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Dynamical Shape Control in Non-cylindrical Navier-Stokes Equations

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This paper deals with a dynamical shape control problem. The state equations are the non-cylindrical Navier-Stokes equations with a non-homogenous Dirichlet condition. The goal is to compute a necessary optimality condition for the considered functional (kinetic energy). Our work is based on the "transformation" of the domain functional on a field functional and the consideration of transverse fields in the application of the velocity method. The originality of this approach is the introduction of an extra adjoint equation in relation with the initial transverse field to obtain the final form of the necessary condition.

1. Introduction

We consider the evolution of an incompressible viscous flow in a moving domain. The displacement of the fluid is due to a force f and a non-autonomous velocity V acting through a non homogeneous Dirichlet boundary condition at the lateral boundary of the domain. Let

$$Q = \bigcup_{t_0<t<\tau} (\{t\} \times \Omega_t) \text{ and } \Sigma = \bigcup_{t_0<t<\tau} (\{t\} \times \Gamma_t)$$

be the tube described by the flow during the time-interval (t_0, τ); Ω_t (resp. Γ_t) being the domain (resp. the boundary of the domain) occupied by the fluid at time t. The domain Ω_t, $t \in [t_0, \tau]$, is assumed to be contained in a fixed smooth and bounded three dimensional hold-all D. Its volume $|\Omega_t|$ is constant with the time t since we deal with incompressible fluids. The lateral boundary Σ being smooth enough, let ν be its unitary normal field (out-going to Q). It can be written in a unique way as $\nu = \{\sqrt{1+v_\nu^2}\}^{-1}(-v_\nu, n_t) \in \mathbb{R}\times\mathbb{R}^3$ where n_t is the usual normal on Γ_t and $-\{\sqrt{1+v_\nu^2}\}^{-1} v_\nu(t,x)$ is the time-component of ν.

Notice that for any smooth tube Q there exists a vector field W that "builds" Q as follows: $\Omega_t = T_t(W)(\Omega_0)$, where $T_t(W)$ is the flow mapping of W. To match that property it is sufficient for W to verify, at each time t, the condition $\langle W(t,.), n_t(.)\rangle = v_\nu(t)(.)$ on Γ_t, where v_ν is the term in the time-component of the normal field ν (see [13] and [4]).

Any divergence free field verifying that condition builds the same tube Q so that the tangential component $W(t)_{\Gamma_t} := W(t)|_{\Gamma_t} - \langle W(t)|_{\Gamma_t}, n_t\rangle\, n_t$ is a non geometrical data.

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We assume that V builds the tube Q, that is $\langle V(t,.), n_t(.)\rangle = v_\nu(t)(.)$.

The vector field V also satisfies $\operatorname{div} V = 0$ in D, $V.n_D = 0$ on ∂D and is given in $H^1((0,\tau); \mathrm{H}^m(D,\mathbb{R}^3))$, $(m > 5/2)$. The flow verifies the so-called non-cylindrical Navier-Stokes equations

$$\partial_t u - \eta\Delta u + Du.u + \nabla p = f \quad \text{in } Q(V) \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } Q(V) \tag{1.2}$$

$$u = V \quad \text{on } \Sigma(V), \tag{1.3}$$

$$\text{at} \quad t = t_0, \quad \Omega_0 = \Omega \quad \text{and} \quad u(t_0) = u_0 \text{ in } \Omega, \tag{1.4}$$

where η is the coefficient of kinematic viscosity of the fluid and

$$Q(V) = \bigcup_{t_0<t<\tau} (\{t\} \times T_t(V)(\Omega)) \;\text{ and }\; \Sigma(V) = \bigcup_{t_0<t<\tau} (\{t\} \times T_t(V)(\Gamma)),$$

the domain Ω and the function u_0 are given.

We shall give existence and uniqueness result for the non-cylindrical problem (1.1)–(1.4).

If the fluid was not sticking at the boundary, the Dirichlet boundary condition (1.3) would take the form $u(t).n_t = v(t)$ on the moving boundary Γ_t. Now as the fluid is viscous the sticking condition implies that at a point $x(t)$ of the boundary Γ_t the "two particles" of boundary and of fluid located at $x(t)$ have the same velocity. Then the solution $u = u(t_0, u_0, \Omega, V)$ of the flow problem (1.1)–(1.4) is the sticking one.

The dynamical shape control problem is then, given (t_0, Ω, u_0), to move the boundary Γ_t in order to decrease a functional J governed by $u(t_0, u_0, \Omega, V)$. As first example we take here the functional J as the total kinetic energy

$$J(t_0, u_0, \Omega, V) = \int_{t_0}^{\tau} \int_{\Omega_t(V)} |u|^2 \, dxdt \;\;+\;\; \alpha\, ||V||^2, \;\; \alpha > 0.$$

The optimal control problem is then to solve the optimization problem

$$\mathcal{V}(t_0, u_0, \Omega) = \min\{J(t_0, u_0, \Omega, V),\; V \in \mathcal{A}d(D)\} \tag{$\mathcal{P}$}$$

where $\mathcal{A}d(D)$ is the set of admissible fields. In the present paper we study the optimization problem $(\mathcal{P})$. That is for a given time t_0 (we take $t_0 = 0$), given u_0 and Ω we shall derive necessary conditions for the optimality in V and present the first results concerning Lyapunov trajectory, which in the present choice of the functional J will be an optimal choice of the velocity V in order to bring the fluid to rest. According to [14], the sensitivity analysis of J (with respect to the field V) approach is based on the concept of *Transverse field* $\mathbf{Z}$ associated with the field $\mathcal{Z}^t$ such that

$$\Omega_t(V + \sigma W) = T_\sigma(\mathcal{Z}^t)(\Omega_t(V)).$$

The term $\frac{\partial}{\partial\sigma} J(V + \sigma W)|_{\sigma=0}$ only depends on $\mathcal{Z}^t(0,.)$. The field $\mathbf{Z}(t,.) = \mathcal{Z}^t(0,.)$ is solution of a linear evolution problem governed by the *Lie brackets* $[V, Z]$. By introduction of a new adjoint problem derived by transposition of that problem, we derive the gradient of J.

The properties of the *Value function* $\mathcal{V}$ in connection with the real *time control problem* will be considered in a further work in which the control variable will be in fact the acceleration field $\zeta(t,.) = \dot{V} = \partial V(t,.)/\partial t \in L^2(0,\tau; H^m(D,\mathbb{R}^3))$ which is the obvious extension of the "speed flow" setting of the framework of [3]. Throughout this paper the control variable V is lying in $H^1([0,\tau], H^m(D,\mathbb{R}^3))$.

If the functional J was taken as the total energy (which would imply some extra technicalities), following [8], we would derive existence results and algorithms for the corresponding dynamical free boundary problem.

2. The Non Cylindrical Navier-Stokes Flow

The incompressibility of the fluid (u is a divergence free field) implies the compatibility condition on the field V:

$$\int_{\partial\Omega_t} \langle V(t,x), n_t(x)\rangle \; d\Gamma_t(x) \;=\; 0.$$

That condition is fulfilled as we take $\operatorname{div} V(t,.) = 0$ throughout the universe D.

By making the change of variable $U = u - V$, the system of equations (1.1)–(1.4) is transformed in a system of equations on (U,p) with a homogeneous Dirichlet boundary condition.

$$\partial_t U - \eta\Delta U + DU.U + DU.V + DV.U + \nabla p = f - \partial_t V - DV.V + \eta\Delta V \quad \text{in } Q \tag{2.1}$$

$$\operatorname{div} U = 0 \quad \text{in } Q \tag{2.2}$$

$$U = 0 \quad \text{on } \Sigma. \tag{2.3}$$

The initial condition (1.4) should be replaced by

$$\text{At} \quad t = 0,\;\; U(0) = u_0 - V(0) \;\text{ in } \Omega. \tag{2.4}$$

For simplicity we take $V(0) = 0$. Since the domain occupied by the fluid is time-dependent, It would not be possible, as it is generally done in cylindrical problems, to treat separately, the spatial and temporal aspects of the problem. So, a priori, the Faedo-Galerkin method should be applied with a time-dependent "basis".

Then to solve such problem, one can consider either a family (P_ϵ) of penalized problems formulated in the cylindrical hold-all $(0,\tau)\times D$ or else a domain transformation $T_t(W)$ to come-back to a cylindrical problem in $(0,\tau)\times\Omega$, where Ω is the initial domain.

Whatever the view-point considered in the investigation of weak solutions for the system of equations (2.1)–(2.4), the following two results are needed.

Lemma 2.1. *Let E be a reflexive Banach space and F a Hilbert space, E' being the dual of E, such that*

$$E \hookrightarrow F \hookrightarrow E'$$

with continuous and dense injections. Assume that

$$u \in L^p(a,b;E),\; u' = \frac{du}{dt} \in L^q(a,b;E');\;\; \frac{1}{p} + \frac{1}{q} = 1,\; 1 \le p,q \le \infty.$$

Then for each $\eta \in F$,

$$\frac{d}{dt}(u,\eta) =< u', \eta > \ \ in\ \mathcal{D}'(a,b).$$

Moreover u *is a.e. equal to a continuous function* $\overline{u} \in \mathcal{C}(a,b;F)$.

Lemma 2.2. *Let* E_0, E, E_1 *be three Banach spaces such that*

(i) E_0 *and* E_1 *are reflexive.*
(ii) *The injections* $E_0 \hookrightarrow E \hookrightarrow E_1$ *are continuous.*
(iii) $E_0 \hookrightarrow E$ *is compact.*

The space

$$\mathcal{V} = \{v \in L^{\alpha_0}(0,\tau;E_0), v' = \frac{dv}{dt} \in L^{\alpha_1}(0,\tau;E_1)\},\ \alpha_i > 1, i = 0,1,$$

endowed with the norm

$$\|v\|_{\mathcal{V}} = \|v\|_{L^{\alpha_0}(0,\tau;E_0)} + \|v'\|_{L^{\alpha_1}(0,\tau;E_1)}$$

is a Banach space and

$$\mathcal{V} \hookrightarrow L^{\alpha_0}(0,\tau;E),\ \ \textit{the injection is continuous and compact.}$$

For proofs, see for example [12] or [9].

The following functional spaces will be used throughout this paper:

$$\begin{aligned} H &= \{v \in L^2(D)^3, \operatorname{div} v = 0 \text{ in } D,\ v.n_D = 0 \text{ on } \partial D\},\\ H(\Omega) &= \{v \in L^2(\Omega)^3, \operatorname{div} v = 0 \text{ in } \Omega,\ v.n_\Omega = 0 \text{ on } \partial\Omega\},\\ H^1_0(\operatorname{div}, O) &= \{v \in H^1_0(O)^3, \operatorname{div} v = 0 \text{ in } O\} \end{aligned}$$

where O is an open subset of D.

We also define the space $L^p(0,\tau;L^2(\Omega_t))$ as the set of restrictions to Q of elements of $L^p(0,\tau;L^2(D))$ $(1 \leq p \leq \infty)$. Then $L^p(0,\tau;H^1(\Omega_t)) = \{\varphi \in L^p(0,\tau;L^2(\Omega_t)) | \nabla_x \varphi \in L^p(0,\tau;L^2(\Omega_t))\}$.

As Σ is smooth enough, it is also the set of restrictions to Q of elements of $L^p(0,\tau;H^1(D))$.

