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Variational Problems with Pointwise Constraints on the Derivatives*

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This paper is concerned with the solvability of a class of nonlinear variational inequalities involving pointwise unilateral constraints on the laplacian. We describe the set of the pairs (ψ, h) of the right hand sides h and the obstacles ψ for which the problem has solutions and study the structure of the set of solutions. The existence and multiplicity results we obtain point out that the presence of the obstacle gives rise to some phenomena which are typical of the semilinear elliptic equations with "jumping" nonlinearities.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a given Carathéodory function.

Let us set

$$K_\psi = \{u \in H_0^1(\Omega) \mid \Delta u \leq \Delta\psi \text{ (in weak sense)}\}$$

and consider the following problem: find $u \in K_\psi$ such that

$$\int_\Omega [DuD(v-u) - g(x,u)(v-u) + h(v-u)]dx \geq 0 \quad \forall v \in K_\psi.$$

The solutions of this variational inequality (whose pointwise meaning is discussed in Remark 2.2) correspond to the critical points of the functional

$$f_h(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega dx \int_0^u g(x,t)dt + \int_\Omega hu\ dx,$$

constrained on the convex cone K_ψ.

Several papers have been devoted to variational inequalities involving unilateral pointwise constraints on the function u (see, for example, [8, 11, 12, 13, 14, 15]). A constraint on the

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laplacian of u, as K_ψ, has been considered by Brezis and Stampacchia in [4] for variational inequalities involving the biharmonic operator. Also, unilateral pointwise constraints on the gradient arise, for example, in the problem of the elastic-plastic torsion of a bar (see [3, 7]).

The aim of this paper is to study the solvability and the multiplicity of solutions of our problem for a generic pair (ψ, h), under suitable assumptions on the asymptotic behaviour of the function $g(x, \cdot)$ with respect to the eigenvalues $(\lambda_i)_i$ of the Laplace operator in $H^1_0(\Omega)$. The results we obtain show that the presence of the obstacle ψ produces some phenomena which make evident a deep analogy with well known results, firstly stated by Ambrosetti and Prodi in [1], concerning semilinear elliptic differential equations with "jumping" nonlinearities (see Remark 5.5 for more details about this analogy).

In [10] this problem has been studied in the particular case $g(x, u) = \lambda u$, with $\lambda \leq \lambda_2$. In that case the solutions can be obtained as mini-max points of the functional f_h, constrained on K_ψ, with respect to a suitable pair of orthogonal subspaces of $H^1_0(\Omega)$. On the contrary, in this paper we consider the general case in which $g(x, t)$ is not necessarily a linear function; therefore we need to apply general topological methods of the Calculus of Variations to find critical points of the related functional, which does not satisfy the usual smoothness conditions (we refer the reader, for example, to [2] and [5], for the general tools of nonsmooth analysis we shall use in this paper).

Finally, let us mention that a new notion of supersolution (see Definition 3.1), which is natural when we handle constraints on the laplacian, turns out to be very useful to analyse the solvability of our problem and the structure of the set of solutions.

The paper is organized as follows: In Section 2 we introduce our problem and, under suitable assumptions on g, characterize its solutions as lower critical points of a suitable functional; in Section 3 we define the supersolutions and state their main properties; in Section 4 we prove the main existence and multiplicity results; in Section 5 we specify these results under additional assumptions on the function $g(x, u)$.

2. Setting of the problem

Let Ω be a bounded domain of $\mathbb{R}^n$ and ψ a function in $H^1_0(\Omega)$; set

$$K_\psi = \left\{ u \in H^1_0(\Omega) \;\middle|\; \int_\Omega DuDw\ dx \geq \int_\Omega D\psi Dw\ dx\ \forall w \in C^\infty_0(\Omega)\ w \geq 0 \right\} \tag{2.1}$$

(note that K_ψ is a convex cone with vertex in ψ).

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for almost all $x \in \Omega$,

$$|g(x, t)| \leq a(x) + b|t|^{p-1} \quad \forall t \in \mathbb{R}, \tag{2.2}$$

for suitable $p, b \in \mathbb{R}$, $p > 1$, $a \in L^{p/(p-1)}(\Omega)$, with $p \leq 2^* = 2n/(n-2)$ if $n > 2$ (2^* is the critical Sobolev exponent).

Definition 2.1. Assume that the function g satisfies condition (2.2). Then, for all $h \in L^2(\Omega)$ and $\psi \in H^1_0(\Omega)$, we say that u is a solution of problem $P_\psi(h)$ if:

$$P_\psi(h) \begin{cases} u \in K_\psi \\ \int_\Omega [DuD(v-u) - g(x, u)(v-u) + h(v-u)]dx \geq 0 \quad \forall v \in K_\psi. \end{cases}$$

Remark 2.2. Notice that, if we can apply the Gauss-Green formula, the inequality of problem $P_\psi(h)$ becomes:

$$\int_\Omega [u + \Delta^{-1}(g(x,u) - h)]\Delta(v-u)dx \le 0 \quad \forall v \in K_\psi,$$

whose pointwise meaning would be:

$$\begin{cases} u + \Delta^{-1}(g(x,u) - h) = 0 & \text{a.e. where } \Delta u < \Delta\psi \\ u + \Delta^{-1}(g(x,u) - h) \ge 0 & \text{a.e. where } \Delta u = \Delta\psi \end{cases}$$

or, equivalently,

$$\begin{cases} u \ge -\Delta^{-1}(g(x,u) - h) & \text{a.e. in } \Omega \\ u > -\Delta^{-1}(g(x,u) - h) & \Rightarrow \quad \Delta u = \Delta\psi. \end{cases}$$

This remark is specified in the following lemma, which can be proved arguing as in the proof of Lemma 1.2 in [10].

Lemma 2.3. *Assume $\psi \in H_0^1(\Omega)$, $k \in L^{p/(p-1)}(\Omega)$, with $p > 1$ and $p \le 2n/(n-2)$ if $n > 2$.*

Let us set

$$\bar{K} = \{\ u \in H_0^1(\Omega)\ :\ u \ge \Delta^{-1}k \ \text{a.e. in } \Omega\ \}.$$

Then a function $u \in H_0^1(\Omega)$ solves the problem

$$\begin{cases} u \in K_\psi \\ \int_\Omega DuD(v-u)dx + \int_\Omega k(v-u)dx \ge 0 \quad \forall v \in K_\psi \end{cases} \tag{2.3}$$

if and only if it is a solution of the variational inequality:

$$\begin{cases} u \in \bar{K} \\ \int_\Omega DuD(w-u)dx - \int_\Omega D\psi D(w-u)dx \ge 0 \quad \forall w \in \bar{K}. \end{cases} \tag{2.4}$$

Now let us introduce some notions of nonsmooth analysis (see, for example, [2, 5, 6]), we shall use to describe the variational nature of problem $P_\psi(h)$.

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

For all $f : H \to \mathbb{R} \cup \{+\infty\}$ let us define the domain of f to be the set

$$\mathcal{D}(f) = \{u \in H \mid f(u) < +\infty\}.$$

As usual, for all $c \in \mathbb{R}$ we denote by f^c the sublevel of f

$$f^c = \{u \in H \mid f(u) \le c\}.$$

For all $u \in \mathcal{D}$, we define the subdifferential of f at u to be the set $\partial^- f(u)$ consisting of all α in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha, v-u)}{\|v-u\|} \ge 0.$$

If $\partial^- f(u) \neq \emptyset$, then we define the subgradient of f at u, denoted by $\mathrm{grad}^- f(u)$, to be the element of $\partial^- f(u)$ having minimal norm (it is easy to check that $\partial^- f(u)$ is a closed and convex subset of H).

