Journal of Convex Analysis
Volume 6 (1999), No. 2, 387–393

A Note on the Closedness of the Convex Hull and Its Applications

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Received June 2, 1998
Revised manuscript received March 19, 1999

This paper answers the following question motivated by the problem of spannability of functions. When is the convex hull of an unbounded (closed) set closed? We provide necessary and sufficient conditions for the closedness of the convex hull. Then we apply these results to the problem of spannability of functions playing an important role in mathematical economics and variational calculus. Resulting characterizations of spannability of functions imply previously known sufficiency conditions for spannability.

1991 Mathematics Subject Classification: 52A20

1. Introduction

It is a well-known, elementary fact that the convex hull of a closed bounded set in finite-dimensional space is closed. When is the convex hull of an unbounded (closed) set closed? This question is motivated by the problem of spannability of functions, which plays an important role in various applications. The spannability problem appeared first in mathematical economics. Shapley and Shubik [9] found the spannability of utility functions to be important for the existence of quasi-cores of economies with nonconvex preferences. Later, the spannability of integrands of variational problems proved to be of paramount importance in the study of relaxation of variational problems (see Ekeland-Temam [1]). Thereafter its use in variational calculus and nonsmooth analysis became common (see e.g. [2], [3] and [4]).

Though the question under consideration is motivated by the spannability problem, it is certainly of general interest and may potentially find other applications.

We will freely use standard notations and concepts of convex analysis, most of which can be found in Rockafellar [8]. We recall some of them. All sets considered in the sequel are subsets of $n-$dimensional real coordinate space R^n. The dimension of a set A is the dimension of its affine hull. Following Klee [6] we say that a set A is *coterminal* with a ray $\rho = \{x + ry;\ r > 0\}$ provided $\sup\{r :\ r > 0 \text{ and } x + ry \in A\} = \infty$. When M is an affine subspace, an *embracing subset* of M is a set whose convex hull is M. The *relative interior* of a convex set C is defined as the interior of C when C is regarded as a subset of its affine hull. It will be denoted by $\operatorname{ri} C$. A point of a convex set C is called an *extreme point* if it can not be represented as a convex combination of two points of C different from that point. An *exposed point* is an extreme point through which there

*I am grateful to an anonymous referee for helpful comments

ISSN 0944-6532 / $ 2.50

is a supporting hyperplane which contains no other points of C. A *face* of a convex set C is a subset $C' \subset C$ such that every line segment in C with a relative interior point in C' has both end-points in C'. An *extreme ray* is a face which is a closed half-line. The set of all extreme points of C will be denoted by $\operatorname{ext} C$ and the union of all extreme rays by $\operatorname{rext} C$. The *lineality space* of a convex set C is defined as the intersection of asymptotic cones of C and $-C$. Specifically, it is a subspace consisting of vectors x such that $C + \lambda x \subset C$ for all $\lambda \in R$. $\bar{R} = R \cup \{-\infty, \infty\}$ is the extended real line. The graph and the epigraph of a function $f : R^n \to \bar{R}$ are denoted by $\operatorname{gr} f$ and $\operatorname{epi} f$, respectively. Recall that $\operatorname{epi} f = \{(x, a) \in R^n \times R : \ a \geq f(x)\}$.

For a subspace $L \subset R^n$, $L^\perp$ and Pr_L will denote respectively its orthogonal complement and the orthogonal projection operator to L.

We start with a simple example. Consider the set $A = \{(x, y) \in R^2 : \ x = 0, \ 0 \leq y \leq 1 \text{ or } x > 0, \ y = 1\}$. Clearly, $\operatorname{co} A = \{(x, y) \in R^2 : \ x \geq 0, \ 0 < y \leq 1\} \cup \{0\}$, and it is not closed. Notice that $\bar{\operatorname{co}} A$ contains two extreme rays. For one of them $e = \{(x, y) \in R^2 : \ y = 0, \ x \geq 0\}$ we have $e \cap \operatorname{co} A = \{0\}$. The convex hull of the set A', obtained by adding any bounded part of e to A, still will not be closed. It is equally easy to see that the convex hull of any set obtained from A by an addition of any unbounded part of e will be closed.

Actually, this simple observation is crucial. Consider a closed set in R^n, the convex hull of which contains no line. It turns out that this convex hull is closed, if and only if the set itself is coterminal with every extreme ray of its closed convex hull. This result, which is central to the present paper easily implies the theorem characterizing the spannability of convexification of a function of several variables from [5]. Recall that this theorem implies all previously known results on sufficient conditions for spannability. We consider also the case of a set whose convex hull does contain a line.