We finally introduce the operators

$$A : H^1_0(\operatorname{div}, D) \to H^1_0(\operatorname{div}, D)',\ \langle Av, w\rangle = \int_D Dv..Dw\,dx$$

$$B : H^1_0(\operatorname{div}, D) \to H^1_0(\operatorname{div}, D)',\ \langle B(v), w\rangle = \int_D \langle Dv.v, w\rangle dx.$$

2.1. Penalty Method

The force f is assumed to be in $L^2(0,\tau;H^{-1}(D)^3)$. Let $F \in L^2(0,\tau;H^{-1}(D)^3)$ defined by: $\forall \psi \in L^2(0,\tau;H^1_0(D)^3)$,

$$\langle F,\psi\rangle_{L^2(H^{-1}),L^2(H^1_0)} = \\ \langle f,\psi\rangle_{L^2(H^{-1}),L^2(H^1_0)} - \int_{(0,\tau)\times D} \partial_t V.\psi + \eta\, DV..D\psi + \langle DV.V,\psi\rangle dxdt. \quad (2.5)$$

Theorem 2.3. *Let $\eta > 0$, $F \in L^2(0,\tau; H_0^1(\mathrm{div}, D)')$ and $u_0 \in H$. For all $\epsilon > 0$, there exists at least a function U^ϵ such that*

$$U^\epsilon \in L^2(0,\tau; H_0^1(\mathrm{div}, D)) \cap L^\infty(0,\tau; H),\ \partial_t U^\epsilon \in L^{\frac{4}{3}}(0,\tau; H_0^1(\mathrm{div}, D)')$$

and

$$\begin{aligned}\partial_t U^\epsilon - \eta\Delta U^\epsilon + DU^\epsilon.U^\epsilon + DU^\epsilon.V + DV.U^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c} U^\epsilon + \nabla P^\epsilon &\\ = F \quad in \quad L^{\frac{4}{3}}(H_0^1(\mathrm{div}, D)')&\end{aligned} \tag{2.6}$$

$$\mathrm{div}\, U^\epsilon = 0 \quad in \quad (0,\tau)\times D \tag{2.7}$$

$$U^\epsilon(0) = u_0 \quad in \quad D. \tag{2.8}$$

Before proving Theorem 2.3, we recall the existence of a family of eigenfunctions φ_i in $H_0^1(\mathrm{div}, D)$ and scalars $\lambda_i > 0$, $i \geq 1$, $\lambda_i \to \infty$ as $i \to \infty$ such that

$$((\varphi_i, v)) = \lambda_i(\varphi_i, v),\ \forall v \in H_0^1(\mathrm{div}, D), \tag{2.9}$$

where $((\,,\,))$ denotes the inner product in $H_0^1(D)^3$, and

$$\begin{aligned}(\varphi_i, \varphi_j) &= \delta_{ij}\\ ((\varphi_i, \varphi_j)) &= \lambda_i\delta_{ij},\ \forall i, j \geq 1.\end{aligned}$$

The eigenfunctions, φ_i, $i \geq 1$, form an Hilbertian basis of $H_0^1(\mathrm{div}, D)$.

Proof. The constant of Poincaré being denoted C_D, in a first step we assume that

$$\eta' = \eta - C_D\|V\|_{L^\infty(W^{1,\infty})} > 0. \tag{2.10}$$

We apply the Faedo-Galerkin method (see for example [12] or [9]) with the Hilbertian basis $\{\varphi_i, i \in \mathbb{N}^*\}$ of $H_0^1(\mathrm{div}, D)$ defined above. Let U_{0m} in $H_{0,m}^1(\mathrm{div}, D) = \mathrm{span}\{\varphi_1, \ldots, \varphi_m\}$ such that $U_{0m} \to u_0$ strongly in H. We look for

$$U_m^\epsilon(t) = \sum_{i=1}^m \alpha_i^{m,\epsilon}(t)\,\varphi_i \ \text{ such that } \ U_m^\epsilon(0) = U_{0m},\ \ m \in \mathbb{N}^*,$$

such that for all $\varphi \in H_{0,m}^1(\mathrm{div}, D)$, we have

$$\begin{aligned}\int_D \partial_t U_m^\epsilon.\,\varphi\,dx + \eta\int_D DU_m^\epsilon..\,D\varphi\,dx + \int_D \langle DU_m^\epsilon.\,U_m^\epsilon, \varphi\rangle dx + \int_D \langle DU_m^\epsilon.V, \varphi\rangle dx +&\\ \int_D \langle DV.U_m^\epsilon, \varphi\rangle dx + \frac{1}{\epsilon}\int_D \chi_{\Omega_t^c} U_m^\epsilon.\,\varphi\,dx = \langle F, \varphi\rangle \text{ in } \mathcal{D}'(0,\tau).&\end{aligned} \tag{2.11}$$

For $\varphi = U_m^\epsilon$ and $t \in (0,\tau]$, we have

$$\begin{aligned}\frac{1}{2}|U_m^\epsilon(t)|_{L^2(D)}^2 + \eta\int_0^t |DU_m^\epsilon(s)|_{L^2(D)}^2\,ds - \|V\|_{L^\infty(W^{1,\infty})}\int_0^t |U_m^\epsilon(s)|_{L^2(D)}^2\,ds&\\ + \frac{1}{\epsilon}\int_0^t |\chi_{\Omega_s^c} U_m^\epsilon(s)|_{L^2(D)}^2\,ds \leq \int_0^t \langle F, U_m^\epsilon(s)\rangle ds + \frac{1}{2}|U_m^\epsilon(0)|_{L^2(D)}^2.&\end{aligned}$$

$$\frac{1}{2}|U_m^\epsilon(t)|^2_{L^2(D)} + \eta' \int_0^t |DU_m^\epsilon(s)|^2_{L^2(D)}\,ds + \frac{1}{\epsilon}\int_0^t |\chi_{\Omega_s^c} U_m^\epsilon(s)|^2_{L^2(D)}\,ds$$
$$\leq \int_0^t \langle F, U_m^\epsilon(s)\rangle ds + \frac{1}{2}|U_m^\epsilon(0)|^2_{L^2(D)}.$$

$$|U_m^\epsilon(t)|^2_{L^2(D)} + \eta' \int_0^t |DU_m^\epsilon(s)|^2_{L^2(D)}\,ds + \frac{2}{\epsilon}\int_0^t |\chi_{\Omega_s^c} U_m^\epsilon(s)|^2_{L^2(D)}\,ds$$
$$\leq \frac{1}{\eta'}\|F\|^2_{L^2(H^{-1}(D)^3)} + |U_m^\epsilon(0)|^2_{L^2(D)}.$$

Since $|U_m^\epsilon(0)|_{L^2(D)} \leq |U(0)|_{L^2(D)}$, we deduce the uniform boundedness, on m and ϵ, of

$$U_m^\epsilon \quad \text{in} \quad L^\infty(0,\tau;L^2(D)^3) \cap L^2(0,\tau;H_0^1(D)^3) \tag{2.12}$$
$$\frac{1}{\sqrt{\epsilon}}(\chi_{\Omega_t^c} U_m^\epsilon) \quad \text{in} \quad L^2(0,\tau;L^2(D)^3). \tag{2.13}$$

Let us introduce the projection operator $P_m : H \to \mathrm{span}\{\varphi_1, \cdots, \varphi_m\}$ such that

$$\text{for} \;\; v \in H, \;\; P_m(v) = \sum_{i=1}^m (v,\varphi_i)_D\, \varphi_i. \tag{2.14}$$

$(\,,\,)_D$ is the inner product of $L^2(D)$.

We denote by ${}^*P_m$ the adjoint operator of P_m seen as an element of $\mathcal{L}(H_0^1(\mathrm{div}, D))$. Then,

$$\partial_t U_m^\epsilon = -\frac{1}{\epsilon} P_m(\chi_{\Omega_t^c} U_m^\epsilon) - {}^*P_m(AU_m^\epsilon) - {}^*P_m(B(U_m^\epsilon)) + {}^*P_m(F).$$

It implies that

$$\partial_t U_m^\epsilon \text{ is bounded, uniformly on } m, \text{ in } L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)'). \tag{2.15}$$

Indeed, let v be any function in $L^2(0,\tau;H_0^1(\mathrm{div},D)^3) \cap L^\infty(0,\tau;L^2(D)^3)$, then

$$\|v(t)\|_{L^3(D)} \leq \|v(t)\|^{\frac{1}{2}}_{L^6(D)} |v(t)|^{\frac{1}{2}}_{L^2(D)}$$
$$\int_0^\tau \|v(t)\|^4_{L^3(D)}\,dt \leq \|v\|^2_{L^\infty(L^2(D))} \int_0^\tau \|v(t)\|^2_{L^6(D)} dt$$

which implies that $v \in L^4(0,\tau;L^3(D)^3)$.

Moreover
$$\|B(v)(t)\|_{H_0^1(\mathrm{div},D)'} \leq c\|v(t)\|_{L^3(D)}\|v(t)\|_{L^6(D)}.$$

Then

$$\int_0^\tau \|B(v)(t)\|^{\frac{4}{3}}_{H_0^1(\mathrm{div},D)'}\,dt \leq c\int_0^\tau \|v(t)\|^{\frac{4}{3}}_{L^3(D)}\,\|v(t)\|^{\frac{4}{3}}_{L^6(D)}\,dt$$
$$\leq c\left(\frac{1}{3}\int_0^\tau \|v(t)\|^4_{L^3(D)}\,dt + \frac{2}{3}\int_0^\tau \|v(t)\|^2_{L^6(D)} dt\right).$$

Thus $B(v) \in L^{\frac{4}{3}}(H_0^1(\mathrm{div}, D)')$. On the other-hand note that $\|{}^*P_m\|_{\mathcal{L}(H_0^1(\mathrm{div},D)')} \leq c_0$, ($c_0 > 0$ is a constant) and take $v = U_m^\epsilon$, we deduce that $B(U_m^\epsilon)$ is bounded, independently of m and ϵ, in $L^{\frac{4}{3}}(H_0^1(\mathrm{div}, D)')$.

Hence, there exists a subsequence (still denoted U_m^ϵ) and a function $U^\epsilon \in L^\infty(0,\tau;H) \cap L^2(0,\tau;H_0^1(\mathrm{div}, D))$ such that, when $m \to \infty$,

$$
\begin{aligned}
U_m^\epsilon &\to U^\epsilon \text{ in } L^2(0,\tau;L^4(D)^3) \text{ strong,} && (2.16)\\
U_m^\epsilon &\overset{*}{\rightharpoonup} U^\epsilon \text{ in } L^\infty(0,\tau;L^2(D)^3) \text{ weak-star,} && (2.17)\\
U_m^\epsilon &\rightharpoonup U^\epsilon \text{ in } L^2(0,\tau;H_0^1(D)^3) \text{ weak.} && (2.18)
\end{aligned}
$$

So by passing to the limit in (2.11) we obtain

$$U^\epsilon \in L^\infty(0,\tau;H) \cap L^2(0,\tau;H_0^1(\mathrm{div}, D)),$$

$$
\begin{aligned}
\partial_t U^\epsilon - \eta\Delta U^\epsilon + DU^\epsilon.U^\epsilon + DU^\epsilon.V + DV.U^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c}U^\epsilon &= F \text{ in } (0,\tau)\times D\\
\mathrm{div}\, U^\epsilon &= 0 \text{ in } (0,\tau)\times D
\end{aligned}
$$

and that $\partial_t U^\epsilon \in L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div}, D)')$. In a second step the extra-hypothesis on the viscosity can be avoided by making the change of variable: $\tilde{U} = e^{\lambda t}U$, where λ is suitably chosen. Then its approximation $\tilde{U}_m$ solves the problem, in $(0,\tau)\times D$,

$$
\begin{aligned}
\partial_t\tilde{U}_m^\epsilon + \lambda\,\tilde{U}_m^\epsilon - \eta\Delta\tilde{U}_m^\epsilon + e^{\lambda t}\, D\tilde{U}_m^\epsilon.\tilde{U}_m^\epsilon + DU_m^\epsilon.V + DV.\tilde{U}_m^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c}\tilde{U}_m^\epsilon&\\
= e^{-\lambda t}F& \qquad (2.19)
\end{aligned}
$$

$$\mathrm{div}\,\tilde{U}_m^\epsilon = 0 \qquad (2.20)$$

$$\tilde{U}_m^\epsilon(0) = u_0 \text{ in } D. \qquad (2.21)$$

As $\int_D \langle D\tilde{U}_m^\epsilon.\tilde{U}_m^\epsilon, \tilde{U}_m^\epsilon\rangle dx = 0$ we get the previous estimates for the sequence $\tilde{U}_m^\epsilon$ where $(\eta - C_D\|V\|)$ is replaced by η, if $\lambda > \|V\|$. Then, given η and V, we choose λ such that $(\lambda - \|V\|) > 0$. $\square$

Now, we are interested in the behavior of $\{U^\epsilon\}$ in the tube Q when ϵ goes to 0. We shall prove the existence of a subsequence of $\{U^\epsilon\}$ converging to a solution of the non-cylindrical problem (2.1)–(2.4).