We say that u is a lower critical point for f if $0 \in \partial^- f(u)$, that is if $\mathrm{grad}^- f(u) = 0$.

Now let us assume that the function g satisfies condition (2.2) and, for all $\psi \in H_0^1(\Omega)$ and $h \in L^2(\Omega)$, consider the functionals $f_h,\ f_{h,\psi} : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as follows:

$$f_h(u) = \frac{1}{2}\int_\Omega (|Du|^2 - G(x,u))dx + \int_\Omega hudx, \tag{2.5}$$

where $G(x,t) = \int_0^t g(x,s)ds$ and $f_{h,\psi} = f_h + I_{K_\psi}$ (in this paper we denote by I_E the indicator function of any given set E, i.e. $I_E(u) = 0$ if $u \in E$ and $I_E(u) = +\infty$ if $u \notin E$).

Notice that condition (2.2) implies that there exist $A \in L^1(\Omega)$ and $B \in \mathbb{R}$ such that, for almost all $x \in \Omega$,

$$|G(x,t)| \leq A(x) + B|t|^p.$$

Hence, under this condition, f_h is a C^1 functional and $f_{h,\psi}$ is lower semicontinuous (because K_ψ is a closed subset of $H_0^1(\Omega)$). Moreover it is easy to verify that a function $u \in H_0^1(\Omega)$ solves problem $P_\psi(h)$ if and only if $0 \in \partial^- f_{h,\psi}(u)$.

Finally, let us introduce some notations we shall use in this paper:

- We denote by $(\lambda_i)_i$ the eigenvalues of the Laplace operator with zero boundary condition on Ω and by e_1 the eigenfunction related to λ_1, positive and satisfying $\int_\Omega e_1^2 dx = 1$.
- For all $t \in \mathbb{R}$, we set

$$P_t = \{u \in L^2(\Omega) \mid \int_\Omega ue_1dx = t\} \text{ and } S_t = \{u \in L^2(\Omega) | \int_\Omega ue_1dx \leq t\}.$$

- Usually, the Hilbert space we shall consider in this paper will be $H_0^1(\Omega)$. For simplicity (when this does not give rise to any ambiguity), we shall denote by $\|\cdot\|$ and $\|\cdot\|_2$ the usual norms in $H_0^1(\Omega)$ and in $L^2(\Omega)$ respectively.

3. Supersolutions and related properties

In this section we give the definition of a new type of supersolution, different from the classical one, which seems to be appropriate and useful to handle constraints as K_ψ (a similar notion has been used in [10]). We prove that these supersolutions can be used as "upper fictitious obstacles" in our problem (see Proposition 3.5). This property allows us to study the structure of the set of the pairs (ψ, h) for which problem $P_\psi(h)$ has solutions and to describe the main properties of the set of solutions of $P_\psi(h)$.

Definition 3.1. Let $h \in L^2(\Omega)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying condition (2.2). We say that $\bar\psi \in H_0^1(\Omega)$ is a supersolution for the operator $I + \Delta^{-1}(g(x,\cdot) - h)$ if

$$\int_\Omega (D\bar\psi Dw - g(x,\bar\psi)w + hw)dx \geq 0 \quad \forall w \in K_0.$$

Remark 3.2. It is evident that every solution of problem $P_\psi(h)$ is a supersolution for the operator $I+\Delta^{-1}(g(x,\cdot)-h)$. Moreover, it is easy to verify that, if $\bar\psi$ is a supersolution for the operator $I+\Delta^{-1}(g(x,\cdot)-h)$ and $h' \geq h$, then $\bar\psi$ is a supersolution for the operator $I+\Delta^{-1}(g(x,\cdot)-h')$ too.

Proposition 3.3. *The function $\bar\psi$ is a supersolution for the operator $I+\Delta^{-1}(g(x,\cdot)-h)$ if and only if $\bar\psi+\Delta^{-1}(g(x,\bar\psi)-h)\geq 0$ a.e. in Ω.*

Proof. If $\bar\psi$ is a supersolution, then Definition 3.1 (with $w=-\Delta^{-1}\varphi$) implies

$$\int_\Omega [\bar\psi+\Delta^{-1}(g(x,\bar\psi)-h)]\varphi \geq 0 \quad \forall \varphi\in L^2(\Omega) \text{ such that } \varphi\geq 0;$$

so $\bar\psi+\Delta^{-1}(g(x,\bar\psi)-h)\geq 0$ a.e. in Ω.

Conversely, if $\bar\psi+\Delta^{-1}(g(x,\psi)-h)\geq 0$, then it is easily seen that

$$\int_\Omega (D\bar\psi Dw - g(x,\bar\psi)w + hw)dx \geq 0$$

for every $w\in H_0^1(\Omega)$ such that $\Delta w\leq 0$ in weak sense, i.e. for every $w\in K_0$. □

Proposition 3.4. *Assume that the function g satisfies condition (2.2) and that*

$$g(x,\cdot) \ \text{ is a nondecreasing function for a.a. } x\in\Omega. \tag{3.1}$$

Let $\bar\psi_1$, $\bar\psi_2$ be supersolutions for the operator $I+\Delta^{-1}(g(x,\cdot)-h)$. Then $\bar\psi_1\wedge\bar\psi_2$ is a supersolution too.

Proof. Proposition 3.3 and assumption (3.1) guarantee that

$$\bar\psi_1 \geq -\Delta^{-1}(g(x,\bar\psi_1)-h) \geq -\Delta^{-1}(g(x,\bar\psi_1\wedge\bar\psi_2)-h) \qquad \text{a.e. in } \Omega$$

$$\bar\psi_2 \geq -\Delta^{-1}(g(x,\bar\psi_2)-h) \geq -\Delta^{-1}(g(x,\bar\psi_1\wedge\bar\psi_2)-h) \qquad \text{a.e. in } \Omega.$$

Therefore

$$\bar\psi_1\wedge\bar\psi_2 \geq -\Delta^{-1}(g(x,\bar\psi_1\wedge\bar\psi_2)-h) \qquad \text{a.e. in } \Omega$$

that is, again by Proposition 3.3, $\bar\psi_1\wedge\bar\psi_2$ is a supersolution. □

Proposition 3.5. *Let g satisfy condition (2.2) and $\bar\psi\in K_\psi$ be a supersolution for the operator $I+\Delta^{-1}(g(x,\cdot)-h)$. Set $K=\{u\in K_\psi \mid u\leq\bar\psi \text{ a.e. in } \Omega\}$ and assume that w is a lower critical point for f_h+I_K. If we assume in addition that g satisfies condition (3.1), then w is a solution of problem $P_\psi(h)$.*

Proof. First of all, let us remark that $\bar\psi \geq -\Delta^{-1}(g(x,\bar\psi)-h)$ a.e., because $\bar\psi$ is a supersolution. Moreover $g(x,w)-h\leq g(x,\bar\psi)-h$, because $w\in K$ and (3.1) holds. So we obtain:

$$-\Delta^{-1}(g(x,w)-h) \leq -\Delta^{-1}(g(x,\bar\psi)-h) \leq \bar\psi. \tag{3.2}$$

The function w verifies

$$\int_\Omega [DwD(v-w) - g(x,w)(v-w) + h(v-w)]dx \geq 0 \qquad \forall v \in K;$$

therefore, if we put

$$\tilde{f}(u) = \frac{1}{2}\int_\Omega |Du|^2 dx + \int_\Omega [h - g(x,w)]u dx,$$

w is a lower critical point for $\tilde{f} + I_K$. The functional $\tilde{f} + I_{K_\psi}$ is strictly convex, lower semicontinuous and coercive; so there exists only one minimum point for $\tilde{f}$ on K_ψ; let us call it $\widetilde{w}$.

