0 + C'.$$

Analogously follows

$$B_0 + C' \;\subset\; A_0 + D'$$

and hence we have:

$$(A_0, B_0) \sim (C_0, D_0) \sim (C', D').$$

Since the pair (C_0, D_0) is minimal (see [6]), it follows that

$$C - c_0 = C' \quad \text{and} \quad D - d_0 = D'.$$

Hence we have that $C - c_0 \subset X_0$ and $D - d_0 \subset X_0$. These inclusions imply that

$$C \subset X_0 + c_0 = (X_0 - b_0) + b_0 + c_0 = X_0 + (b_0 + c_0)$$

and

$$D \subset X_0 + d_0 = (X_0 - a_0) + a_0 + d_0 = X_0 + (a_0 + d_0).$$

Hence

$$C \subset X_0 + x_0, \quad D \subset X_0 + x_0 \quad \text{with} \quad x_0 = (b_0 + c_0).$$

□

Remark 3.2. Now let (X,τ) be a locally convex vector space and C be a nonempty compact convex subset. Then for every $y \in C$ the set

$$C_y = \operatorname{span}(C - y) = \operatorname{cl}(\{z \in X \mid z = \sum_{i=1}^{n} \lambda_i (c_i - y), \quad c_1, ..., c_n \in C, \quad \lambda_1, ..., \lambda_n \in \mathbf{R}, \; n \in \mathbf{N}\})$$

is the smallest closed linear subspace containing $C - y$ or equivalently the intersection of all closed linear subspace containing $C - y$. The affine hull of C is given by

$$\text{aff hull}(C) = C_y + y$$

and is independent of the special choice of $y \in C$. The affine dimension and codimension is defined by:

$$\dim \text{aff}(C) = \dim(C_y)$$

and

$$\text{codim aff}(C) = \text{codim}(C_y) = \dim(X/_{C_y})$$

As a generalization of [7], Theorem 3.1 we get:

Corollary 3.3. *Let (X, τ) be a topological vector space and and $(A, B), (C, D) \in \mathcal{K}^2(X)$ be equivalent minimal pairs.*

Then

$$\dim \text{aff}(A \cup B) = \dim \text{aff}(C \cup D)$$

and

$$\text{codim aff}(A \cup B) = \text{codim aff}(C \cup D)$$

4. Invariant Convexificators

We begin with the following observation:

Proposition 4.1. *Let (X, τ) be a topological vector space and $(A, B), (C, D) \in \mathcal{K}^2(X)$ be equivalent pairs, i.e. $(C, D) \in [A, B]$. Then $C \cup D$ is convex if and only if $A \vee B$ is a summand of $A + D$.*

Proof. *Necessity.* If $C \cup D$ is a convex set, then from Lemma 3.3 in [14] it follows that

$$A + D \;=\; B + C \;=\; A \vee B + C \cap D.$$

Hence $A \vee B$ is a summand of $A + D$.

Sufficiency. Let us assume that $A + D \;=\; B + C \;=\; A \vee B + S$, for some $S \in \mathcal{K}(X)$. Then

$$B + C \supset B + S \quad \text{and} \quad A + D \supset A + S$$

and hence from the cancellation law we have $S \subset C \cap D$. Now observe that

$$A + C \cap D \subset A + D \quad \text{and} \quad B + C \cap D \subset B + C.$$

Hence from the Pinsker rule follows that

$$A \vee B + C \cap D \subset A + D \subset A \vee B + C \cap D.$$

Therefore we have:

$$A + D \;=\; B + C \;=\; A \vee B + C \cap D.$$

But

$$\begin{aligned} C\ +\ A\ +\ D &= C + A \vee B + C \cap D \\ &= (A+C) \vee (B+C) + C \cap D \\ &= (A+C) \vee (A+D) + C \cap D \\ &= A + (C \vee D) + C \cap D. \end{aligned}$$

Hence the law of cancellation gives

$$C + D \ = \ C \vee D + C \cap D$$

and therefore from Theorem 1.1 follows that $C \cup D$ is convex. □

Proposition 4.2. *Let (X,τ) be a topological vector space and $(A,B),(C,D) \in \mathcal{K}^2(X)$ be equivalent pairs, i.e. $(C,D) \in [A,B]$. Then $(C + A \vee B) \cup (D + A \vee B)$ is convex.*

Proof. We have

$$\begin{aligned} (C + A \vee B) + (D + A \vee B) &= (A+C) \vee (B+C) + D + A \vee B \\ &= (A+C) \vee (A+D) + D + A \vee B \ \text{(since } A+D = B+C\text{)} \\ &= A + (C \vee D) + D + A \vee B \ \text{(by the Pinsker rule)} \\ &= (C \vee D) + A \vee B + A + D \\ &= (C + A \vee B) \vee (D + A \vee B) + A + D, \end{aligned}$$

since

$$(C + A \vee B) \vee (D + A \vee B) = (C \vee D) + A \vee B.$$

Hence by Proposition 4.1 the set $(C + A \vee B) \cup (D + A \vee B)$ is convex. □

Corollary 4.3. *Let (X,τ) be a topological vector space and $(A,B) \in \mathcal{K}^2(X)$. Then $C = A \vee B$ is a convexificator for the class $[A,B]$.*

5. Examples

Let X be a normed vector space and X^* its topological dual space, endowed with the weak-*-topology. For every $A \in \mathcal{K}(X)$ let us denote by

$$p_A : X^* \longrightarrow \mathbf{R} \ \text{ with } \ p_A(v) = \sup_{x \in A} \langle v, x \rangle$$

its support function, where $\langle .,. \rangle$ denotes the dual pairing.