Theorem 2.4. *The open set Ω is assumed to be piecewise $\mathcal{C}^1$. Let $F \in L^2(0,\tau; H_0^1(\mathrm{div}, D)')$ and $u_0 \in H$ such that $u_{0|\Omega} \in H(\Omega)$. There exists at least a function $U \in L^2(0,\tau;H_0^1(\mathrm{div}, D)) \cap L^\infty(0,\tau;H)$ satisfying (2.1) in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},\Omega_t))$ and also (2.2)–(2.4).*

According to (2.12), we have the boundedness, independently of ϵ, of

$$U^\epsilon \text{ in } L^\infty(0,\tau;L^2(D)^3) \cap L^2(0,\tau;H_0^1(D)^3). \qquad (2.22)$$

This implies the existence of $U \in L^\infty(0,\tau;L^2(D)^3) \cap L^2(0,\tau;H^1_0(D)^3)$ and of a subsequence, still denoted U^ϵ, such that

$$U^\epsilon \rightharpoonup U \ \text{ in } \ L^2(0,\tau;H^1_0(D)^3) \text{ weak} \tag{2.23}$$

$$U^\epsilon \overset{*}{\rightharpoonup} U \ \text{ in } \ L^\infty(0,\tau;L^2(D)^3) \text{ weak-star.} \tag{2.24}$$

Moreover (2.13) implies that

$$\chi_{\Omega_t^c} U^\epsilon \rightharpoonup \chi_{\Omega_t^c} U = 0 \ \text{ in } L^2(0,\tau;L^2(D)^3) \text{ weak}, \epsilon \searrow 0. \tag{2.25}$$

Then $U(t) \in H^1_0(\Omega_t)^3$ for almost every $t \in (0,\tau)$.

On the other-hand, since $\partial_t U^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c} U^\epsilon = \eta \Delta U^\epsilon - DU^\epsilon.U^\epsilon - DU^\epsilon.V - DV.U^\epsilon + F$ belongs to $L^{\frac{4}{3}}(0,\tau;H^1_0(\mathrm{div},D)')$ estimation (2.22) gives the boundedness of

$$\partial_t U^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c} U^\epsilon \ \text{ in } L^{\frac{4}{3}}(0,\tau;H^1_0(\mathrm{div},D)') \text{ independently of } \epsilon. \tag{2.26}$$

The fact that the limit $U(t)$ shall be in $H^1_0(\mathrm{div},\Omega_t)$ a.e. allows us to restrict the space of spatial test functions to $H^1_0(\mathrm{div},\Omega_t)$. This has the advantage to eliminate the contribution of the term $\frac{1}{\epsilon}\chi_{\Omega_t^c} U^\epsilon$ and makes the search for uniform boundedness for the derivative $\partial_t U^\epsilon$ easier.

To be able to express estimations in fixed spaces, we need to introduce some domain transformations.

Lemma 2.5. *Let $W \in \mathcal{C}^0([0,\tau],\mathcal{C}^2(\overline{D}))$ such that $W.n_D = 0$ on ∂D, n_D unit normal vector field on ∂D, outward to D. Then, the mapping $T_t(W)$ defined as follows*

$$X \longmapsto T_t(W)(X) = x(t;W) : \overline{D} \to \overline{D},$$

where $x(.;W)$ is the solution of

$$\frac{dx}{dt}(t) = W(t,x(t)), \ \ t \in [0,\tau], x(0) = X \in \overline{D},$$

is an homeomorphism. Moreover, if

$$\langle W(t,x), n_t(x)\rangle = v_\nu(t,x), \ \forall (t,x) \in [0,\tau] \times \partial\Omega_t \ \ (v_\nu \text{ is the time-component of } \nu). \tag{2.27}$$

Then $T_t(W)$ builds the tube Q i.e. $\forall t \in [0,\tau]$, $T_t(W)(\Omega) = \Omega_t$.

Proof. Cf. [11], [13]. □

We have the following properties (see [6]).

Lemma 2.6. *Let $T_t(W)$ be the transformation defined in* Lemma 2.5. *We have*

(i) $D(T_t(W)^{-1}) \circ T_t(W) = (DT_t(W))^{-1}$,

(ii) $(Dv) \circ T_t(W) = D(v \circ T_t(W)).(DT_t(W))^{-1}$, $v \in H^1_0(\Omega_t)^3$,

(iii) $(\partial_t v) \circ T_t(W) = \partial_t(v \circ T_t(W)) - (Dv) \circ T_t(W).W \circ T_t(W)$, $\ v \in \mathcal{C}^1(0,\tau;H^1_0(D)^3)$.

Let W be a vector field satisfying hypotheses of Lemma 2.5 notably property (2.27), then

Lemma 2.7. *The mapping* $H_0^1(\mathrm{div},\Omega) \longrightarrow H_0^1(\mathrm{div},\Omega_t)$; $\varphi \to (DT_t(W)\varphi)\circ T_t(W)^{-1}$ *is an isomorphism.*

Now, we can state

Lemma 2.8. $\partial_t[{}^*DT_t(W)(U^\epsilon \circ T_t(W))]$ *is bounded in* $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},\Omega)')$.

Proof. Indeed, set

$$\omega^\epsilon = \partial_t U^\epsilon + \frac{1}{\epsilon}\chi_{\Omega_t^c} U^\epsilon$$

and let $v \in H_0^1(\mathrm{div},\Omega)$, $(DT_t v)\circ T_t^{-1} \in H_0^1(\mathrm{div},\Omega_t)$. Then,

$$\langle \omega^\epsilon, (DT_t v)\circ T_t^{-1}\rangle = \langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle + \frac{1}{\epsilon}\int_D \chi_{\Omega_t^c} U^\epsilon.(DT_t v)\circ T_t^{-1} dx.$$

It is clear that

$$\int_D \chi_{\Omega_t^c} U^\epsilon.(DT_t v)\circ T_t^{-1} dx = 0.$$

Then, we deduce that

$$\omega^\epsilon \equiv \partial_t U^\epsilon \quad \text{in } L^{\frac{4}{3}}(H_0^1(\mathrm{div},\Omega_t)').$$

Hence, the boundedness of $\partial_t U^\epsilon$ in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},\Omega_t)')$ follows from the boundedness of ω^ϵ in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$. Moreover since

$$\begin{aligned}
&\langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle = \frac{d}{dt}(U^\epsilon, (DT_t v)\circ T_t^{-1}) - (U^\epsilon, \partial_t((DT_t v)\circ T_t^{-1}))\\
&= \frac{d}{dt}(U^\epsilon, (DT_t v)\circ T_t^{-1}) - (U^\epsilon, [D(W\circ T_t)v]\circ T_t^{-1} - [D(DT_t v)W]\circ T_t^{-1})\\
&= \frac{d}{dt}({}^*DT_t(U^\epsilon\circ T_t), v) - (U^\epsilon\circ T_t, D(W\circ T_t)v) + (U^\epsilon\circ T_t, D(DT_t v)W)\\
&= \langle \partial_t({}^*DT_t(U^\epsilon\circ T_t)), v\rangle - (U^\epsilon\circ T_t, D(W\circ T_t)v) + (U^\epsilon\circ T_t, D(DT_t v)W).
\end{aligned} \tag{2.28}$$

Thus,

$$\langle \partial_t({}^*DT_t(U^\epsilon\circ T_t)), v\rangle = \langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle + (U^\epsilon\circ T_t, D(W\circ T_t)v) - (U^\epsilon\circ T_t, D(DT_t v)W).$$

It follows that $\partial_t({}^*DT_t(U^\epsilon\circ T_t))$ is bounded in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},\Omega)')$. □

By the compactness result stated in Lemma 2.2 applied for $E_0 = H_0^1(\mathrm{div},\Omega)$, $E_1 = H_0^1(\mathrm{div},\Omega)'$, $E = L^4(\Omega)^3$, we obtain, when $\epsilon \to 0$,

$$ {}^*DT_t(U^\epsilon\circ T_t) \longrightarrow {}^*DT_t(U\circ T_t) \quad \text{strongly in } L^2(0,\tau;L^4(\Omega)^3) \tag{2.29}$$

which is equivalent to

$$U^\epsilon \longrightarrow U \quad \text{strongly in } L^2(0,\tau;L^4(\Omega_t)^3). \tag{2.30}$$

To show that the limit U satisfies (2.1)–(2.4), we need to compute, first, the weak limit of (2.6) as $\epsilon \to 0$.

Let $\theta \in \mathcal{D}(0,\tau)$ and $v \in H_0^1(\mathrm{div},\Omega)$,

$$\int_0^\tau \langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dt = \int_0^\tau \frac{d}{dt}(U^\epsilon, (DT_t v)\circ T_t^{-1})\,\theta(t)\,dt$$
$$- \int_0^\tau (U^\epsilon, [D(W\circ T_t)v]\circ T_t^{-1})\,\theta(t)\,dt + \int_0^\tau (U^\epsilon, D(DT_t v)(W\circ T_t^{-1}))\,\theta(t)\,dt.$$

Then, the limit

$$\lim_{\epsilon\to 0}\int_0^\tau \langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dt = -\int_0^\tau (U, (DT_t v)\circ T_t^{-1})\,\theta'(t)\,dt$$
$$- \int_0^\tau (U, [D(W\circ T_t)v]\circ T_t^{-1})\,\theta(t)\,dt + \int_0^\tau (U, D(DT_t v)(W\circ T_t^{-1}))\,\theta(t)\,dt.$$

The same calculus as in (2.28) gives

$$\int_0^\tau \langle \partial_t U, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dt = -\int_0^\tau (U, (DT_t v)\circ T_t^{-1})\,\theta'(t)\,dt$$
$$- \int_0^\tau (U, [D(W\circ T_t)v]\circ T_t^{-1})\,\theta(t)\,dt + \int_0^\tau (U, D(DT_t v)(W\circ T_t^{-1}))\,\theta(t)\,dt.$$

It means that

$$\lim_{\epsilon\to 0}\int_0^\tau \langle \partial_t U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dt = \int_0^\tau \langle \partial_t U, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dt. \tag{2.31}$$

Concerning the nonlinear term, note that

$$\int_Q \langle DU^\epsilon.U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dxdt = -\int_Q \langle D[(DT_t v)\circ T_t^{-1}].U^\epsilon, U^\epsilon\rangle\,\theta(t)\,dxdt$$

and using(2.30), it is obvious to see that

$$\lim_{\epsilon\to 0}\int_Q \langle DU^\epsilon.U^\epsilon, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dxdt =$$
$$-\int_Q \langle D[(DT_t v)\circ T_t^{-1}].U, U\rangle\,\theta(t)\,dxdt = \int_Q \langle DU.U, (DT_t v)\circ T_t^{-1}\rangle\,\theta(t)\,dxdt. \tag{2.32}$$