The function $\widetilde{w}$ verifies

$$\int_\Omega D\widetilde{w}D(v-\widetilde{w})dx - \int_\Omega (g(x,w) - h)(v-\widetilde{w})dx \geq 0 \quad \forall v \in K_\psi. \tag{3.3}$$

The functional $\tilde{f} + I_K$ admits only one lower critical point (its unique minimum point), because it is strictly convex; so, if we show that $\widetilde{w} \leq \bar{\psi}$, then we have $\widetilde{w} = w$ and (3.3) gives us the desired conclusion.

Applying Lemma 2.3 with $k = h - g(x,w)$, we have that $\widetilde{w}$ is a lower critical point for the functional

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega D\psi D u dx$$

constrained on the set

$$\bar{K} = \{u \in H_0^1(\Omega) \mid u \geq \Delta^{-1}(h - g(x,w)) \text{ a.e. in } \Omega\}.$$

The function $\bar{\psi}$ is in the set $\bar{K}$ by (3.2) and it verifies $\Delta\bar{\psi} \leq \Delta\psi$ (in weak sense) by assumption; then it is a supersolution for the operator F' (in the usual sense: see, for example, [11, 12]).

Therefore, as it is stated in [11], the functional $F + I_{\bar{K}}$ has a lower critical point, that we call w', satisfying $w' \leq \bar{\psi}$; but $F + I_{\bar{K}}$ has only one critical point, because it is strictly convex, so $\widetilde{w} = w'$. This implies that $\widetilde{w} \leq \bar{\psi}$ and so $\widetilde{w} = w$, which completes the proof. □

Proposition 3.6. *Let g satisfy conditions* (2.2) *and* (3.1)*; let $\bar{\psi} \in K_\psi$ be a supersolution for the operator $I + \Delta^{-1}(g(x,\cdot) - h)$. Then problem $P_\psi(h)$ has at least one solution w such that $w \leq \bar{\psi}$ a.e. in Ω.*

Proof. In this proof we use the notations introduced in Proposition 3.5. Clearly it is sufficient to find a minimum point w for the functional $f_h + I_K$ and then apply Proposition 3.5. In order to prove the existence of such a minimum point, observe that $K \subset \{u \in H_0^1(\Omega) \mid \psi \leq u \leq \bar{\psi}\}$ with ψ and $\bar{\psi}$ in $H_0^1(\Omega)$; taking also into account condition (2.2), this fact implies that the sublevels of $f_h + I_K$ are bounded in $H_0^1(\Omega)$ and that $f_h + I_K$ is weakly lower semicontinuous (even if $p = 2^*$). It follows that every minimizing sequence for $f_h + I_K$ is weakly convergent in $H_0^1(\Omega)$ (up to a subsequence) and the weak limit is a minimum point for $f_h + I_K$. □

Proposition 3.7. *Under the same assumptions as in* Proposition 3.6, *we have:*

(i) *if problem $P_\psi(h)$ has solution, then there exists a solution for every problem $P_{\psi'}(h')$ such that $h' \geq h$ a.e. in Ω and $\Delta\psi' \geq \Delta\psi$ in weak sense;*

(ii) *if u_1 and u_2 are solutions of problem $P_\psi(h)$, then there exists a solution u such that $u \leq u_1 \wedge u_2$.*

The proof follows easily from Propositions 3.6 and 3.4, taking into account Remark 3.2.

In order to prove the existence of a minimal solution of $P_\psi(h)$, we need the following result.

Lemma 3.8 (see, for example, [5]). *Let H be a Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. Suppose that there exists $c \in \mathbb{R}$ such that*

$$f(v) \geq f(u) + (\alpha, v-u) - c\|v-u\|^2 \quad \forall u, v \in \mathcal{D}(f) \;\; \forall \alpha \in \partial^- f(u).$$

Let $(u_m)_m$ and $(\alpha_m)_m$ be two sequences such that $u_m \in \mathcal{D}(f)$, $\alpha_m \in \partial^- f(u_m)$ for every $m \in \mathbb{N}$, $\lim_{m\to\infty} u_m = u$ and $\alpha_m \rightharpoonup \alpha$ weakly in H.

Then $u \in \mathcal{D}(f)$, $\lim_{m\to\infty} f(u_m) = f(u)$ and $\alpha \in \partial^- f(u)$.

Proposition 3.9. *Assume that g satisfies conditions* (2.2) *and* (3.1) *and, in addition, there exists $\lambda \in \mathbb{R}$ such that, for almost all $x \in \Omega$,*

$$\frac{g(x,t_1) - g(x,t_2)}{t_1 - t_2} \leq \lambda \quad \text{for } t_1 \neq t_2. \tag{3.4}$$

If problem $P_\psi(h)$ has solution, then there exists a minimal solution $\bar{u}$ (that is $\bar{u} \leq u$ a.e. in Ω, for every u solution of $P_\psi(h)$).

Proof. First let us remark that condition (3.4) implies

$$f_{h,\psi}(v) \geq f_{h,\psi}(u) + (\alpha, v-u) + \frac{1}{2}\|v-u\|^2 - \frac{\lambda}{2}\|v-u\|_2^2$$
$$\forall u, v \in K_\psi \;\; \forall \alpha \in \partial^- f_{h,\psi}(u).$$

Therefore, since

$$\|v-u\|_2^2 \leq c(\Omega)\|v-u\|^2 \quad \forall u, v \in H_0^1(\Omega)$$

for a suitable constant $c(\Omega)$, we can apply Lemma 3.8 with $H = H_0^1(\Omega)$ and $f = f_{h,\psi}$.

Let $(u_m)_m$ be a sequence of solutions of $P_\psi(h)$ such that

$$\lim_{m\to\infty} \int_\Omega u_m dx = \inf\left\{ \int_\Omega u dx \;\middle|\; u \text{ solution of } P_\psi(h) \right\}$$

(notice that this infimum is finite because there exists at least one solution and $K_\psi \subset \{u \in H_0^1(\Omega) \mid u \geq \psi\}$).