2. Main Results

We formulate now the central result.

Theorem 2.1. *Let A be a closed set in R^n, the convex hull of which contains no line. Then the convex hull of A is closed, if and only if A is coterminal with every extreme ray of the convex closure of A.*

Some helpful results for the proof of this theorem are in order.

Lemma 2.2. *For an arbitrary set A in R^n, each face F of its convex hull is the convex hull of the intersection $A \cap F$.*

Proof. Let F be a face of the convex hull $C = \operatorname{co} A$. Let H_1 be a hyperplane supporting C at some point of $\operatorname{ri} F$, and let H_1^+ be the closed half-space with the boundary H_1 and containing C. Clearly, a convex combination of points from A, at least one of which belongs to $A \setminus H_1$, is contained in $C \setminus H_1$, and therefore not in $F \subset H_1$. Since $F \subset \operatorname{co} A$, it follows that $C_1 = \operatorname{co}(A \cap H_1) \supset F$. Two cases are possible: (i) $F = C_1$. Then, $A \cap H_1 \subset F \subset H_1$, and therefore $A \cap F = A \cap H_1$, so that $\operatorname{co}(A \cap F) = F$.

(ii) $F \neq C_1$. Since F and C_1 are the faces of C and $F \subset C_1$, then F is the face of C_1. Let H_2 be a hyperplane in H supporting C_1 at some point of $\operatorname{ri} F$. As above, we obtain

$C_2 = \mathrm{co}(A \cap H_2) \supset F$. Again two cases are possible: $F = C_2$ or $F \neq C_2$. Since R^n is finite-dimensional and $\dim C_{k+1} < \dim\ C_k$ for $k = 0, 1, ...$, proceeding in this manner, after a finite number (say m) of steps we will come to the coincidence $F = C_m = \mathrm{co}(A \cap H_m)$. Then $A \cap H_m \subset F \subset H_m$, and therefore $A \cap F = A \cap H_m$, and thus $\mathrm{co}(A \cap F) = F$. □

Proposition 2.3. *Let A be a nonempty convex closed set in R^n and x be an exposed point of A. Let H_x be an arbitrary hyperplane support to A at x, such that $H_x \cap A = \{x\}$. Also, let H_x^δ be a hyperplane lying in the half-space containing A, and parallel to H_x. Then, $A_x^\delta = \Pi_x^\delta \cap A$, where $\Pi_x^\delta = \mathrm{co}(H_x \cup H_x^\delta)$ is bounded and diameter d^δ of A_x^δ tends to zero, when δ tends to zero.*

Proof. Suppose A_x^δ is unbounded. Let e be a ray from the asymptotic cone of A_x^δ. Then, $x + e \subset A_x^\delta$ and clearly, e is parallel to the hyperplane H_x. Hence, $x + e \in H_x$. Since, $x + e \subset A_x^\delta$ it follows that $x + e \subset A_x^\delta \cap H_x \subset A \cap H_x$. This contradicts the assumption $A \cap H_x = \{x\}$.

Next, we show that $d^\delta \to 0$ for $\delta \to 0$. Suppose otherwise. Let δ_k be a decreasing sequence which tends to zero, $\delta_1 < 1$, and $d^{\delta_k} \geq \varepsilon_0 > 0$. Then, there exists $x_k \in A$ $(k \in N)$ such that $x_k \in \Pi_x^{\delta_k}$ $(k \in N)$ and $||x_k - x|| \geq \varepsilon_0$. As proved above, A_x^1 and thus the sequence $\{x_k\}$ is bounded. Without loss of generality, assume that $x_k \to x_0$. Clearly, $x_0 \in A \cap H_x$ and $x_0 \neq x$. This contradicts the assumption $A \cap H_x = \{x\}$. □

The following lemma is a direct consequence of Theorem 3.5 from Klee [6].

Lemma 2.4. *Let A be a nonempty closed set in R^n. Then, every extreme point of the convex closed hull $\overline{\mathrm{co}}A$ belongs to A.*

Proof. Obviously if $\overline{\mathrm{co}}A$ has an extreme point, then it contains no line. Since A is closed, one direction of Theorem 3.5 from Klee [6] gives $A \supset \mathrm{ext}(\overline{\mathrm{co}}A)$. □

The following result is proved by Klee [6, point 2.8]: *Suppose C is a finite-dimensional closed convex set which contains no line and $X \subset C$. Then $\mathrm{co}\, X = C$ if and only if $X \supset \mathrm{ext}\, C$, and X is coterminal with every extreme ray of C.*

We paraphrase this result in the following way.