The other terms of the weak form of (2.6) do not present any special difficulties in the limit process. Then, we obtain, for all $w = (DT_t v)\circ T_t^{-1}$ ($\in H_0^1(\mathrm{div},\Omega_t)$), v (resp. θ) describing $H_0^1(\mathrm{div},\Omega)$ (resp. $\mathcal{D}(0,\tau)$),

$$\int_0^\tau \langle \partial_t U, w\rangle\,\theta(t)\,dt + \eta\int_Q DU..Dw\;\theta(t)\,dxdt + \int_Q \langle DU.U, w\rangle\,\theta(t)\,dxdt$$
$$+ \int_Q \langle DU.V + DV.U, w\rangle\,\theta(t)\,dxdt = \int_0^\tau \langle F, w\rangle\,\theta(t)\,dt \tag{2.33}$$

This implies equation (2.1) in $\mathcal{D}'(0,\tau; H_0^1(\mathrm{div},\Omega)')$. By density and continuity, we obtain it in $L^{\frac{4}{3}}(0,\tau; H_0^1(\mathrm{div},\Omega)')$. Now, let $\theta \in \mathcal{C}^1([0,\tau])$ such that $\theta(\tau)=0$. We obtain from (2.1)

$$-\int_0^\tau (U,w)\,\theta'(t)\,dt - \int_0^\tau (U,\partial_t w)\,\theta(t)\,dt - (U(0),v)\,\theta(0) + \eta\int_Q DU..Dw\,\theta(t)\,dxdt \\ + \int_Q \langle DU.U + DU.V + DV.U, w\rangle\,\theta(t)\,dxdt = \int_0^\tau \langle F,w\rangle\,\theta(t)\,dt.$$

Doing the same with equation (2.6) and passing to the limit ($\epsilon \to 0$), we get

$$-\int_0^\tau (U,w)\,\theta'(t)\,dt - \int_0^\tau (U,\partial_t w)\,\theta(t)\,dt - (u_0 - V(0),v)\,\theta(0) + \eta\int_Q DU..Dw\,\theta(t)\,dxdt \\ + \int_Q \langle DU.U + DU.V + DV.U, w\rangle\,\theta(t)\,dxdt = \int_0^\tau \langle F,w\rangle\,\theta(t)\,dt.$$

By comparison, we deduce that, $\forall v \in H_0^1(\mathrm{div},\Omega)$,

$$(U(0),v)\,\theta(0) = (u_0 - V(0),v)\,\theta(0) \ \text{ and thus } \ U(0) = u_0 - V(0) = u_0.$$

Then, $u = U + V$ is a solution of the initial problem (1.1)–(1.4).

Remark 2.9. The uniqueness of U is obtained if we assume

$$u_0 \in H^2(D,\mathbb{R}^3) \cap H_0^1(\mathrm{div},D) \tag{2.34}$$
$$f \in L^\infty(0,\tau;L^2(D)),\ f' \in L^2((0,\tau)\times D) \tag{2.35}$$

and if η is large enough or when f and u_0 are "small enough":

$$\|f\|^2_{L^\infty(L^2(D))} + \mu^{1/2}(1+d_0^2)(\mu|u_0|^2 \\ + \tau\|f\|^2_{L^\infty(L^2(D))})^{1/2}.\exp\left(\int_0^\tau |f'(s)|^2 ds\right) < \frac{\mu^4}{c^2} \tag{2.36}$$

where $d_0 = |f(0)| + \mu c_0\|u_0\|_{H^2} + c_1\|u_0\|^2_{H^2}$, and c, c_0, c_1 are constants.

Under the previous assumptions

$$\partial_t U \in L^2(0,\tau;\Omega_t) \cap L^\infty(0,\tau;H(\Omega_t)).$$

Moreover, if Ω is assumed to be of class $\mathcal{C}^2$, $U \in L^\infty(0,\tau;H^2(\Omega_t,\mathbb{R}^3))$. For more details see D. N. Bock [1], R. Temam [12] or J. L. Lions [9]. The previous conditions give uniqueness for the penalized problem too.

From now on we assume (2.34)–(2.36) satisfied. The well-posedness of the considered non-cylindrical motion being established, we can now turn our attention to the main point of this work which is the minimization of the kinetic energy with respect to the "shape" of the non-cylindrical domain followed by the fluid. Practically the minimization process will work with a family of admissible vector fields V satisfying conditions of Lemma 2.5 and such that $u = V$ on $\Sigma(V)$.

3. Optimization on the speed field V

Let $\mathbf{H}_0^m(D) = \{v \in H^m(D, \mathbb{R}^3), \operatorname{div} v = 0 \text{ in } D, v.n = 0 \text{ on } \partial D\}$. Assume $m > (5/2)$ so that $\mathbf{H}_0^m(D) \hookrightarrow \mathcal{C}^2(\overline{D})$. This section is devoted to the study of the minimization problem:

Find V solution of

$$\min_{V \in H^1(0,\tau;\mathbf{H}_0^m(D))} \frac{1}{2}\left\{\int_{Q(V)} |u(V)|^2 \, dxdt + \alpha \|V\|^2_{H^1(0,\tau;\mathbf{H}_0^m(D))}\right\} \tag{3.1}$$

where $Q(V) = \bigcup_{0\le t\le\tau} \{t\} \times T_t(V)(\Omega)$, $u(V)$ is the solution of (1.1)–(1.4), $\alpha > 0$ is a constant.

The existence for that problem shall be obtained via the cylindrical penalized problem and its shape continuity properties.

3.1. Continuity Properties

Let $\{V_n\}_n$ and V in $H^1(0, \tau; \mathbf{H}_0^m(D))$.

Lemma 3.1. *Assume that $V_n \rightharpoonup V$, as $n \to \infty$, weakly in $H^1(0, \tau; \mathbf{H}_0^m(D))$. Then $V_n \to V$ strongly in $\mathcal{C}([0, \tau]; \mathbf{H}_0^{m'}(D))$, $m' = m - \mu$, $\mu > 0$ arbitrarily small.*

Proof. By hypothesis, the sequence $\{\partial_t V_n\}$ is bounded in $L^2(0, \tau; \mathbf{H}_0^m(D))$ and

$$V_n \text{ is bounded in } L^\infty(0, \tau; \mathbf{H}_0^m(D)). \tag{3.2}$$

Using the compactness result stated in Lemma 2.2 and the compact injection $H^{m'}(D) \hookrightarrow H^m(D)$, we obtain

$$V_n \to V \text{ strongly in } \mathcal{C}([0, \tau]; \mathbf{H}_0^{m'}(D)). \tag{3.3}$$

□

As $m > 5/2$, we obtain

Corollary 3.2. *Assume that $V_n \rightharpoonup V$ as $n \to \infty$ weakly in $H^1(0, \tau; \mathbf{H}_0^m(D))$. Then,*

$$V_n \to V \text{ strongly in } \mathcal{C}([0, \tau]; W^{k,\infty}(D))$$

for all k such that $1 < k < m - (3/2)$.

Proof. It suffices to see that under the above condition on k, the embedding $H^{m'}(D) \hookrightarrow W^{k,\infty}(D)$ holds and is compact. □

From now on, we will assume that $m > 5/2$ and $1 < k < m - (3/2)$.

Proposition 3.3. *Assume that $V_n \to V$ strongly in $\mathcal{C}([0, \tau]; W^{k,\infty}(D))$ as $n \to \infty$. Then, $T(V_n)\ (= T^n) \longrightarrow T(V)\ (= T)$ strongly in $\mathcal{C}^1([0, \tau]; W^{k-1,\infty}(D))$.*

Before proving this convergence result, recall

Lemma 3.4. *Let $F \in W^{m,\infty}(D)$, $m \ge 1$, be a homeomorphic transformation such that F^{-1} is Lipschitz-continuous in $\overline{D}$. Then, for any $V \in W^{m,\infty}$, $V \circ F \in W^{m,\infty}(D)$ and there exists a constant $c > 0$ such that $\|V \circ F\|_{W^{m,\infty}(D)} \le c\|V\|_{W^{m,\infty}(D)}$.*

Proof. See [10]. □

Proof of Proposition 3.3. Note that

$$\begin{aligned} T_t^n(x) - T_t(x) &= \int_0^t V_n(s, T_s^n(x)) - V(s, T_s(x))\, ds \\ &= \int_0^t V_n(s, T_s^n(x)) - V(s, T_s^n(x))\, ds + \int_0^t V(s, T_s^n(x)) - V(s, T_s(x))\, ds. \end{aligned}$$

Set $r_n(t) = \|T_t^n - T_t\|_{W^{k-1,\infty}(D)}$ and $R_n(t) = \int_0^t r_n(s)\, ds$. Since $V(s)$ belongs to $W^{k,\infty}(D)$, $k > 1$, $V_i(s, T_s^n(x)) - V_i(s, T_s(x)) = DV_i(s, T_s(x) + \theta_i(T_s^n(x) - T_s(x)))(T_s^n(x) - T_s(x))$, $\theta_i = \theta(n, s, x)$, $i = 1, 2, 3$. Then

$$\begin{aligned} r_n(t) &\le c_V \int_0^t \|V_n(s) - V(s)\|_{W^{k-1,\infty}}\, ds + K_V\, R_n(t), \\ K_V &= \|DV\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))}. \\ R_n^{'}(t) - K_V R_n(t) &\le c_V\, \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} \\ (\exp(-K_V t) R_n(t))' &\le c_V\, \tau \exp(-K_V t) \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))}. \end{aligned}$$

By integration, we get

$$\begin{aligned} \exp(-K_V\, t) R_n(t) &\le c_V \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} \int_0^t \exp(-K_V s)\, ds \\ R_n(t) &\le c_V\, \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} \frac{\exp(K_V t) - 1}{K_V}. \end{aligned}$$

Then,

$$\begin{aligned} &r_n(t) \le \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} + \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} (\exp(K_V t) - 1) \\ &r_n(t) \le \tau \|V_n - V\|_{\mathcal{C}([0,\tau];W^{k-1,\infty}(D))} \exp(K_V \tau). \end{aligned}$$

This last estimation gives the convergence of T^n, strongly in $\mathcal{C}([0,\tau]; W^{k-1,\infty}(D))$, to T. Moreover, T^n and T are in $\mathcal{C}^1([0,\tau]; W^{k-1,\infty}(D))$ and

$$\frac{d}{dt} T^n = V_n \circ T^n.$$

Therefore

$$T^n \longrightarrow T,\ n \to \infty, \quad \text{strongly in } \ \mathcal{C}^1([0,\tau]; W^{k-1,\infty}(D)). \tag{3.4}$$

□

Proposition 3.5. *The mapping*

$$V \longrightarrow T(V)^{-1}$$

is continuous from $\mathcal{C}([0,\tau], W^{k,\infty}(D))$ *in* $\mathcal{C}([0,\tau] \times D)$.