Let us fix $v \in K_\psi$. By (3.4) we obtain

$$f_h(v) \geq f_h(u_m) - \frac{\lambda}{2}\|v - u_m\|_2^2 \geq \frac{1}{2}\int_\Omega |Du_m|^2 dx - \\ \int_\Omega |g(x,0)||u_m| dx - \frac{\lambda}{2}\int_\Omega u_m^2 dx + \int_\Omega h u_m dx - \frac{\lambda}{2}\|v - u_m\|_2^2, \tag{3.5}$$

with $g(\cdot, 0) \in L^{p/(p-1)}(\Omega)$ because of condition (2.2)

We claim that $\sup_{m\in\mathbb{N}} \|u_m\|_2 < +\infty$. In fact, otherwise, we could assume that, up to a subsequence, $\lim_{m\to\infty} \|u_m\|_2 = +\infty$. Let us set $z_m = u_m/\|u_m\|_2$; from (3.5) we deduce that $\sup_{m\in\mathbb{N}} \|z_m\| < +\infty$ and, consequently, $(z_m)_m$ (or a subsequence) converges in $L^2(\Omega)$ and a.e. in Ω to a function $z \in H_0^1(\Omega)$ with the properties that $\|z\|_2 = 1$ and $z \geq 0$, because $u_m \geq \psi$. Hence

$$\lim_{m\to\infty} \int_\Omega z_m dx = \int_\Omega z dx > 0.$$

But this is impossible, since

$$\lim_{m\to\infty} \int_\Omega z_m dx = \lim_{m\to\infty} \frac{1}{\|u_m\|_2} \int_\Omega u_m dx \leq 0$$

because $\lim_{m\to\infty} \int_\Omega u_m dx < +\infty$ and $\lim_{m\to\infty} \|u_m\|_2 = +\infty$.

So $(u_m)_m$ must be bounded in $L^2(\Omega)$ and, from (3.5), it follows that $(u_m)_m$ is bounded in $H_0^1(\Omega)$ too; therefore, up to a subsequence, it converges in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to a function $\bar{u}$ that, by Lemma 3.8, is a solution for $P_\psi(h)$. Let us remark that

$$\int_\Omega \bar{u} dx = \min\left\{ \int_\Omega u dx \;\middle|\; u \text{ solution of } P_\psi(h) \right\}. \tag{3.6}$$

Hence we deduce that $\bar{u}$ is the minimal solution. In fact if, by contradiction, there exists a solution u such that $\bar{u} \wedge u \neq \bar{u}$, then there exists another solution $w \leq \bar{u} \wedge u$, because of Proposition 3.7. Therefore $\int_\Omega w dx < \int_\Omega \bar{u} dx$, in contradiction with (3.6). □

Another consequence of the properties of the supersolutions is the following result, that holds if $g(x, \cdot)$ is a convex function.

Proposition 3.10. *Let g satisfy conditions* (2.2) *and* (3.1) *and assume, in addition, that $g(x,\cdot)$ is a convex function for a.a. $x \in \Omega$. Then the set of the pairs (ψ, h) such that $P_\psi(h)$ has solution is convex.*

Proof. It suffices to observe that, if u_i is a solution of problem $P_{\psi_i}(h_i)$, $i = 1, 2$, then $tu_1 + (1-t)u_2$ is in $K_{t\psi_1 + (1-t)\psi_2}$ and is a supersolution for the operator $I + \Delta^{-1}(g(x,\cdot) - th_1 - (1-t)h_2)$, for all $t \in [0,1]$, because of the convexity of $g(x,\cdot)$. Hence we can complete the proof using Proposition 3.6. □

Now we shall use the supersolutions to describe some closure properties of the set of data ψ and h for which $P_\psi(h)$ has solution.

Lemma 3.11. *Let g satisfy conditions* (2.2) *and* (3.4) *and assume, in addition, that there exist $\bar\lambda \in \mathbb{R}$ and $c \in L^2(\Omega)$ such that, for almost all $x \in \Omega$,*

$$g(x,t) \geq \bar\lambda t - c(x) \quad \forall t \geq 0. \tag{3.7}$$

Then for all $u \in K_\psi$ and $\alpha \in \partial^- f_{h,\psi}(u)$ we have:

$$\int_\Omega [(\lambda_1 - \bar\lambda)u^+ - (\lambda_1 - \lambda)u^- + c + h]e_1 dx \geq (\alpha, e_1).$$

Proof. Set $v = u + e_1$ (note that $v \in K_\psi$). Then, for every $\alpha \in \partial^- f_{h,\psi}(u)$, it holds

$$\begin{gathered}(\alpha, v-u) \leq f'_h(u)[v-u] = \\ \int_\Omega [DuDe_1 - g(x,u)e_1 + he_1]dx = \int_\Omega [\lambda_1 u - g(x,u) + h]e_1 dx.\end{gathered} \tag{3.8}$$

Furthermore, if $\Omega^+ = \{x \in \Omega \mid u(x) \geq 0\}$, then

$$\begin{gathered}\int_\Omega g(x,u)e_1 dx = \int_{\Omega^+} g(x,u)e_1 dx + \int_{\Omega\setminus\Omega^+} g(x,u)e_1 dx \geq \\ \int_{\Omega^+} (\bar\lambda u - c)e_1 dx + \int_{\Omega\setminus\Omega^+} (\lambda u - c)e_1 dx = \int_\Omega (\bar\lambda u^+ - \lambda u^- - c)e_1 dx,\end{gathered}$$

which, together with (3.8), completes the proof. □

Lemma 3.12. *Let g satisfy conditions* (2.2) *and* (3.4)*. Moreover, let* (3.7) *hold with $\bar\lambda > \lambda_1$ (see also* Remark 3.13*). Let $(\psi_m)_m$ and $(h_m)_m$ be two sequences such that, for all $m \in \mathbb{N}$, $\psi_m \in H_0^1(\Omega)$, $h_m \in L^2(\Omega)$ and $P_{\psi_m}(h_m)$ has at least one solution u_m. Furthermore assume that $\psi_m \to \psi$ in $H_0^1(\Omega)$ and $h_m \to h$ in $L^2(\Omega)$ as $m \to \infty$. Then:*

(i) *the sequence $(u_m)_m$ is bounded in $H_0^1(\Omega)$;*
(ii) *if $(u_m)_m$ (or a subsequence) converges to u in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, then u solves problem $P_\psi(h)$;*
(iii) *problem $P_\psi(h)$ has at least one solution.*

Proof. Taking into account (3.4), we obtain

$$\begin{gathered}f_{h_m}(\psi_m) \geq f_{h_m}(u_m) - \frac{\lambda}{2}\|\psi - u_m\|_2^2 \geq \frac{1}{2}\int_\Omega |Du_m|^2 dx - \\ \int_\Omega |g(x,0)||u_m| dx - \frac{\lambda}{2}\int_\Omega u_m^2 dx + \int_\Omega h_m u_m dx - \frac{\lambda}{2}\|\psi - u_m\|_2^2,\end{gathered} \tag{3.9}$$

with $\int_\Omega |g(x,0)||u_m|dx \leq c_1\|u_m\|$, for a suitable $c_1 \in \mathbb{R}$, because of condition (2.2).

Let us prove that the sequence $(u_m)_m$ is bounded in $L^2(\Omega)$. In fact, arguing by contradiction, assume that, up to a subsequence, $\lim_{m\to\infty}\|u_m\|_2 = +\infty$. If we put $z_m = u_m/\|u_m\|_2$, from (3.9) we deduce that $(z_m)_m$ is bounded in $H_0^1(\Omega)$; so, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω, to a function $z \in H_0^1(\Omega)$. The function z verifies:

$$\|z\|_2 = 1 \quad \text{and} \quad z \geq 0 \text{ in } \Omega \tag{3.10}$$

(because $u_m \geq \psi_m$ a.e. in Ω and $\psi_m \to \psi$ in $H_0^1(\Omega)$).