Lemma 2.5. *For a subset A in R^n, $\mathrm{co}\, A$ is closed if and only if $A \supset \mathrm{ext}(\overline{\mathrm{co}}A)$, and A is coterminal with every extreme ray of $\overline{\mathrm{co}}A$.*

Proof of Theorem 2.1. Since an extreme ray of $\overline{\mathrm{co}}A$ is its face, the necessity of the condition of Theorem 2.1 for the convex hull of A to be closed follows from Lemma 2.2. By Lemma 2.4, every extreme point of $\overline{\mathrm{co}}A$ belongs to A, that is, $\mathrm{ext}(\overline{\mathrm{co}}A) \subset A$. By the assumption, A is coterminal with every extreme ray of $\overline{\mathrm{co}}A$. Then, by Lemma 2.5, $\mathrm{co}\, A = \overline{\mathrm{co}}A$, i.e., the convex hull of A is closed. This proves the theorem. □

Theorem 2.6. *Let A be a set in R^n such that the lineality space L of its convex hull is not trivial. Then, the convex hull of A is closed, if and only if*

(i) *the projection $Pr_{L^\perp}(A)$ is coterminal with every extreme ray of $\overline{\mathrm{co}}A \cap L^\perp$, and*
(ii) *for every extreme point z of $\overline{\mathrm{co}}A \cap L^\perp$, $A \cap (z + L)$ is an embracing subset of $z + L$.*

Proof. Necessity: Let $\operatorname{co} A$ be closed, i.e., $\operatorname{co} A = \overline{\operatorname{co}} A$. Let $z \in \operatorname{ext}(\overline{\operatorname{co}} A \cap L^{\perp})$. Then, since $z + L$ is the face of $\operatorname{co} A = \overline{\operatorname{co}} A$ by Lemma 2.2, $z + L = \operatorname{co}[A \cap (z + L)]$. That is, $A \cap (z + L)$ is an embracing subset of $z + L$.

Let e be an extreme ray of $\operatorname{rext} \overline{\operatorname{co}} A \cap L^{\perp}$. Since e is a face of $\overline{\operatorname{co}} A \cap L^{\perp}$, then $e + L$ is a face of $\overline{\operatorname{co}} A$. By assumption (i) $\overline{\operatorname{co}}\ A = \operatorname{co} A$, $e + L$ is the face of $\operatorname{co} A$. By Lemma 2.2 then, $e + L = \operatorname{co}(A \cap (e + L))$. It follows that $Pr_{L^{\perp}}(A)$ is coterminal with e.

Sufficiency: The following simple fact will be exploited in the sequel: if A and B are subsets of R^n, $B \subset \operatorname{co} A$, then $\operatorname{co}(A \cup B) = \operatorname{co} A$. So, by assumption (ii), we can assume that for each $z \in \operatorname{ext} C$, where $C = \overline{\operatorname{co}} A \cap L^{\perp}$,

$$z + L \subset A. \tag{2.1}$$

Note that set $\operatorname{ext} C$ is nonempty (see Rockafellar [7, Corollary 18.5.3]). Moreover, by Theorem 2.5 from Klee [6],

$$C = \operatorname{co}(\operatorname{ext} C \cup \operatorname{rext} C). \tag{2.2}$$

Since the endpoint of an arbitrary extreme ray e of C is an extreme point of C, it follows easily from (2.1) and the condition (i) of the theorem that,

$$e + L \subset \operatorname{co} A. \tag{2.3}$$

Again by the fact above, (2.3) can be replaced by

$$e + L \subset A. \tag{2.4}$$

Fix $x \in L$, and denote $C_x = \overline{\operatorname{co}} A \cap (x + L^{\perp})$. It follows from (2.1) and (2.4) that

$$\operatorname{ext} C_x \cup \operatorname{rext} C_x \subset A.$$

Then by Theorem 2.5 from Klee [6], $C_x = \operatorname{co}(\operatorname{ext} C_x \cup \operatorname{rext} C_x)$, and hence $C_x \subset \operatorname{co} A$. Since $\overline{\operatorname{co}} A = \cup\{C_x : x \in L\}$, it follows that $\overline{\operatorname{co}}\ A \subset \operatorname{co} A$, i.e., $\operatorname{co} A$ is closed. This completes the proof. □

Remark 2.7. Assumption (i) of Theorem 2.6 is analogous to the "coterminality" assumption in Theorem 2.1, while assumption (ii) requiring that for every extreme point z of $\overline{\operatorname{co}} A \cap L^{\perp}$, $A \cap (z + L)$ be an embracing subset of $z + L$, substitutes, in a sense, for the "closedness" assumption in Theorem 2.1. Note that by Lemma 2.4 the closedness of A implies that every extreme point of its closed convex hull belongs to A. The following simple example shows that assumption (ii) of Theorem 2.6 cannot be replaced by the closedness of A.