Proof. Let

$$\begin{aligned}\mathcal{Q}^*(t,x) &= T_t^{-1}(V+W)(x) - T_t^{-1}(V)(x) = \\ &= \int_0^t -(V+W)(t-\mu, T_\mu^{-1}(V+W)(x)) + V(t-\mu, T_\mu^{-1}(V)(x))\, d\mu\end{aligned}$$

$$|\mathcal{Q}^*(t,x)| \leq c\int_0^t \|DV\|_{L^\infty(I\times D)} |\mathcal{Q}^*(\mu,x)|\, d\mu + \tau \|W\|_{L^\infty(I\times D)}.$$

The generalized Gronwall inequality gives

$$\|\mathcal{Q}^*(t,x)\|_{L^\infty(I\times D)} \leq b_\tau \exp(\tau a). \tag{3.5}$$

So that we obtain the desired result. □

Proposition 3.6. *Assume $V_n \longrightarrow V$ in $\mathcal{C}([0,\tau], \mathcal{C}^{k-1}(D,\mathbb{R}^3)), k > 1$, and let Σ_V and Σ_{V_n} be the lateral boundaries of the tubes built respectively by V and V_n. Then Σ_{V_n} converges to Σ_V in the Hausdorff distance:*

$$\max_{X\in\Sigma_{V_n}} \min_{Y\in\Sigma_V} |X-Y| + \max_{Y\in\Sigma_V} \min_{X\in\Sigma_{V_n}} |X-Y| \longrightarrow 0.$$

Proof. From propositions 3.3 and 3.5 we deduce that $T_t(V_n)\circ T_t(V)^{-1}$ and $T_t(V)\circ T_t(V_n)^{-1}$ converges to the identity in $\mathcal{C}(I\times \overline{D})$. Let $(t,x)\in\Sigma_V$ then

$$\min_{(s,y)\in\Sigma_{V_n}} |(s,y)-(t,x)| \leq |(t,x) - (t, T_t(V_n)\circ T_t(V)^{-1}(x))|$$

goes to zero uniformly with n then also $\max_{(t,x)\in\Sigma_V} \min_{(s,y)\in\Sigma_{V_n}} |(s,y)-(t,x)|$. Similarly we consider the mapping $T_t(V)\circ T_t(V_n)^{-1}$ for the second term. □

Corollary 3.7. *The tube $Q(V)$ has the compactivorous property: For any compact K, $K\subset Q(V)$, $\exists n(K)$ s.t. $n \geq n(K)$ implies $K\subset Q(V_n)$.*

Now, we shall study the continuity of the solution of the penalized Navier-Stokes equations with respect to the field V. Let U_n^ϵ, $n\in\mathbb{N}^*$, be the unique solution of

$$\begin{aligned}\partial_t U_n^\epsilon - \eta\Delta U_n^\epsilon + DU_n^\epsilon.U_n^\epsilon + DV_n.U_n^\epsilon + DU_n^\epsilon.V_n + \frac{1}{\epsilon}\chi_{\Omega_{t,n}^c} U_n^\epsilon& \\ = F_n \text{ in } L^{\frac{4}{3}}(0,\tau; H_0^1(\mathrm{div}, D)')& \\ \mathrm{div}\, U_n^\epsilon = 0 \text{ in } (0,\tau)\times D& \\ U_n^\epsilon(0) = u_0 \text{ in } D&\end{aligned}$$

where $F_n = f - \partial_t V_n + \eta\Delta V_n - DV_n.V_n$ and

$$\left|\begin{array}{lcl} T_t^n(\Omega) &=& \Omega_{t,n},\ T_t^n(\Gamma) = \Gamma_{t,n};\ t\in(0,\tau) \\ T_0^n(\Omega) &=& \Omega. \end{array}\right.$$

Lemma 3.8. *Assume that $V_n \longrightarrow V$ strongly in $\mathcal{C}([0,\tau]; \mathbf{H}_0^{m'}(D))$ as $n\to\infty$. Then,*

$$U_n^\epsilon \rightharpoonup U^\epsilon \quad \text{weakly in } L^2(0,\tau; H_0^1(D)^3), \tag{3.6}$$

where U^ϵ is the unique solution of (2.6)–(2.8).

Proof. It is easy to see that $F_n \rightharpoonup F$ weak in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$. Moreover, we have the boundedness uniformly on n of (see proof of Theorem 2.3):

- U_n^ϵ in $L^2(0,\tau;H_0^1(D)^3)\cap L^\infty(0,\tau;H)$
- $\partial_t U_n^\epsilon$ in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$
- $\frac{1}{\epsilon}\chi_{\Omega_{t,n}^c}(U_n^\epsilon)$ in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$.

Then, there exists a function U^ϵ and a subsequence, still denoted U_n^ϵ, such that

$$\begin{aligned} U_n^\epsilon &\rightharpoonup U^\epsilon \text{ in } L^2(0,\tau;H_0^1(D)^3) - \text{weak} \\ U_n^\epsilon &\overset{*}{\rightharpoonup} U^\epsilon \text{ in } L^\infty(0,\tau;L^2(D)^3) - \text{weak star} \\ U_n^\epsilon &\to U^\epsilon \text{ in } L^2(0,\tau;L^4(D)^3) - \text{strong.} \end{aligned} \tag{3.7}$$

To get the continuity result, we should have

$$\chi_{\Omega_{t,n}^c}U_n^\epsilon \rightharpoonup \chi_{\Omega_t^c}U^\epsilon,\ n\to\infty,\ \text{in } L^2((0,\tau)\times D). \tag{3.8}$$

Let $\psi\in L^2((0,\tau)\times D)$,

$$\int_0^\tau\int_D \chi_{\Omega_{t,n}^c}U_n^\epsilon.\,\psi\,dxdt = \int_0^\tau\int_{\Omega^c} U_n^\epsilon\circ T_t^n\,\psi\circ T_t^n\,dxdt.$$

At the limit ($n\to\infty$) and thanks to (3.7), we have

$$\int_0^\tau\int_D \chi_{\Omega_{t,n}^c}U_n^\epsilon\,\psi\,dxdt \longrightarrow \int_0^\tau\int_D \chi_{\Omega_t^c}U^\epsilon\,\psi\,dxdt.$$

Indeed $U_n^\epsilon\circ T_t^n - U^\epsilon\circ T_t = (U_n^\epsilon\circ T_t^n - U^\epsilon\circ T_t^n) + (U^\epsilon\circ T_t^n - U^\epsilon\circ T_t)$. The first term of the right hand side goes to zero as $n\to\infty$. For the second term, it suffices to prove that

$$\psi\circ T_t^n \to \psi\circ T_t \text{ in } L^2((0,\tau)\times D).$$

Let $\psi\in\mathcal{D}((0,\tau)\times D)$, then

$$\psi(T_t^n(x))\to\psi(T_t(x))\ \forall(t,x)\in(0,\tau)\times D \ \text{ and } \ |\psi\circ T_t^n|\le \max_{(0,\tau)\times D}|\psi|.$$

The Lebesgue convergence theorem gives the strong convergence in $L^2((0,\tau)\times D)$ of the considered sequence. Thus (3.8) holds. Finally, we deduce easily that U^ϵ is solution of

$$\begin{aligned} \partial_t U^\epsilon - \eta\Delta U^\epsilon + DU^\epsilon.U^\epsilon + DV.U^\epsilon + DU^\epsilon.V + \frac{1}{\epsilon}\chi_{\Omega_t^c}U^\epsilon &= F \quad \text{in } L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)') \\ \mathrm{div}\,U^\epsilon &= 0 \quad \text{in } (0,\tau)\times D \\ U^\epsilon(0) &= u_0 \quad \text{in } D. \end{aligned}$$

□

In order to derive an existence result for the functional J we also need a continuity result when $(V_n,\epsilon_n)\to(V,0)$ as $n\to\infty$.

Lemma 3.9. *Assume that* $V_n \longrightarrow V$ *strongly in* $\mathcal{C}([0,\tau];\mathbf{H}_0^{m'}(D))$ *and* $\epsilon_n \to 0$ *as* $n \to \infty$. *Then, there exists a function* U *such that*

$$U_n^{\epsilon_n} \rightharpoonup U \quad \textit{weakly in } L^2((0,\tau)\times D), \tag{3.9}$$

and $U_{|Q(V)}$ *is the unique solution of* (2.1)–(2.4).

Proof. It is easy to see that $F_n \rightharpoonup F$ weak in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$. Moreover, we have the boundedness uniformly on n of (see proof of Theorem 2.3).

- $U_n^{\epsilon_n}$ in $L^2(0,\tau;H_0^1(D)^3)\cap L^\infty(0,\tau;H)$
- $\frac{1}{\epsilon_n}\chi_{\Omega_{t,n}^c}(U_n^{\epsilon_n}) + \partial_t U_n^{\epsilon_n}$ in $L^{\frac{4}{3}}(0,\tau;H_0^1(\mathrm{div},D)')$
- $\frac{1}{\sqrt{\epsilon_n}}\chi_{\Omega_{t,n}^c}(U_n^{\epsilon_n})$ in $L^2((0,\tau)\times D)$.

Then, there exists a function U and a subsequence, still denoted $U_n^{\epsilon_n}$, such that

$$\begin{aligned} U_n^{\epsilon_n} &\rightharpoonup U \text{ in } L^2(H_0^1(D)^3) - \text{weak} \\ U_n^{\epsilon_n} &\overset{*}{\rightharpoonup} U \text{ in } L^\infty(0,\tau;L^2(D)^3) - \text{weak star.} \end{aligned} \tag{3.10}$$

From the equation satisfied by $U_n^{\epsilon_n}$, we deduce that

$$\chi_{\Omega_{t,n}^c} U_n^{\epsilon_n} \to 0,\ n\to\infty,\ \text{in } L^2((0,\tau)\times D). \tag{3.11}$$

Using the same technic as for the proof of (3.8), we get $\forall \psi \in L^2((0,\tau)\times D)$,

$$\int_0^\tau \int_D \chi_{\Omega_{t,n}^c} U_n^{\epsilon_n}.\psi\, dxdt \longrightarrow \int_0^\tau \int_D \chi_{\Omega_t^c} U\,\psi\, dxdt$$

since $\psi\circ T_t^n$ (resp. $U_n^{\epsilon_n}\circ T_t^n$) converges strongly (resp. weakly) in $L^2((0,\tau)\times D)$. All this implies that $U(t)=0$ in $D\setminus\Omega_t(V)$ for a.e. $t\in(0,\tau)$.To establish that $U_{|Q(V)}$ is solution of (2.1)–(2.4) we proceed as following. Let $\psi\in\mathcal{D}(Q(V))$. The tube $Q(V)$ has the compactivorous property. So there exists $n_\psi\in\mathbb{N}$ such that the compact support of ψ is in $Q(V_n)$, $\forall n\geq n_\psi$. Hence

$$\begin{aligned} &\int_{Q(V)} \partial_t U_n^{\epsilon_n}.\psi\, dxdt + \eta\int_{Q(V)} DU_n^{\epsilon_n}..D\psi\, dxdt + \int_{Q(V)} \langle DU_n^{\epsilon_n}.U_n^{\epsilon_n},\psi\rangle dxdt + \\ &\quad + \int_{Q(V)} \langle DU_n^{\epsilon_n}.V_n,\psi\rangle dxdt + \int_{Q(V)} \langle DV_n.U_n^{\epsilon_n},\psi\rangle dxdt + \frac{1}{\epsilon_n}\int_{Q(V)} \chi_{\Omega_{t,n}^c} U_n^{\epsilon_n}.\psi\, dxdt \\ &\qquad = \int_0^\tau \langle F_n,\psi\rangle_{\mathcal{D}'(Q(V)),\mathcal{D}(Q(V))} dt. \end{aligned}$$

But the penalizing term vanishes so we can pass to the limit on n without any difficulty. □

3.2. Existence of an optimal tube

The initial domain $\Omega\subset\subset D$ being fixed, we have the following existence result.

Proposition 3.10. *There exists a vector field denoted* $\overline{V} \in H^1(0,\tau;\mathbf{H}_0^m(D))$ *solution of* (3.1).

Proof. From the previous continuity result on the penalized problem we get that for each positive ϵ the following problem has a solution.