By Lemma 3.11 we have

$$\frac{1}{\|u_m\|_{L^2}} \int_\Omega [(\lambda_1 - \bar{\lambda})u_m^+ - (\lambda_1 - \lambda)u_m^- + c + h_m]e_1 dx \geq 0,$$

from which, as $m \to \infty$, we obtain $(\lambda_1 - \bar{\lambda}) \int_\Omega z e_1 dx \geq 0$, which is impossible because $\bar{\lambda} > \lambda_1$ and (3.10) holds. So the sequence $(u_m)_m$ has to be bounded in $L^2(\Omega)$ and then, from (3.9), it follows that it is bounded in $H_0^1(\Omega)$ too. Thus (i) is proved.

Let us prove (ii): for all $v \in K_\psi$, set $v_m = v + \psi_m - \psi$. Then $v_m \in K_{\psi_m}$ $\forall m \in \mathbb{N}$ and so, by (3.4),

$$f_{h_m}(v_m) \geq f_{h_m}(u_m) - \frac{\lambda}{2}\|v_m - u_m\|_2^2 \qquad \forall m \in \mathbb{N};$$

letting $m \to \infty$, since $v_m \to v$ in $H_0^1(\Omega)$, we get

$$f_h(v) \geq f_h(u) - \frac{\lambda}{2}\|v - u\|_2^2 \qquad \forall v \in K_\psi,$$

which gives (ii).

Assertion (iii) is a direct consequence of (i) and (ii). □

Remark 3.13. Notice that in Lemma 3.12 the condition $\bar{\lambda} > \lambda_1$ cannot be removed, as showed by the following example.

Let $g(x,t) = \lambda_1 t$; choose $\psi \in H_0^1(\Omega)$ such that $\sup_\Omega(\psi/e_1) = +\infty$ and set $\psi_m = \psi$ and $h_m = (e_1/m)$ for all $m \in \mathbb{N}$. Then Theorem 6.1 of [10] guarantees that $P_{\psi_m}(h_m)$ has a unique solution u_m for all $m \in \mathbb{N}$, while the limit problem $P_\psi(0)$ has no solution. In fact, by Theorem 6.1 of [10], every solution of $P_\psi(0)$ should be an eigenfunction for the first eigenvalue λ_1 (which cannot belong to K_ψ under our assumption). In this case the sequence of solutions $(u_m)_m$ is not bounded in $H_0^1(\Omega)$.

4. Existence and multiplicity results

In this section we use the topological methods of Calculus of Variations to study the existence and multiplicity of solutions for our problem, i.e. we analyse the topological properties of the sublevels of the functional $f_{h,\psi}$ in order to evaluate the number of solutions of $P_\psi(h)$.

Theorem 4.1. *Let g satisfy conditions (2.2), (3.1) and (3.4); let $\psi \in H_0^1(\Omega)$ and $\bar{h} \in L^2(\Omega)$. Then there exists $\tau_1 \in [-\infty, +\infty[$ such that problem $P_\psi(\bar{h} + \tau e_1)$ has at least one solution for every $\tau > \tau_1$, while it has no solution if $\tau < \tau_1$.*

Furthermore, if we assume in addition that condition (3.7) holds with $\bar{\lambda} > \lambda_1$, then $\tau_1 > -\infty$, $P_\psi(\bar{h} + \tau_1 e_1)$ has solution and there exists $\tau_2 \geq \tau_1$ such that problem $P_\psi(\bar{h} + \tau e_1)$ has at least two solutions for every $\tau > \tau_2$.

To prove this theorem, we need some preliminary results. In particular Lemma 4.3 gives us a compactness property for the (nonsmooth) functional $f_{h,\psi}$, analogous to Palais-Smale condition.

Lemma 4.2. *Let $\psi \in H^1_0(\Omega)$, $\bar h \in L^2(\Omega)$ and g satisfy conditions (2.2) and (3.4). Then, for all $t \in \mathbb{R}$ such that $K_\psi \cap S_t \neq \emptyset$, the sublevels of the functional $f_{\bar h,\psi} + I_{S_t}$ are bounded in $H^1_0(\Omega)$ (see the notations introduced in* Section 2*).*

Proof. Condition (3.4) implies

$$f_{\bar h}(u) \geq \frac{1}{2}\int_\Omega |Du|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega |g(x,0)u| dx + \int_\Omega \bar h u\; dx \quad \forall u \in H^1_0(\Omega) \tag{4.1}$$

with $\int_\Omega |g(x,0)u| dx \leq c_1 \|u\|$ by (2.2).

Let us prove that the sublevels of $f_{\bar h,\psi} + I_{S_t}$ are bounded in $L^2(\Omega)$. Suppose, contrary to our claim, that there exists a sequence $(u_m)_m$ in a sublevel of $f_{\bar h,\psi} + I_{S_t}$ such that $\lim_{m\to\infty} \|u_m\|_2 = +\infty$ and set $z_m = u_m/\|u_m\|_2$. Inequality (4.1) implies

$$\limsup_{m\to\infty} \int_\Omega |Dz_m|^2 dx < +\infty.$$

Hence there exists a function $z \in H^1_0(\Omega)$ such that (up to a subsequence) $z_m \to z$ in $L^2(\Omega)$ and a.e. in Ω. Furthermore

$$\|z\|_2 = 1 \quad \text{and} \quad z \geq 0 \quad (\text{because } u_m \geq \psi). \tag{4.2}$$

On the other hand we have

$$\lim_{m\to\infty} \frac{1}{\|u_m\|_2} \int_\Omega u_m e_1 dx = \lim_{m\to\infty} \int_\Omega z_m e_1 dx = \int_\Omega z e_1 dx$$

which is not possible, because $\int_\Omega u_m e_1 dx \leq t$ implies

$$\lim_{m\to\infty} \frac{1}{\|u_m\|_2} \int_\Omega u_m e_1 dx = 0,$$

while $\int_\Omega z e_1 dx > 0$ by (4.2). □

Lemma 4.3. *Let g satisfy conditions (2.2) and (3.4); assume that condition (3.7) holds with $\bar\lambda > \lambda_1$. If $(u_m)_m$ is a sequence in K_ψ such that $\partial^- f_{h,\psi}(u_m) \neq \emptyset$ $\forall m \in \mathbb{N}$ and $\sup_{m\in\mathbb{N}} \|\operatorname{grad}^- f_{h,\psi}(u_m)\| < +\infty$, then $(u_m)_m$ is bounded in $H^1_0(\Omega)$.*

If we assume in addition that $\lim_{m\to\infty} \|\operatorname{grad}^- f_{h,\psi}(u_m)\| = 0$, then the sequence $(u_m)_m$ is relatively compact in $H^1_0(\Omega)$.

Proof. Set $\alpha_m = \operatorname{grad}^- f_{h,\psi}(u_m)$ and fix $\bar u \in K_\psi$. Then we have

$$\begin{gathered} f_{h,\psi}(\bar u) \geq f_{h,\psi}(u_m) + (\alpha_m, \bar u - u_m) - \frac{\lambda}{2}\|\bar u - u_m\|_2^2 \geq \frac{1}{2}\int_\Omega |Du_m|^2 dx - \\ \frac{\lambda}{2}\int_\Omega u_m^2 dx - \int_\Omega |g(x,0)u_m| dx + \int_\Omega h u_m dx + (\alpha_m, \bar u - u_m) - \frac{\lambda}{2}\|\bar u - u_m\|_2^2. \end{gathered} \tag{4.3}$$

Let us prove that $(u_m)_m$ is bounded in $L^2(\Omega)$. Arguing by contradiction, assume that (up to a subsequence) $\lim_{m\to\infty} \|u_m\|_2 = +\infty$.