Example 2.8. Put $f(x, y) = \max\{\exp(-x^2), |y|\}$ for $(x, y) \in R^2$, and $A = \operatorname{epi} f$. It is easily seen that $\overline{\operatorname{co}} A = \{(x, y, z) \in R^3 : z \geq |y|\}$, with the lineality space $L = \{(x, 0, 0) : x \in R\}$. Clearly $\operatorname{co} A \cap L = \emptyset$. So, $\operatorname{co} A$ is not closed and for an extreme point $z = 0$ of $\overline{\operatorname{co}} A \cap L^{\perp}$, $A \cap (z + L)$ is empty, and therefore it is not an embracing subset of $z + L$.

3. Applications to Spannability

Recall that a function $f : R^n \to \bar{R}$ is said to be spannable if for each point $x \in R^n$ there exist points $x_1, \dots, x_n \in R^n$ and nonnegative numbers $\lambda_1, \dots, \lambda_n$ with $\lambda_1 + \ldots + \lambda_n = 1$ such that

$$x = \sum_{i=1}^{m} \lambda_i x_i \text{ and } f^{**}(x) = \sum_{i=1}^{m} \lambda_i f(x_i),$$

that is, the graph of function f^{**} is contained in the convex hull of the graph of function f.

It is easily seen that the spannability of an affinely bounded from below function f, i.e., the inclusion

$$\operatorname{gr} f^{**} \subset \operatorname{co}(\operatorname{gr} f). \tag{3.1}$$

is equivalent to the inclusion

$$\operatorname{epi} f^{**} \subset \operatorname{co}(\operatorname{epi} f). \tag{3.2}$$

To see that (3.1) implies (3.2), assume (3.1) holds, and let $(\alpha, x) \in \operatorname{epi} f^{**}$. Since $f^{**}(x) \leq \alpha$, where $\alpha \in R$, $f^{**}(x)$ is finite. By assumption (3.1), $(f^{**}(x), x) \in \operatorname{co}(\operatorname{gr} f)$, i.e., $(f^{**}(x), x) = (\sum_{i=1}^{m} \lambda_i f(x_i), \sum_{i=1}^{m} \lambda_i x_i)$ for some $x_i \in R^n$, $\lambda_i > 0$ $(i = 1, ..., m)$ and $\lambda_1 + ... + \lambda_m = 1$. Clearly, $(f(x_1) + \frac{\alpha - f^{**}(x)}{\lambda_1}, x_1) \in \operatorname{epi} f$. Then,

$$(\alpha, x) = (\lambda_1 f_1(x_1) + (\alpha - f^{**}(x)) + \sum_{i=2}^{m} \lambda_i f(x_i), \sum_{i=1}^{m} \lambda_i x_i),$$

and therefore $(\alpha, x) \in \operatorname{co}(\operatorname{epi} f)$. A similar argument shows that inclusion (3.2) implies inclusion (3.1).

The following theorem which gives necessary and sufficient conditions for the spannability of function f such that $\operatorname{epi} f^{**}$ contains no line, easily follows from Theorem 2.1. Recall that this theorem implies all known results on sufficient conditions for the spannability (see [5]).

Theorem 3.1. *Let $f : R^n \to \bar{R}$ be a lower semi-continuous function such that* $\operatorname{epi} f^{**}$ *contains no line. Then f is spannable, if and only if its graph is coterminal with every nonvertical extreme ray of the epigraph of f^{**}.*