Find V *solution of*

$$\min_{V\in H^1(0,\tau;\mathbf{H}_0^m(D))} J_\epsilon(V) = \frac{1}{2}\left\{\int_{Q(V)} |U^\epsilon(V)+V|^2\,dxdt + \alpha\|V\|^2_{H^1(0,\tau;\mathbf{H}_0^m(D))}\right\} \tag{3.12}$$

where $U^\epsilon(V)$ is the solution of (2.6)–(2.8).

Let ϵ_n be a sequence which converges to zero and $\overline{V}_n$ the corresponding minimizing fields. So

$$\forall\, V \in H^1(0,\tau,\mathbf{H}_0^m(D)),\quad J_{\epsilon_n}(\overline{V}_n) \;\leq\; J_{\epsilon_n}(V). \tag{3.13}$$

It is easy to see that $\{\overline{V}_n\}$ is bounded in $H^1(0,\tau;\mathbf{H}_0^m(D))$. So there exists a subsequence still denoted $\{\overline{V}_n\}$ converging weakly to a vector field $\overline{V}$. To each $\overline{V}_n$ we associate the vector field $U^{\epsilon_n}(\overline{V}_n)$ solution of (2.6)–(2.8) which from Lemma 3.9 converges weakly in $L^2((0,\tau)\times D)$ to $\overline{U}$ such that $\overline{U}_{|Q(\overline{V})}$ is the solution of (2.1)–(2.4). We conclude by going to the limit on n in inequality (3.13). As the right hand-side is continuous

$$J_{\epsilon_n}(V) \to J(V).$$

For the left hand-side we notice that

$$\chi_{Q(\overline{V_n})}\, U^{\epsilon_n}(\overline{V_n}),\ \text{ weakly converges in } \ L^2((0,\tau)\times D) \text{ to } \chi_{Q(\overline{V})}\, U(\overline{V})$$

since $U^{\epsilon_n}(\overline{V_n})\circ T(\overline{V_n})$, weakly converges in $L^2((0,\tau)\times D)$ to $U(\overline{V})\circ T(\overline{V_n})$. □

4. Sensitivity Analysis with respect to the Field V

Let $W_0^{k,\infty}(D)$ be the space $W^{k,\infty}(D)\cap H_0^1(D)$ and $V\in H^1(0,\tau;\mathbf{H}_0^m(D))$. According to [5], we can define a family of transformations by

$$(t,X)\longmapsto T(t,X) \stackrel{def}{=} x(t,X) : [0,\tau]\times\overline{D} \longrightarrow \mathbb{R}^3, \tag{4.1}$$

where $x(.,X)$ is the solution of

$$\frac{dx}{dt}(t) = V(t,x(t)),\quad t\in[0,\tau],\quad x(0)=X \tag{4.2}$$

with the following properties

$$\begin{aligned}
&\forall X\in\overline{D}, T(.,X)\in\mathcal{C}^1([0,\tau];\mathbb{R}^3)\\
&\exists c>0, \forall X,Y\in\overline{D}, \|T(.,X)-T(.,Y)\|_{\mathcal{C}^1([0,\tau];\mathbb{R}^3)} \leq c|X-Y|,\\
&\forall t\in[0,\tau], X\longmapsto T_t(X) \stackrel{def}{=} T(t,X) : \overline{D}\to\overline{D} \text{ is bijective},\\
&\forall X\in\overline{D}, T^{-1}(.,X)\in\mathcal{C}^0([0,\tau];\mathbb{R}^3)\\
&\exists c>0, \forall x,y\in\overline{D}, \|T^{-1}(.,x)-T^{-1}(.,y)\|_{\mathcal{C}^0([0,\tau];\mathbb{R}^3)} \leq c|x-y|.
\end{aligned}$$

The transformation T involved by the previous ordinary differential equation (4.2) is generally denoted $T(V)$.

Remark 4.1. Notice that by induction, one can prove that

$$T_t(V) \in W^{k,\infty}(D) \hookrightarrow \mathcal{C}^{k-1}(D),\ \forall t \in [0,\tau].$$

4.1. The Transverse Field Z

At each fixed time $t \in [0,\tau]$ and any σ sufficiently small, we consider the moving domain

$$\Omega_t(V+\sigma W) = T_t(V+\sigma W)(\Omega),\ \Omega \subset\subset D \text{ is given.}$$

At fixed $t \in [0,\tau]$, we should consider a map $\mathcal{T}_\sigma^t$ which maps $\Omega_t(V)$ onto $\Omega_t(V+\sigma W)$(and $\overline{D}$ onto $\overline{D}$). A quite natural choice is

$$\mathcal{T}_\sigma^t = T_t(V+\sigma W) \circ T_t(V)^{-1}.$$

Under some assumptions, see for instance [5], that transformation can be considered as the flow of the vector field

$$\mathcal{Z}^t(\sigma,.) = \left(\frac{\partial}{\partial\sigma}\mathcal{T}_\sigma^t\right) \circ \mathcal{T}_\sigma^t(\ .\ )^{-1}. \tag{4.3}$$

Proposition 4.2. *The mapping $\sigma \longrightarrow T(V+\sigma W)$, $I_0 \longrightarrow \mathcal{C}^0([0,\tau]; W^{k-1,\infty}(D))$, is continuously derivable and $\partial_\sigma(T_t(V+\sigma W))$ satisfies*

$$\partial_\sigma[T_t(V+\sigma W)] = \int_0^t D(V+\sigma W)(\mu, T_\mu(V+\sigma W))\mathcal{S}^\mu(\sigma)\, d\mu + \int_0^t W(\mu, T_\mu(V+\sigma W))\, d\mu, \forall t \in [0,\tau].$$

Proof. For any given $t \in [0,\tau]$ and $x \in D$, the mapping

$$\sigma \longrightarrow T_t(V+\sigma W)(x) \text{ is } \mathcal{C}^\infty(I_0, \mathbb{R}^3)$$

since $\sigma \longrightarrow V+\sigma W$ is affine in σ. We know that $\partial[T_t(V+\sigma W)(x)]/\partial\sigma$ is the solution of

$$\begin{aligned} &\dot{\mathcal{S}}^t(\sigma,x) = D(V+\sigma W)(t, T_t(V+\sigma W)(x))\mathcal{S}^t(\sigma,x) + W(t, T_t(V+\sigma W)(x)) \\ &\mathcal{S}^0(\sigma,x) = 0 \end{aligned}$$

or equivalently that

$$\mathcal{S}^t(\sigma,x) = \frac{\partial}{\partial\sigma}[T_t(V+\sigma W)(x)] = \int_0^t D(V+\sigma W)(\mu, T_\mu(V+\sigma W)(x))\mathcal{S}^\mu(\sigma,x)\, d\mu + \int_0^t W(\mu, T_\mu(V+\sigma W)(x))\, d\mu.$$

We introduce the functions $\mathcal{Q}_\varepsilon^t(\sigma,x) = \varepsilon^{-1}\left(T_t(V+(\sigma+\varepsilon)W)(x) - T_t(V+\sigma W)(x)\right)$ and $\mathcal{R}_\epsilon^t(\sigma,x) = \mathcal{Q}_\epsilon^t(\sigma,x) - \mathcal{S}^t(\sigma,x)$.

More explicitly, we have

$$\begin{aligned}\mathcal{R}^t_\epsilon(\sigma,x) = &\int_0^t W(\mu, T_\mu(V+(\sigma+\epsilon)W)(x)) - W(\mu, T_\mu(V+\sigma W)(x))\, d\mu \\ &+ \int_0^t DV+\sigma W(\mu, T_\mu(V+\sigma W)(x))\mathcal{R}^\mu_\epsilon(\sigma,x) d\mu \\ &+ \int_0^t [D(V+\sigma W)\,(\mu, T_\mu(V+\sigma W)x + \theta\,\epsilon \mathcal{Q}^\mu_\epsilon(\sigma,x)) - \\ &\qquad D(V+\sigma W)(\mu, T_\mu(V+\sigma\xi)x)]\,\mathcal{Q}^\mu_\epsilon(\sigma,x)\, d\mu.\end{aligned}$$

So,

$$\begin{aligned}\|\mathcal{R}^t_\epsilon(\sigma)\|_{W^{k-1,\infty}} \leq &\int_0^t \|DV+\sigma W(\mu)\|_{W^{k-1,\infty}} \|\mathcal{R}^\mu_\epsilon(\sigma)\|_{W^{k-1,\infty}} d\mu \\ &+ \int_0^t \|W(\mu, T_\mu(V+(\sigma+\epsilon)W)(x)) - W(\mu, T_\mu(V+\sigma W)(x))\|_{W^{k-1,\infty}} d\mu \\ &+ \int_0^t \|[D(V+\sigma W)\,(\mu, T_\mu(V+\sigma W)x + \theta\,\epsilon \mathcal{Q}^\mu_\epsilon(\sigma,x)) - \\ &\qquad D(V+\sigma W)(\mu, T_\mu(V+\sigma W)x)]\|_{W^{k-1,\infty}}. \\ &\quad \|\mathcal{Q}^\mu_\epsilon(\sigma)\|_{W^{k-1,\infty}} d\mu.\end{aligned}$$

For simplicity, we set

$$\begin{aligned}b^t_\epsilon(\sigma) = &\int_0^t \|W(\mu, T_\mu(V+(\sigma+\epsilon)W)) - W(\mu, T_\mu(V+\sigma W))\|_{W^{k-1,\infty}} + \\ &+ \|D(V+\sigma W)(\mu, T_\mu(V+\sigma W) + \theta\,\epsilon\,\mathcal{Q}^\mu_\epsilon(\sigma)) - D(V+\sigma W)(\mu, T_\mu(V+\sigma W))\|_{W^{k-1,\infty}}. \\ &\|\mathcal{Q}^\mu_\epsilon(\sigma)\|_{W^{k-1,\infty}}\, d\mu\end{aligned}$$

and $a^t(\sigma) = \|D(V+\sigma W)(t, T_t(V+\sigma W))\|_{W^{k-1,\infty}}$. The Generalized Gronwall inequality gives

$$\|\mathcal{R}^t_\epsilon(\sigma)\|_{W^{k-1,\infty}} \leq b^t_\epsilon(\sigma) + \int_0^t a^\mu(\sigma) b^\mu_\epsilon(\sigma) \left(\exp\int_\mu^t a^u(\sigma) du\right) d\mu. \tag{4.4}$$

Thus,

$$\max_{t\in[0,\tau]} \|\mathcal{R}^t_\epsilon(\sigma)\|_{W^{k-1,\infty}} \longrightarrow 0,\ \varepsilon \longrightarrow 0.$$

□

Remark 4.3. The above result remains true when $\mathcal{C}^0([0,\tau]; W^{k-1,\infty}(D))$ is replaced by $\mathcal{C}^0([0,\tau]; \mathcal{C}^{k-1}(D))$.

The derivability results, we are interested in, shall use only the vector field $\mathbf{Z}(t,x) = \mathcal{Z}^t(0,x)$ which can be characterized by the following result.

Proposition 4.4. *The vector field* $\mathbf{Z}$ *belongs to* $\mathcal{C}([0,\tau], W^{k-1}(D))$ *and is the unique solution of*

$$\begin{aligned} \partial_t \mathbf{Z} + [\mathbf{Z}, V] &= W \quad in\ (0,\tau)\times D && (4.5)\\ \mathbf{Z}(0,.) &= 0 \quad in\ D && (4.6) \end{aligned}$$

where [,] *denotes the Lie Brackets.*

First, let us give the next uniqueness result.