Set $z_m = u_m/\|u_m\|_2$. From (4.3) it follows that $(z_m)_m$ is bounded in $H_0^1(\Omega)$; so there is a subsequence (still denoted by $(z_m)_m$) converging in $L^2(\Omega)$ and a.e. in Ω to a function $z \in H_0^1(\Omega)$. Furthermore

$$\|z\|_2 = 1 \quad \text{and } z \geq 0 \ \text{ in } \ \Omega \quad (\text{because } u_m \geq \psi). \tag{4.4}$$

On the other hand, from Lemma 3.11 it follows that

$$(\lambda_1 - \bar{\lambda}) \int_\Omega z e_1 dx \geq 0$$

which is not possible because $\bar{\lambda} > \lambda_1$ and (4.4) holds.

Hence $(u_m)_m$ is bounded in $L^2(\Omega)$ and, by (4.3), in $H_0^1(\Omega)$.

Now assume in addition that $\alpha_m \to 0$ in $H_0^1(\Omega)$ and prove that $(u_m)_m$ is relatively compact in $H_0^1(\Omega)$. Since $(u_m)_m$ is bounded in $H_0^1(\Omega)$, up to a subsequence, we have $u_m \to u$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ and $L^p(\Omega)$, for a suitable $u \in H_0^1(\Omega)$. Arguing as in (4.3) we obtain

$$f_{h,\psi}(u) \geq f_{h,\psi}(u_m) + (\alpha_m, u - u_m) - \frac{\lambda}{2}\|u - u_m\|_2^2$$

with $(\alpha_m, u - u_m) \to 0$ (as $\|\alpha_m\| \to 0$ and $(\|u_m\|)_m$ is bounded).

Therefore

$$\limsup_{m\to\infty} f_{h,\psi}(u_m) \leq f_{h,\psi}(u).$$

Taking into account (2.2) and (3.4), by Fatou's Lemma, we infer

$$\limsup_{m\to\infty} \int_\Omega |Du_m|^2 dx \leq \int_\Omega |Du|^2 dx,$$

which implies $u_m \to u$ in $H_0^1(\Omega)$. □

Proof of Theorem 4.1. Taking into account Proposition 3.7, we have only to prove that there exists $\tau \in \mathbb{R}$ such that problem $P_\psi(\bar{h} + \tau e_1)$ has solution.

Choose $t \in \mathbb{R}$ such that $t > \int_\Omega \psi e_1 dx$. Since the functional $f_{h,\psi}$ is weakly lower semicontinuous, Lemma 4.2 implies that, for all $h \in L^2(\Omega)$, the minimum of $f_{h,\psi}$ constrained on the subsets S_t and P_t is achieved. Moreover, for τ sufficiently large, we have

$$\min\{f_{\bar{h}+\tau e_1,\psi}(u) \mid u \in S_t\} \leq f_{\bar{h}+\tau e_1,\psi}(\psi) = f_{\bar{h}}(\psi) + \tau \int_\Omega \psi e_1 dx <$$

$$\min\{f_{\bar{h},\psi}(u) \mid u \in P_t\} + \tau t = \min\{f_{\bar{h}+\tau e_1,\psi}(u) \mid u \in P_t\}. \tag{4.5}$$

Hence a solution of problem $P_\psi(\bar{h} + \tau e_1)$ can be obtained as minimum point of the functional $f_{\bar{h}+\tau e_1,\psi}$ on the open subset $S_t \setminus P_t$.

Now let us prove the second part of the theorem.

Assume, contrary to our claim, that $\tau_1 = -\infty$. Then problem $P_\psi(\bar{h} - me_1)$ has a solution u_m for every $m \in \mathbb{N}$.

Let us fix $\bar{u} \in K_\psi$. By (2.2) and (3.4) we have

$$f_{\bar{h}}(\bar{u}) - m\int_\Omega \bar{u}e_1 dx = f_{\bar{h}-me_1}(\bar{u}) \geq$$
$$f_{\bar{h}-me_1}(u_m) - \frac{\lambda}{2}\|\bar{u} - u_m\|_2^2 \geq \frac{1}{2}\int_\Omega |Du_m|^2 dx - \int_\Omega |g(x,0)u_m| dx -$$
$$\frac{\lambda}{2}\int_\Omega u_m^2 dx + \int_\Omega (\bar{h} - me_1)u_m dx - \frac{\lambda}{2}\|\bar{u} - u_m\|_2^2.$$

Therefore there exists a constant $c_2 > 0$ such that, if we set $z_m = u_m/m$, we have

$$\int_\Omega |Dz_m|^2 dx \leq c_2\left(1 + \int_\Omega z_m^2 dx\right) \qquad \forall m \in \mathbb{N}. \tag{4.6}$$

Now, if $(z_m)_m$ is bounded in $L^2(\Omega)$, it follows from (4.6) that it is bounded in $H_0^1(\Omega)$ too. So, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω to a function $z \in H_0^1(\Omega)$ such that $z \geq 0$ (because $u_m \geq \psi$). Then Lemma 3.11 yields

$$(\lambda_1 - \bar{\lambda})\int_\Omega z e_1 dx - \int_\Omega e_1^2\ dx \geq 0,$$

which is not possible because $\bar{\lambda} > \lambda_1$, $z \geq 0$ and $\int_\Omega e_1^2\ dx = 1$.

Hence let us consider the other case, i.e. $(z_m)_m$ not bounded in $L^2(\Omega)$; up to a subsequence we can assume that $\lim_{m\to\infty}\|z_m\|_2 = +\infty$.

Set $z'_m = z_m/\|z_m\|_2$; from (4.6) it follows that $(z'_m)_m$ is bounded in $H_0^1(\Omega)$; so, up to a subsequence, it converges in $L^2(\Omega)$ and a.e. in Ω to a function $z' \in H_0^1(\Omega)$ such that $\|z'\|_2 = 1$ and $z' \geq 0$. Then Lemma 3.11 yields

$$(\lambda_1 - \bar{\lambda})\int_\Omega z' e_1 dx \geq 0,$$

which is not possible because $\bar{\lambda} > \lambda_1$ and $\int_\Omega z' e_1 dx > 0$ (since $z' \geq 0$ and $\|z'\|_2 = 1$).

So it must be $\tau_1 > -\infty$.

The solvability of problem $P_\psi(\bar{h} + \tau_1 e_1)$ is a straightforward consequence of Lemma 3.12.

Our next claim is the existence of two distinct solutions, for τ sufficiently large.

Choose τ_2 large enough in such a way that (4.5) holds for all $\tau > \tau_2$ and set $h = \bar{h} + \tau e_1$ for a fixed $\tau > \tau_2$. Let $\bar{u}$ be a minimum point for the functional $f_{h,\psi}$ on the set $K_\psi \cap S_t$. Since $\Delta e_1 < 0$ on Ω, $\bar{u} + se_1 \in K_\psi$ for all $s \geq 0$. Let us prove that

$$\lim_{s\to\infty} f_{h,\psi}(\bar{u} + se_1) = -\infty. \tag{4.7}$$

In fact condition (3.7) implies

$$
\begin{aligned}
f_{h,\psi}(\bar{u}+se_1) \le \frac{1}{2}\int_\Omega |D(\bar{u}+se_1)|^2 dx - \frac{\bar{\lambda}}{2}\int_\Omega (\bar{u}+se_1)^2 dx + \\
\int_\Omega c(\bar{u}+se_1)^+ dx + \int_\Omega \bar{u}h\ dx + s\int_\Omega he_1 dx + k_1 \le \\
\frac{\lambda_1}{2}s^2 - \frac{\bar{\lambda}}{2}s^2 + k_2 s + k_3 \quad \forall s \ge 0,
\end{aligned}
$$

where k_1, k_2 and k_3 are suitable positive numbers, which do not depend on s. Hence (4.7) follows since $\bar{\lambda} > \lambda_1$.