Proof. Let function f be spannable, i.e. $\operatorname{gr} f^{**} \subset \operatorname{co}(\operatorname{gr} f)$, or equivalently $\operatorname{co}(\operatorname{epi} f) = \operatorname{epi} f^{**}$. Since f is lower semi-continuous, $\operatorname{epi} f$ is closed and by the assumption $\operatorname{co}(\operatorname{epi} f)$ contains no line. Then, by Theorem 2.1, $\operatorname{epi} f$ is coterminal with every extremal ray of $\operatorname{epi} f^{**}$. Clearly, for every nonvertical extreme ray e of $\operatorname{epi} f^{**}$, $(\operatorname{epi} f) \cap e = (\operatorname{gr} f) \cap e$. Hence, $\operatorname{gr} f$ is coterminal with every extreme ray of $\operatorname{epi} f^{**}$. We now show that if a function f satisfies the assumptions of the theorem, then it is spannable. Since $\operatorname{epi} f$ is closed and $\overline{\operatorname{co}}(\operatorname{epi} f)$ contains no line, by Theorem 2.1, $\operatorname{epi} f$ is coterminal with every extreme ray of $\overline{\operatorname{co}}(\operatorname{epi} f) = \operatorname{epi} f^{**}$. But as noted above, for nonvertical extreme ray e of $\operatorname{epi} f^{**}$, $(\operatorname{epi} f) \cap e = (\operatorname{gr} f) \cap e$. So, $\operatorname{gr} f$ is coterminal with every nonvertical extreme ray of $\operatorname{epi} f^{**}$. This completes the proof. □

Theorem 3.2. *Let $f : R^n \to \bar{R}$ be an affinely bounded from below function with the nontrivial lineality space L of epif^{**}. Then, f is spannable if and only if the projection $Pr_{L^\perp}(\mathrm{gr}\, f)$ is coterminal with every nonvertical extreme ray of epi$f^{**} \cap L^\perp$, and its graph* $\mathrm{gr}\, f$ *contains an embracing subset of $z+L$ for every extreme point $\bar{z}$ of the set* $(\mathrm{epi}\, f^{**}) \cap L^\perp$.

Proof. Necessity: Let f be spannable, affinely bounded from below, and suppose epi f^{**} has nontrivial lineality space L. Then,

$$\bar{\mathrm{co}}(\mathrm{epi}\, f) = \mathrm{epi}\, f^{**} \subset \mathrm{co}(\mathrm{epi}\, f)$$

i.e., co(epi f) is closed. By Theorem 2.6, $Pr_{L^\perp}(\mathrm{epi}\, f)$ is coterminal with every extreme ray of epi $f^{**} \cap L^\perp$, and for every extreme point z of epi $f^{**} \cap L^\perp$, epi $f \cap (z + L)$ is an embracing subset of $z + L$. As we note in the proof of Theorem 3.1, for every function g with epi g^{**} containing no line, $(\mathrm{epi}\, g) \cap e = (\mathrm{gr}\, g) \cap e$ for every nonvertical extreme ray e of epi g^{**}. Similarly, in this case $(Pr_{L^\perp}(\mathrm{epi}\, f)) \cap e = (Pr_{L^\perp}(\mathrm{gr}\, f)) \cap e$ for every extreme ray e of $(\mathrm{epi}\, f^{**}) \cap L^\perp$. So, it follows that $Pr_{L^\perp}(\mathrm{gr}\, f)$ is coterminal with every extreme ray of $(\mathrm{epi}\, f^{**}) \cap L^\perp$. Since for an extreme point z of $(\mathrm{epi}\, f^{**}) \cap L^\perp$, $z + L \subset \mathrm{gr}\, f^{**}$ and $f^{**} \leq f$, it follows that $(\mathrm{gr}\, f) \cap (z + L) = (\mathrm{epi}\, f) \cap (z + L)$, so $(\mathrm{gr}\, f) \cap (z + L)$ is an embracing subset of $z + L$.

Sufficiency of the assumptions (i) and (ii) for spannability of function f is a direct consequence of the inclusion gr $f \subset$ epi f and Theorem 2.6. This completes the proof. □

The relationship between Theorem 3.1 and Theorem 3.2 is quite similar to that between Theorem 2.1 and Theorem 2.6. The remark following Theorem 2.6 clarifies the latter relationship. The convexification of the function given in this remark is $f^{**}(x, y) = |y|$, and by the arguments in the remark, f is not spannable.

In conclusion we bring two very simple examples of spannable functions illustrating theorems 3.1 and 3.2 which are not encompassed by previous sufficiency results on spannability.

Example 3.3. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} -[x] & \text{for } x \geq 0, \\ -[2x] & \text{for } x < 0, \end{cases}$$

where $[x]$ denotes the integer part of x. Clearly

$$f^{**}(x) = \begin{cases} -x & \text{for } x \geq 0, \\ -2x & \text{for } x < 0, \end{cases}$$

and f is spannable.

Example 3.4. Let $f : R^2 \to R$ be equal to zero over the sequences $(k, 0)$ and $(0, k)$ for $k = \pm 1, \pm 2, ...,$ and positive elsewhere. Clearly $f^{**}(x) \equiv 0$ and f is spannable.

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