Lemma 4.5. *The function* S, $S(t,.) = \mathcal{S}^t(0,.)$, *is the unique function in* $\mathcal{C}^0([0,\tau]; W^{k-1,\infty}(D))$ *satisfying*

$$S(t) = \int_0^t W(\mu, T_\mu(V))d\mu + \int_0^t DV(\mu, T_\mu(V))S(\mu)\,d\mu. \tag{4.7}$$

Proof. Assume that $S_i, i = 1,2$, are two solutions of (4.7). Then, $S = S_1 - S_2$ satisfies

$$S(t) = \int_0^t DV(\mu, T_\mu(V))S(\mu)\,d\mu$$

$$\begin{aligned} \|S(t)\|_{W^{k-1,\infty}} &\le c\int_0^t \|DV(\mu)\|_{W^{k-1,\infty}}\|S(\mu)\|_{W^{k-1,\infty}}\,d\mu && (4.8)\\ &\le c\max_{t\in[0,\tau]}\|DV(t)\|_{W^{k-1,\infty}}\int_0^t \|S(\mu)\|_{W^{k-1,\infty}}\,d\mu. && (4.9) \end{aligned}$$

Set

$$r(t) = \int_0^t \|S(\mu)\|_{W^{k-1,\infty}}d\mu \quad\text{and}\quad K = c\max_{t\in[0,\tau]}\|DV(t)\|_{W^{k-1,\infty}}.$$

The previous inequality can be written as follows

$$\frac{d}{dt}r(t) \le Kr(t)$$

or equivalently

$$\frac{d}{dt}(\exp(-Kt)r(t)) \le 0.$$

It implies that $r(.)$ and $\|S(.)\|_{W^{k-1,\infty}}$ vanish in $[0,\tau]$. □

Proof of Proposition 4.4. We know that $S(t,.) = \mathcal{S}^t(0,.)$ satisfies

$$S(t,x) = \int_0^t W(\mu, T_\mu(V))\,d\mu + \int_0^t DV(\mu, T_\mu(V))S(\mu,x)\,d\mu. \tag{4.10}$$

Equivalently

$$\begin{aligned} \partial_t S - DV(t, T_t(V))\,S &= W(t, T_t(V))\\ S(0,.) &= 0 \end{aligned}$$

in a classical sense since S is C^1 in the variable t. We compute, in a distribution sense, the partial derivative $\partial_t \mathbf{Z}$ using the fact that $\mathbf{Z}(t,.) = S(t,.) \circ T_t(V)^{-1}$.

Let $\varphi \in \mathcal{D}((0,\tau) \times D)$, we have

$$
\begin{aligned}
&\int_0^\tau \int_D \mathbf{Z}(t,x) \,.\, \partial_t \varphi \, dxdt \\
&= \int_0^\tau \int_D S(t) \circ T_t(V)^{-1} \,.\, \partial_t \varphi \, dxdt = \int_0^\tau \int_D S(t).\, (\partial_t \varphi) \circ T_t(V) \, dxdt \\
&= \int_0^\tau \int_D S(t).\, [\partial_t(\varphi \circ T_t(V)) - (D\varphi(t)) \circ T_t(V).V \circ T_t(V)] \; dxdt \\
&= -\langle \partial_t S(t), \varphi \circ T_t(V) \rangle_{\mathcal{D}',\mathcal{D}} - \int_0^\tau \int_D \langle S(t), (D\varphi(t)) \circ T_t(V).\, V \circ T_t(V) \rangle dxdt \\
&= -\int_0^\tau \int_D W(t, T_t(V)) \,.\, \varphi \circ T_t(V) + DV(t, T_t(V)) S(t) \,.\, \varphi \circ T_t(V) \, dxdt \, - \\
&\qquad \int_0^\tau \int_D \langle S(t), D(\varphi(t) \circ T_t)(DT_t)^{-1} \, V \circ T_t \rangle dxdt \\
&= -\int_0^\tau \int_D W(t,x)\varphi + \langle DV(t,x)\mathbf{Z}(t,x), \varphi \rangle dxdt - \int_0^\tau \int_D \langle \mathbf{Z}, D\varphi(t).V \rangle dxdt.
\end{aligned}
$$

Thus,

$$\partial_t \mathbf{Z} + D\mathbf{Z}.V - DV.\mathbf{Z} = W \text{ in } (0,\tau) \times D.$$

Now, assume that $\mathbf{Z}_i, i = 1,2$, are two solutions of problem (4.5)–(4.6) and let $S_i(t) = \mathbf{Z}_i(t) \circ T_t(V)$. We will show that $S_i, i = 1,2$, are two solutions, in $\mathcal{C}^0([0,\tau]; W^{k-1,\infty}(D))$, of problem (4.7). According to the uniqueness result (4.5), we deduce that necessary $\mathbf{Z}_1 = \mathbf{Z}_2$. □

5. Necessary Optimality Condition

5.1. Derivative with respect to V

Here we are interested in the derivability of the mapping

$$I_0 = [0, \sigma_0] \longrightarrow L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V)))$$

$$\sigma \longrightarrow (D\mathcal{T}_\sigma^t)^{-1}.\, (U_\sigma \circ \mathcal{T}_\sigma^t), \;\; \sigma_0 \;\text{ is sufficiently small },$$

where $U_\sigma = U(V + \sigma W)$ is the solution of (2.1)–(2.4) when V is replaced by $(V + \sigma W)$.

Let $\mathcal{V}_0 = \{v \in L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V))), \partial_t v \in L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V)))\}$, $\mathcal{V}_\infty = \{v \in L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V))), \partial_t v \in L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V))) \cap L^\infty(0,\tau; H(\mathrm{div}, \Omega_t(V)))\}$ and $\mathcal{F} = \{F \in L^\infty(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V))'), \partial_t F \in L^1(0,\tau; H(\Omega_t(V))\}$. Assume conditions (2.34)–(2.36). So the Weak Implicit Function Theorem (see [13]) works. Indeed let

$$\Phi = (\Phi_1, \Phi_2) : I_0 \times \mathcal{V}_0 \longrightarrow L^2(0,\tau; H_0^1(\mathrm{div}, \Omega_t(V))') \times H_0^1(\mathrm{div}, \Omega)$$

defined in a weak form by: for all $w \in L^2(0,\tau; H^1_0(\mathrm{div}, \Omega_t(V)))$

$$\begin{aligned}
&\langle \Phi_1(\sigma, v), w\rangle \\
&= \int_{Q(V)} (\partial_t(D\mathcal{T}^t_\sigma v),\ D\mathcal{T}^t_\sigma w)\,dxdt - \int_{Q(V)} (D(D\mathcal{T}^t_\sigma v).(D\mathcal{T}^t_\sigma)^{-1}.\partial_t \mathcal{T}^t_\sigma, (D\mathcal{T}^t_\sigma)w)\,dxdt \\
&+ \eta \int_{Q(V)} D(D\mathcal{T}^t_\sigma v).(D\mathcal{T}^t_\sigma)^{-1}..D(D\mathcal{T}^t_\sigma w).(D\mathcal{T}^t_\sigma)^{-1}\,dxdt + \int_{Q(V)} \langle D(D\mathcal{T}^t_\sigma v).v, D\mathcal{T}^t_\sigma w\rangle dxdt \\
&+ \int_{Q(V)} \langle D(D\mathcal{T}^t_\sigma v)(D\mathcal{T}^t_\sigma)^{-1}.(V+\sigma W)\circ \mathcal{T}^t_\sigma + D((V+\sigma W)\circ \mathcal{T}^t_\sigma).v, D\mathcal{T}^t_\sigma w\rangle\,dxdt \\
&- \int_{Q(V)} \langle F_{V+\sigma W}\circ \mathcal{T}^t_\sigma, D\mathcal{T}^t_\sigma w\rangle\,dxdt
\end{aligned} \tag{5.1}$$

$$\Phi_2(\sigma, v) = v(0) - u_0 \tag{5.2}$$

where $\partial_t \mathcal{T}^t_\sigma = -D\mathcal{T}^t_\sigma.V + (V+\sigma W)\circ \mathcal{T}^t_\sigma$ and $\int_{Q(V)} \langle F_{V+\sigma W}\circ \mathcal{T}^t_\sigma, w\rangle\,dxdt =$

$$\int_{Q(V)} \langle f\circ \mathcal{T}^t_\sigma - (\partial_t(V+\sigma W) + D(V+\sigma W).(V+\sigma W) - \eta\Delta(V+\sigma W))\circ \mathcal{T}^t_\sigma, w\rangle\,dxdt.$$

Lemma 5.1. *We have*

(i) *For any $v, w \in L^2(0,\tau; H^1_0(\mathrm{div}, \Omega_t(V)))$, the mapping*

$$\sigma \longrightarrow \langle \Phi_1(\sigma, v), w\rangle \ \text{is}\ \mathcal{C}^1(I_0) \tag{5.3}$$

and its derivative is given by

$$\begin{aligned}
&\langle \partial_\sigma \Phi_1(\sigma, v), w\rangle = \int_{Q(V)} \langle (\partial_t(D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma), D\mathcal{T}^t_\sigma\, w\rangle + \langle \partial_t(D\mathcal{T}^t_\sigma\, v), D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma)\, w\rangle \\
&+ \langle D(D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).v).V, D\mathcal{T}^t_\sigma\, w\rangle + \langle D(D\mathcal{T}^t_\sigma v).V, D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma)w\rangle\,dxdt \\
&+ \eta \int_{Q(V)} D(D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).v)(D\mathcal{T}^t_\sigma)^{-1} - D(D\mathcal{T}^t_\sigma.v)(D\mathcal{T}^t_\sigma)^{-1} D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).(D\mathcal{T}^t_\sigma)^{-1}.. \\
&D(D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).w)(D\mathcal{T}^t_\sigma)^{-1} - D(D\mathcal{T}^t_\sigma.w)(D\mathcal{T}^t_\sigma)^{-1} D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).(D\mathcal{T}^t_\sigma)^{-1}\,dxdt \\
&+ \int_{Q(V)} \langle D(D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma).v)v, D\mathcal{T}^t_\sigma w\rangle + \langle D(D\mathcal{T}^t_\sigma.v)v, D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma)w\rangle\,dxdt \\
&+ \int_{Q(V)} \langle (D(V+\sigma W)\circ \mathcal{T}^t_\sigma.\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma + D(W\circ \mathcal{T}^t_\sigma)).v, D\mathcal{T}^t_\sigma w\rangle\,dxdt \\
&- \int_{Q(V)} \langle \partial_\sigma(F_\sigma\circ \mathcal{T}^t_\sigma), D\mathcal{T}^t_\sigma w\rangle + \langle F_\sigma\circ \mathcal{T}^t_\sigma, D(\mathcal{Z}^t(\sigma)\circ \mathcal{T}^t_\sigma)w\rangle\,dxdt
\end{aligned}$$

(ii) *The map $I_0 \times L^2(0,\tau; H^1_0(\mathrm{div}, \Omega_t(V))) \longrightarrow L^2(0,\tau; H^1_0(\mathrm{div}, \Omega_t(V))')$, $(\sigma, v) \longrightarrow \partial_\sigma \Phi_1(\sigma, v)$ is weakly continuous.*

Proof. As shown in proposition 4.2 the mapping $\sigma \longrightarrow \mathcal{T}^t_\sigma$ is continuously differentiable for the strong topology and we have

$$\partial_\sigma \mathcal{T}^t_\sigma = \mathcal{Z}^t_\sigma \circ \mathcal{T}^t_\sigma.$$

On the other-hand since $f \in L^2(0, \tau, H)$, the mapping $\sigma \longrightarrow f \circ \mathcal{T}_\sigma^t$ is only weakly differentiable. Thus $\sigma \longrightarrow \Phi_1(\sigma, v)$ is weakly differentiable. We denote by $\partial_\sigma \Phi_1(\sigma, v)$ its weak derivative.