Let us set

$$
d_1 = \min\{f_{h,\psi}(u) \mid u \in P_t\}; \tag{4.8}
$$

(4.5) implies $f_{h,\psi}(\bar{u}) < d_1$; (4.7) allows us to choose $\bar{s}$ such that, if we put $v = \bar{u} + \bar{s}e_1$, then

$$
f_{h,\psi}(v) < f_{h,\psi}(\bar{u}). \tag{4.9}
$$

Now set

$$
d_2 = \sup_{s\in[0,\bar{s}]} f_{h,\psi}(\bar{u}+se_1). \tag{4.10}
$$

Notice that $d_2 \ge d_1$. In fact $\int_\Omega ve_1 dx > t$ by (4.9) and $\int_\Omega \bar{u}e_1 dx < t$ by (4.5) (since $\bar{u}$ is a minimum point for $f_{h,\psi}$ on S_t); hence there exists $\tilde{s} \in]0,\bar{s}[$ such that $\int_\Omega (\bar{u}+\tilde{s}e_1)e_1 dx = t$.

Let us prove that there exists a lower critical value $c \in [d_1, d_2]$ for the functional $f_{h,\psi}$.

Arguing by contradiction, assume that $[d_1, d_2]$ does not contain any lower critical value. Taking into account the Palais-Smale type condition given by Lemma 4.3, it follows that, for all $\varepsilon > 0$ small enough, the sublevel $f_{h,\psi}^{d_1-\varepsilon}$ is a deformation retract of $f_{h,\psi}^{d_2}$ (see, for example, [5, 6, 7, 9, 12]). Then, if we choose $\varepsilon > 0$ small enough in such a way that $f_{h,\psi}(\bar{u}) < d_1 - \varepsilon$, we get a contradiction. In fact $f_{h,\psi}^{d_1-\varepsilon}$ contains $\bar{u}$ and v, but does not contain any continuous path connecting this two points, because $f_{h,\psi}^{d_1-\varepsilon} \cap P_t = \emptyset$. On the contrary $f_{h,\psi}^{d_2}$ contains the segment $\{\bar{u}+se_1 \mid s \in [0,\bar{s}]\}$. Therefore $f_{h,\psi}^{d_1-\varepsilon}$ cannot be a deformation retract of $f_{h,\psi}^{d_2}$, which is a contradiction.

Summarizing, there exist the local minimum point $\bar{u}$ and the lower critical level c such that $f_{h,\psi}(\bar{u}) < d_1 \le c$. This implies the existence of two distinct lower critical points for $f_{h,\psi}$, hence two distinct solutions for $P_\psi(\bar{h}+\tau e_1)$ for all $\tau > \tau_2$. □

Let us remark that, if we take away the assumption (3.7) with $\bar{\lambda} > \lambda_1$ in Theorem 4.1, it is not possible, in general, to describe an analogous "folding type" behaviour for problem $P_\psi(h)$. For example, if $g(x,t) = \lambda t$ with $\lambda < \lambda_1$, then it is easily seen that problem $P_\psi(h)$ has exactly one solution for every h in $L^2(\Omega)$ and ψ in $H_0^1(\Omega)$ (see [10]). When g has such an asymptotic growth, the following existence result holds.

Proposition 4.4. *Let g satisfy condition* (2.2) *and assume in addition that there exist $c_2 \in L^1(\Omega)$ and $\lambda' < \lambda_1$ such that, for almost all $x \in \Omega$,*

$$G(x,t) \leq c_2(x) + \frac{\lambda'}{2} t^2 \quad \text{for } t \geq 0.$$

Then problem $P_\psi(h)$ has at least one solution for all $h \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)$.

Proof. We have only to show that the sublevels of the functional $f_{h,\psi}$ are bounded in $L^2(\Omega)$; in this case, in fact, one solution of $P_\psi(h)$ can be found minimizing $f_{h,\psi}$.

For all $u \in K_\psi$, we have:

$$\begin{aligned} f_{h,\psi}(u) = f_{h,\psi}(u^+) + f_{h,\psi}(-u^-) \geq \\ \frac{1}{2}\int_\Omega |Du^+|^2 dx - \frac{\lambda'}{2}\int_\Omega (u^+)^2 dx - \int_\Omega c_2(x) dx + \int_\Omega hu^+ dx + \\ \frac{1}{2}\int_\Omega |Du^-|^2 dx - \int_\Omega G(x,-u^-) dx - \int_\Omega hu^- dx \geq \\ \frac{1}{2}\left(1 - \frac{\lambda'}{\lambda_1}\right) \int_\Omega |Du^+|^2 dx + \int_\Omega hu^+ dx + \frac{1}{2}\int_\Omega |Du^-|^2 dx - \bar{c} \end{aligned}$$

for a suitable constant $\bar{c}$ independent of u (because $u^- \leq \psi^-$ and (2.2) holds). It follows that the sublevels of $f_{h,\psi}$ are bounded in $H_0^1(\Omega)$. Thus $f_{h,\psi}$ has at least one minimum point (since it is weakly lower semicontinuous), which gives us a solution of $P_\psi(h)$. □

5. Reduction to a finite dimentional problem

In this section we show that, if condition (3.4) holds with $\lambda \leq \lambda_2$, then problem $P_\psi(h)$ is equivalent to find the lower critical points of a suitable function defined in $\mathbb{R}$. This different approach allows us to specify the previous multiplicity results. In particular we shall prove that in Theorem 4.1 we have $\tau_1 = \tau_2$.

Definition 5.1. Let $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ and g satisfy conditions (2.2) and (3.4). Taking into account Lemma 4.2, using the notations introduced in section 2, we can consider the function $S_h : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$S_h(t) = \min_{u \in P_t} f_{h,\psi}(u)$$

(here we set $\min \emptyset = +\infty$).

Lemma 5.2. *Let $\psi \in H_0^1(\Omega)$, $h \in L^2(\Omega)$ and g satisfy conditions* (2.2) *and* (3.4). *Let S_h be the function introduced in* Definition 5.1. *Then:*

(i) *S_h is lower semicontinuous and $\mathcal{D}(S_h) = \left[\int_\Omega \psi e_1 dx, +\infty\right[$;*
(ii) *if $u \in K_\psi$ is a minimum point for $f_{h,\psi} + I_{P_t}$ and if $k \in \partial^- S_h(t)$, then $ke_1 \in \partial^- f_{h,\psi}(u)$;*
(iii) *if condition (3.4) holds with $\lambda \leq \lambda_2$ and $ke_1 \in \partial^- f_{h,\psi}(u)$, then:*
  (a) *u is a minimum point for $f_{h,\psi} + I_{P_{\bar{t}}}$ with $\bar{t} = \int_\Omega u e_1 dx$,*
  (b) *$k \in \partial^- S_h(\bar{t})$.*

Proof. (i) The lower semicontinuity of S_h follows easily from Lemma 4.2 and the weakly lower semicontinuity of $f_{h,\psi}$.