In this notation, P. Diamond, P. Kloeden, A. Rubinov and A. Vladimirov studied in [3] the following norm for the Rådström-Hörmander lattice [5] of equivalence classes of pairs of compact convex sets which is given by:

$$\|[A,B]\| = \inf_{\substack{p_C, p_D \\ (C,D) \in [A,B]}} (\sup\{ \sup_{x \in B(0,1)} p_C(x), \sup_{x \in B(0,1)} p_D(x)\}),$$

where $B(0,1) \subset X^*$ denotes the unit ball of X^*.

In the following two examples we will show that this norm does not characterize the minimal representants of an equivalence class.

Example 5.1. Let $X := \mathbf{R}^2$ and $A, B, A', B' \in \mathcal{K}(X)$ the following compact convex sets:

$$A = \operatorname{conv}\{(1,4),(4,1)\} \qquad B = \operatorname{conv}\{(-4,-1),(-1,-1),(-1,-4)\}$$

and

$$A' = \operatorname{conv}\{(1,4),(4,1),(1-\alpha,4-\beta)\} \qquad B' = \operatorname{conv}\{(-4,-1),(-1,-1),(-1,-4),(-4-\alpha,-1-\beta)\}$$

with $\alpha+\beta \leq 5$ and $\alpha^2+\beta^2-8\alpha-2\beta \leq 0$ (see Figure 2).

Now for every choice of the parameters α and β we have $(A', B') \in [A, B]$ (see [8]) and $\sup_{x \in B(0,1)} p_{A'}(x) = \sup_{x \in B(0,1)} p_A(x)$ and $\sup_{x \in B(0,1)} p_{B'}(x) = \sup_{x \in B(0,1)} p_B(x)$. Since only the pair (A, B) is minimal, the above norm does not characterize this element within its equivalence class (see [4],[12]).

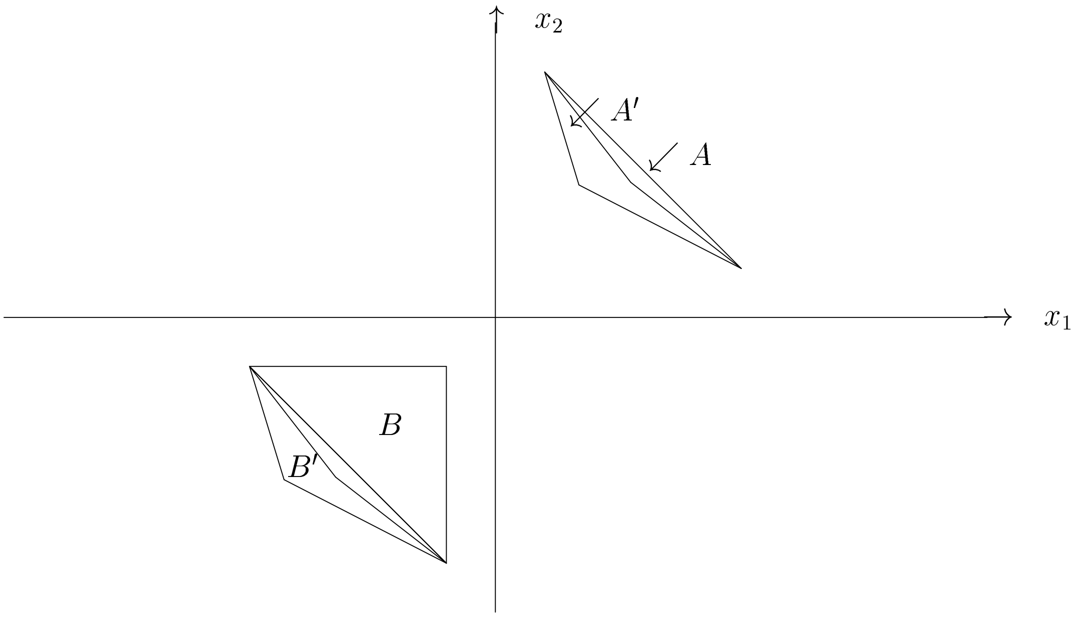

Figure 2

Example 5.2. For $X := \mathbf{R}^3$ the following continuum $(A_\alpha, B_\alpha) \in \mathcal{K}^2(X)$, $\alpha \geq 0$ of equivalent minimal pairs of compact convex sets was constructed [9] namely:

$$A_\alpha = E_\alpha \vee F_\alpha \text{ and } B_\alpha = U_\alpha \vee V_\alpha,$$

with

$$\begin{aligned} E_\alpha &:= \operatorname{conv}\{(0,0,0),\ (1,1,0),\ (1+\alpha,0,0)\} \\ F_\alpha &:= \operatorname{conv}\{(0,1,1),\ (\alpha,0,1),\ (1+\alpha,1,1)\} \\ U_\alpha &:= \operatorname{conv}\{(0,0,0),\ (0,1,0),\ (1,1,0),\ (1+\alpha,0,0)\} \\ V_\alpha &:= \operatorname{conv}\{(0,1,1),\ (\alpha,0,1),\ (1+\alpha,0,1),\ (1+\alpha,1,0)\}. \end{aligned}$$

Now for every $\alpha \geq 0$ we have

$$\sup_{x \in B(0,1)} p_{A_\alpha}(x) = \sup_{x \in B(0,1)} p_{B_\alpha}(x) = \sqrt{3+2\alpha+\alpha^2}$$

which shows, that the function

$$(A, B) \mapsto \sup\{ \sup_{x \in B(0,1)} p_A(x), \sup_{x \in B(0,1)} p_B(x)\},$$

is not constant on the minimal elements of the equivalence class $[A, B]$.

Let us remark, that the Hausdorff metric is constant on the elements of the class $[A, B]$.

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