The continuity of $(\sigma, v) \longrightarrow \partial_\sigma \Phi_1(\sigma, v)$ is easy to check. □

Lemma 5.2. *The mapping*

$$v \longrightarrow \Phi_1(\sigma, v),\ L^2(0, \tau; H_0^1(\mathrm{div}, \Omega_t(V))) \to L^2(0, \tau; H_0^1(\mathrm{div}, \Omega_t(V))') \tag{5.4}$$

is differentiable and

$$(\sigma, v) \longrightarrow \partial_v \Phi_1(\sigma, v) \ \ \textit{is continuous.} \tag{5.5}$$

Moreover,

$$\partial_v \Phi(0, U) \ \textit{is an isomorphism} \tag{5.6}$$

from $\mathcal{V}_\infty$ onto $\mathcal{F}$.

Property (5.6) and the following Lemma are direct consequences of the uniqueness of the solution of the considered Navier-Stokes system.

Lemma 5.3. *The mapping $\sigma \longrightarrow U_\sigma \circ \mathcal{T}_\sigma$ is Lipschitz-continuous.*

Proposition 5.4. *Under hypotheses of* Lemmas 5.1–5.3, *the weak derivative $\dot{U} = \partial_\sigma((D\mathcal{T}_\sigma^t)^{-1} U_\sigma \circ \mathcal{T}_\sigma)_{|\sigma=0}$ exists and is the solution of*

$$\partial_v \Phi(0, U).\dot{U} = -\partial_\sigma \Phi(0, U).$$

From now on we assume Ω of class $\mathcal{C}^2$. Note that since the field V is smooth, $\Omega_t(V)$ has the same regularity than Ω for any $t \in (0, \tau)$. Referring to regularity results for Navier-Stokes equations (see for instance [12]) $U(t) \in H^2(\Omega_t(V), \mathbb{R}^3)$ for a.e. $t \in (0, \tau)$. Then

Proposition 5.5. *The field $U'(t) = \dot{U}(t) - DU(t).\mathbf{Z}(t)$, $t \in (0, \tau)$, satisfies*

$$\begin{aligned} \partial_t U' - \eta \Delta U' &+ DU'.U + DU.U' + DU'.V + DV.U' \\ &= -\partial_t W + \eta \Delta W - DW.U - DU.W - DV.W - DW.V \quad \textit{in} \quad Q(V) \\ \mathrm{div}\, U' &= 0 \qquad \textit{in} \quad Q(V) \\ U'(t) &= -\langle \mathbf{Z}(t), n_t \rangle DU(t).n_t \quad \textit{on} \quad \Sigma(V) \\ U'(0) &= 0 \quad \textit{in} \quad \Omega. \end{aligned}$$

We set $u' = U' + W$ (as $u = U + V$), then

Corollary 5.6. *The field $u'(t) = \dot{U}(t) - DU(t).\mathbf{Z}(t) + W$, $t \in (0, \tau)$, satisfies*

$$\begin{aligned} \partial_t u' - \eta \Delta u' + Du'.u + Du.u' &= 0 \quad \textit{in} \quad Q(V) \\ \mathrm{div}\, u' &= 0 \quad \textit{in} \quad Q(V) \\ u'(t) &= \langle \mathbf{Z}(t), n_t \rangle \,(DV(t) - Du(t)).n_t + W(t) \quad \textit{on} \quad \Sigma(V) \\ u'(0) &= 0 \quad \textit{in} \quad \Omega. \end{aligned}$$

5.2. Adjoint States associated to U' and $\mathbf{Z}$

Let $2\,J_1(V) = \|u(V)\|^2_{Q(V)}$, $u(V) = U(V) + V$. We have

$$dJ_1(V;W) = \lim_{\sigma \searrow 0} \frac{J_1(V+\sigma W) - J_1(V)}{\sigma}$$
$$= \int_{Q(V)} (U+V).\,U' \;+\; (U+V).\,W\,dxdt + \frac{1}{2}\int_{\Sigma(V)} |V|^2 \mathbf{Z}.n_t\, d\Gamma_t dt$$

since $U = 0$ on Σ. That is

$$dJ_1(V;W) = \int_{Q(V)} u\,u'\,dxdt + \frac{1}{2}\int_{\Sigma(V)} |V|^2\mathbf{Z}.n_t\,d\Gamma_t dt$$

5.2.1. The adjoint problem associated to U'

In view of the elimination of U' the adjoint is the following one.

$$\begin{aligned} -\partial_t\mathbf{U} - \eta\Delta\mathbf{U} - D\mathbf{U}.U + {}^*DU.\mathbf{U} + {}^*DV.\mathbf{U} - D\mathbf{U}.V = U+V \quad &\text{in } Q(V)\\ \mathbf{U} = 0 \quad &\text{on } \Sigma(V)\\ \mathbf{U}(\tau) = 0 \quad &\text{in } \Omega_\tau(V). \end{aligned}$$

That is

$$\begin{aligned} -\partial_t\mathbf{U} - \eta\Delta\mathbf{U} - D\mathbf{U}.u + {}^*Du.\mathbf{U} = u \quad &\text{in } Q(V)\\ \mathbf{U} = 0 \quad &\text{on } \Sigma(V)\\ \mathbf{U}(\tau) = 0 \quad &\text{in } \Omega_\tau(V). \end{aligned}$$

Then,

$$\int_{Q(V)} u.u'\,dxdt = \int_{Q(V)} (-\partial_t\mathbf{U}, u') - \eta\,\Delta\mathbf{U}.u' - \langle D\mathbf{U}.u, u'\rangle + \langle {}^*Du\mathbf{U}, u'\rangle\,dxdt$$
$$= \int_{\Sigma(V)} \langle \eta\epsilon(\mathbf{U}).n_t \;-\; \langle V, n_t\rangle\mathbf{U}\,,\, u'\rangle\,d\Gamma_t dt$$

dJ_1 can be expressed as follows

$$dJ_1(V,W) =$$
$$\int_{\Sigma(V)} \langle \eta\,\epsilon(\mathbf{U}).n_t \;-\; \langle V,n_t\rangle\mathbf{U},\; \langle\mathbf{Z}(t), n_t\rangle(DV(t) - Du(t)).n_t + W(t)\rangle \;\; d\Gamma_t dt$$
$$+ \frac{1}{2}\int_{\Sigma(V)} |V|^2\mathbf{Z}.n_t\,d\Gamma_t dt.$$

But $\mathbf{U} = 0$ on Σ then the expression simplifies in the following one:

$$dJ_1(V,W) = \int_{\Sigma(V)} \langle \eta\,\epsilon(\mathbf{U}).n_t\,,\, \langle\mathbf{Z}(t), n_t\rangle(DV(t) - Du(t)).n_t + W(t)\rangle \;\; d\Gamma_t dt$$
$$+ \frac{1}{2}\int_{\Sigma(V)} |V|^2\mathbf{Z}.n_t\,d\Gamma_t dt$$

Lemma 5.7. *As $U = 0$ on the boundary Σ then $DU = DU.n.n^*$*

$$\langle \epsilon(\mathbf{U}).n_t, DU.n_t \rangle = \epsilon(\mathbf{U})..DU \quad on \quad \Gamma_t$$

Proof. □

Also as $\mathbf{Z}$ is a free divergence field we have

$$\forall H, \ \int_{\Sigma(V)} H\mathbf{Z}.n_t \, d\Gamma_t dt = \int_{Q(V)} \langle \nabla(H), \mathbf{Z} \rangle \, dx.$$

So that

$$\begin{aligned}
&\int_{\Sigma(V)} \eta \langle \epsilon(\mathbf{U}).n_t, \ \langle \mathbf{Z}(t), n_t \rangle (DV(t) - Du(t)).n_t \rangle \\
&= \int_{\Sigma(V)} \eta \, \epsilon(\mathbf{U})..(DV(t) - Du(t)) \ \langle \mathbf{Z}(t), n_t \rangle d\Gamma_t dt \\
&= \int_0^\tau \int_{\Omega_t(V)} \eta \ div(\ \epsilon(\mathbf{U})..(DV(t) - Du(t)) \ \mathbf{Z}(t) \ ) \, dx dt \\
&= \int_0^\tau \int_{\Omega_t(V)} (\eta \ \nabla(\ \epsilon(\mathbf{U})..(DV(t) - Du(t)) \,) \, \mathbf{Z}(t) \ dx dt
\end{aligned}$$

and we get

$$\begin{aligned}
dJ(V;W) = \int_0^\tau \int_{\Omega_t(V)} (\chi_{Q(V)} \nabla(\ \eta \, \epsilon(\mathbf{U})..(DV(t) - Du(t)) \ + \ \nabla(|V|^2)) \, \mathbf{Z}(t) \ dx dt \\
+ \int_{\Sigma(V)} \eta \, \langle \epsilon(\mathbf{U}).n_t, W \rangle d\Sigma
\end{aligned}$$

We set

$$\mathbf{L} = \chi_{Q(V)} \nabla(\ \eta \, \epsilon(\mathbf{U})..(DV(t) - Du(t)) \ + \ |V|^2 \ ).$$

Concerning the term independent on Z we have, as $divW(t) = 0$,

$$\begin{aligned}
\int_{\Sigma(V)} \eta \langle \epsilon(\mathbf{U}).n_t, W(t) \rangle d\Gamma_t dt &= \int_0^\tau \int_{\Omega_t(V)} \eta \, div(\, \epsilon(\mathbf{U}).W \,) \, dx dt \\
&= \langle -\eta \, \epsilon(\mathbf{U}).\nabla \chi_{Q_V}, W \rangle
\end{aligned}$$

5.2.2. The adjoint problem associated to Z

Lemma 5.8. *There exists a unique solution $\Lambda \in \mathcal{C}^0(0, \tau; L^2(D, \mathbb{R}^3))$ to the following problem*

$$\begin{aligned}
-\partial_t \Lambda - D\Lambda.V - \ {}^*DV.\Lambda = \mathbf{L} \quad in \ \ (0, \tau) \times D \\
\Lambda(\tau) = 0 \quad in \ \ D.
\end{aligned}$$

Proof. For simplicity we proceed as it was done for the existence result for $\mathbf{Z}$. It means that we study the following ordinary differential equation in $\mathcal{C}^1(0,\tau;L^2(D,\mathbb{R}^3))$.

$$-\partial_t \mathbf{P} - {}^*DV(t)\circ T_t(V).\mathbf{P} = \mathbf{L}(t)\circ T_t(V) \quad \text{in} \quad (0,\tau)\times D$$
$$\mathbf{P}(\tau) = 0 \quad \text{in} \quad D.$$

and take $\Lambda = \mathbf{P}\circ T_t^{-1}(V)$. □

Using that adjoint state Λ associated to $\mathbf{Z}$, dJ_1 can be expressed as follows

$$dJ_1(V;W) = \int_0^\tau \int_D (-\eta\,\epsilon(\mathbf{U}).\nabla\chi_{\Omega_t(V)} + \Lambda).\,W\,dxdt. \tag{5.7}$$

Proposition 5.9. *Let $\mathbf{G}(V)$ be the gradient of J.*

$$\mathbf{G}(V)(t,.) = (-\eta\,\epsilon(\mathbf{U}).\nabla\chi_{\Omega_t(V)} + \Lambda)\,) + \alpha V.$$

Let $\overline{V}$ be an optimal solution of (3.1). *Then there exists a scalar function $\overline{g}$ such that*

$$(-\eta\,\epsilon(\mathbf{U}).\nabla\chi_{\Omega_t(\overline{V})} + \Lambda) + \alpha V = \nabla\overline{g} \quad \textit{in} \quad [0,\tau]\times D.$$

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