To find $\mathcal{D}(S_h)$ it suffices to remark that $u \geq \psi$ $\forall u \in K_\psi$ and that $\psi + te_1 \in K_\psi$ $\forall t \geq 0$.

(ii) It is a straightforward consequence of the definition of S_h.

(iii) To prove (a) it suffices to remark that $f_{h,\psi} + I_{P_{\bar{t}}}$ is convex if $\lambda \leq \lambda_2$.

To prove (b), let us remark that

$$f_h(v) \geq f_h(u) + f_h'(u)[v-u] + \frac{1}{2}\|v-u\|^2 - \frac{\lambda}{2}\|v-u\|_2^2 \quad \forall v \in H_0^1(\Omega).$$

Then, if $\lambda \leq \lambda_2$,

$$f_{h,\psi}(v) \geq f_{h,\psi}(u) + k\int_\Omega (v-u)e_1 dx - \frac{\lambda-\lambda_1}{2}\left(\int_\Omega (v-u)e_1 dx\right)^2 \quad \forall v \in H_0^1(\Omega).$$

Taking into account (a), we obtain

$$S_h(t) \geq S_h(\bar{t}) + k(t-\bar{t}) - \frac{\lambda-\lambda_1}{2}(t-\bar{t})^2 \quad \forall t \in \mathbb{R},$$

which obviously implies (b). □

Notice that Lemma 5.2 shows in particular that, if condition (3.4) holds with $\lambda \leq \lambda_2$, then problem $P_\psi(h)$ is equivalent to find lower critical points for S_h.

In this way we shall prove the following result.

Theorem 5.3. *Let $\psi \in H_0^1(\Omega)$, $\bar{h} \in L^2(\Omega)$ and g satisfy condition* (2.2). *Moreover assume that* (3.4) *holds with $\lambda \leq \lambda_2$ and* (3.7) *with $\bar{\lambda} > \lambda_1$.*

Then there exists $\bar{\tau} \in \mathbb{R}$ such that problem $P_\psi(\bar{h} + \tau e_1)$

(i) *has no solution for $\tau < \bar{\tau}$*
(ii) *has at least one solution for $\tau = \bar{\tau}$*
(iii) *has at least two solutions for all $\tau > \bar{\tau}$.*

To prove Theorem 5.3 we need some properties of the function S_h.

Lemma 5.4. *Let ψ, $\bar{h}$ and g satisfy the same conditions as in* Theorem 5.3 *and, for $\tau \in \mathbb{R}$, set $h = \bar{h} + \tau e_1$. Let S_h be the function introduced in* Definition 5.1. *Then:*

(i) *$S_h(t) + \frac{\lambda-\lambda_1}{2}t^2$ is convex;*
(ii) *S_h is continuous on its domain and $\partial^- S_h(t) \neq \emptyset$ $\forall t > \int_\Omega \psi e_1 dx$;*
(iii) $\lim_{t\to+\infty} S_h(t) = -\infty$.

Proof. (i) This assertion follows easily taking into account that the functional $f_{h,\psi}(u) + \frac{\lambda-\lambda_1}{2}\left(\int_\Omega u e_1 dx\right)^2$ is convex if condition (3.4) holds with $\lambda \leq \lambda_2$.

(ii) It is a straightforward consequence of (i) and (i) of Lemma 5.2.

(iii) It suffices to remark that

$$\lim_{t \to +\infty} f_{h,\psi}(\psi + te_1) = -\infty,$$

which holds if condition (3.7) is satisfied with $\bar{\lambda} > \lambda_1$, as it is proved in the proof of Theorem 4.1. □

Proof of Theorem 5.3. Define

$$\begin{aligned} \bar{\tau} &= \inf\{\tau \mid P_\psi(\bar{h} + \tau e_1) \ \text{ has a solution}\} \\ &= \inf\{\tau \mid S_{\bar{h}+\tau e_1} \ \text{ has a lower critical point}\}. \end{aligned}$$

By Theorem 4.1, $\bar{\tau} > -\infty$ and $P_\psi(\bar{h} + \bar{\tau} e_1)$ has at least one solution, i.e. $S_{\bar{h}+\bar{\tau}e_1}$ has at least one lower critical point $\bar{t}$.

This means that

$$\liminf_{t \to \bar{t}} \frac{S_{\bar{h}+\bar{\tau}e_1}(t) - S_{\bar{h}+\bar{\tau}e_1}(\bar{t})}{t - \bar{t}} \geq 0;$$

hence, for every $l > 0$, there exists $t_l > \bar{t}$ such that

$$S_{\bar{h}+\bar{\tau}e_1}(t_l) - S_{\bar{h}+\bar{\tau}e_1}(\bar{t}) > -l(t_l - \bar{t}). \tag{5.1}$$

From (5.1) it follows that

$$S_{\bar{h}+\bar{\tau}e_1}(\bar{t}) + l\bar{t} < S_{\bar{h}+\bar{\tau}e_1}(t_l) + lt_l,$$

which is equivalent to

$$S_{\bar{h}+(\bar{\tau}+l)e_1}(\bar{t}) < S_{\bar{h}+(\bar{\tau}+l)e_1}(t_l). \tag{5.2}$$

Taking into account Lemma 5.4, from (5.2) we infer that $S_{\bar{h}+(\bar{\tau}+l)e_1}$ has at least two critical points: a local minimum point $t_1 < t_l$ and (because of (iii) of Lemma 5.4) a local maximum point $t_2 \geq t_l$ (where, indeed, $S'_h(t_2) = 0$). □

Remark 5.5. The results proved in this paper (in particular Theorems 4.1 and 5.3) point out a "folding type phenomenon" for the solvability of problem $P_\psi(h)$: the set of the pairs (ψ, h) such that $P_\psi(h)$ has solutions is a region which can be seen as the epigraphic of a suitable function with values in $\{te_1 \mid t \in \mathbb{R}\}$ and, for pairs (ψ, h) lying in the interior of this region, there exist at least two distinct solutions of $P_\psi(h)$ provided $\lim_{t \to +\infty} \frac{g(x,t)}{t} > \lambda_1$.

This behaviour makes evident a surprising analogy with a well known result stated by Ambrosetti and Prodi in [1], concerning semilinear elliptic equations of the form

$$\Delta u + g(u) = h \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where the nonlinear term g satisfies a "jumping" condition involving the first eigenvalue λ_1, namely

$$\lim_{t \to -\infty} \frac{g(t)}{t} < \lambda_1 < \lim_{t \to +\infty} \frac{g(t)}{t} < \lambda_2.$$

Comparing the sublevels of the corresponding functionals, one could see that, roughly speaking, the presence of the obstacle ψ in our problem has the same role played in [1] by the condition

$$\lim_{t\to-\infty} \frac{g(t)}{t} < \lambda_1.$$

However, let us remark that, despite the evident analogy of these results, there is a deep difference between our methods and the ones used in [1], since the latter are based on the analysis of singularities that could not be applied in our problem.

Notice that an analogous "jumping type behaviour" was shown in [8, 11, 12, 13, 14, 15] for some elliptic problems with obstacle on the function u, instead of its laplacian, i.e. with constraints of the form

$$\overline{K}_\varphi = \{u \in H_0^1(\Omega) \mid u \geq \varphi \text{ a.e. in } \Omega\}$$

in place of K_ψ.

On the contrary no "jumping type behaviour" arises if we consider unilateral pointwise constraints on the first derivatives of u.

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