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BV Functions with Respect to a Measure and Relaxation of Metric Integral Functionals

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We introduce and study the space of bounded variation functions with respect to a Radon measure μ on $\mathbb{R}^N$ and to a metric integrand φ on the tangent bundle to μ. We show that it is equivalent to view such space as the class of μ-integrable functions for which a distributional notion of (μ,φ)-total variation is finite, or as the finiteness domain of a relaxed functional. We prove a quite general coarea-type formula and then we focus our attention to the problem of finding an integral representation for the (μ,φ)-total variation.

Keywords: Bounded variation functions, Radon measures, Relaxation, Duality, Integral representation

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1. Introduction

In this paper we define the space of bounded variation functions with respect to a Radon measure μ, and we study some of its properties. Our approach is inspired by [5], where the Sobolev-type spaces $W^{1,p}_\mu$ associated with a measure μ are introduced, and the relaxation of integral functionals on $W^{1,p}_\mu$ is studied for $p > 1$. We focus our attention on the relaxation in L^1_μ of integral functionals with respect to μ, where the integrand is a sublinear function φ. Following the geometric approach proposed in [1], where $\mathbb{R}^N$ is viewed as a Banach space endowed with a Finsler metric, one can look at the integrand φ as a metric: this leads to give, for any $u \in L^1_\mu(\mathbb{R}^N)$, a natural distributional definition of (μ,φ)-total variation $|D_\mu u|_\varphi$. The consistency of our definition with the usual notion of $|Du|$ when μ is the Lebesgue measure and φ is the euclidean metric is proved in Section 3 (see Proposition 3.1). We next introduce the space $BV_{\mu,\varphi}$ as the class of all functions $u \in L^1_\mu(\mathbb{R}^N)$ with $|D_\mu u|_\varphi < +\infty$.

In Section 4 we enlighten the interest of our (μ,φ)-total variation by showing a quite ge-

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neral coarea formula in L^1_μ. When applied to particular measures μ, this formula clearly encompasses previous generalizations (see [13], [3], [15], [21]). We stress that our proof technique, suggested by the use of distributional definitions, is different from the classical methods based on the approximation with smooth functions [12]; namely we use a commutation argument from [7] between supremum and integral. As a corollary, we show that the chain rule holds for functions in the Sobolev space $W^{1,1}_\mu$.

In Section 5 we show that $|D_\mu u|_\varphi$ and $BV_{\mu,\varphi}$ coincide respectively with the relaxation on L^1_μ of an integral functional, and with its domain of finiteness (see Theorem 5.1); furthermore, under suitable regularity assumptions, we prove an integral representation theorem for $|D_\mu u|_\varphi$ on the Sobolev space $W^{1,1}_\mu$ (which is a strict subspace of $BV_{\mu,\varphi}$).

The problem of extending this type of result out of $W^{1,1}_\mu$ is quite delicate (see [6] for the integral representation in the classical space $BV(\Omega)$), and is studied in Section 6. Here we provide some sufficient conditions in order to have an integral representation for $|D_\mu u|_\varphi$ on $BV_{\mu,\varphi}$; we also give a counterexample showing that such a representation does not hold for a general measure μ.

2. Notation

For a positive integer d, let $\mathcal{R}^d$ be the class of all $\mathbb{R}^d$-valued Borel measures with finite total variation on $\mathbb{R}^N$; when $d = 1$, we simply denote by $\mathcal{R}$ the space of signed Borel measures with finite total variation on $\mathbb{R}^N$, and we let $\mathcal{R}_+$ be the subclass of $\mathcal{R}$ given by all positive and finite Borel measures.

For $\mu \in \mathcal{R}_+$ we denote by $\mu \llcorner E$ the restriction measure of μ to a μ-measurable subset E of $\mathbb{R}^N$, and by $\operatorname{spt}\mu$ the support of μ. Whenever dealing with integrals with respect to μ on $\mathbb{R}^N$, we omit the integration domain. For any $k \in [0, N)$, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure and $\mathcal{L}^N$ is the Lebesgue measure. If α is a vector-valued measure with finite total variation, the polar decomposition of α is given by $\alpha = \theta|\alpha|$, where $|\alpha|$ denotes the positive total variation measure of α, and the density $\theta := \frac{d\alpha}{d|\alpha|}$ has unitary norm $|\alpha|$-a.e. ; by writing $\alpha << \mu$, we mean that $|\alpha|$ is absolutely continuous with respect to μ. For $p = 1$ or $p = +\infty$, we set $L^p_\mu := L^p(\mathbb{R}^N, d\mu)$ and $(L^p_\mu)^N := (L^p(\mathbb{R}^N, d\mu))^N$; the subscript μ is omitted when $\mu = \mathcal{L}^N$. The spaces of continuous functions, of continuous vector fields, and of continuous vector fields vanishing at infinity on $\mathbb{R}^N$ are denoted respectively by $\mathcal{C}$, $\mathcal{C}^N$, $\mathcal{C}^N_0$. Unless otherwise specified, the symbol of duality $\langle \cdot, \cdot \rangle$ is used for the pairing between L^∞_μ and L^1_μ, while the euclidean norm and scalar product between two vectors z and z' of $\mathbb{R}^N$ are denoted by $|\cdot|$ and $z \cdot z'$. For a subset E of $\mathbb{R}^N$ we denote by χ_E the characteristic function of E.

We set

$$X_\mu := \{\sigma \in (L^\infty_\mu)^N \ : \ \operatorname{div}(\sigma\mu) \in L^\infty_\mu\} \ ; \tag{2.1}$$

in (2.1) we call $\operatorname{div}(\sigma\mu)$ the distribution whose action on a test function $\psi \in \mathcal{D} := \mathcal{C}^\infty_c(\mathbb{R}^N)$ is given by

$$\langle \operatorname{div}(\sigma\mu), \psi \rangle_{(\mathcal{D}',\mathcal{D})} := -\int \sigma \cdot \nabla\psi \, d\mu \ . \tag{2.2}$$

In other words, an element σ of $(L^\infty_\mu)^N$ belongs to X_μ if and only if there exists a constant $C \in [0, +\infty)$ such that $|\int \sigma \cdot \nabla\psi| \, d\mu \ \leq \ C \, \|\psi\|_{L^1_\mu}$ for every $\psi \in \mathcal{D}$. The explicit expression of $\mathrm{div}(\sigma\mu)$ for a special choice of μ is given in (3.5).

Similarly to [5, Section 2], we define the tangent space to μ at a point $x \in \mathbb{R}^N$ as

$$T_\mu(x) := \mu - \mathrm{ess} \bigcup \{\sigma(x) \ : \ \sigma \in X_\mu\} \ ;$$

in particular, for any $\sigma \in X_\mu$ we have $\sigma(x) \in T_\mu(x)$ for μ-a.e. x.

We always assume that $T_\mu(x)$ is not reduced to zero μ-a.e. , and we set

$$T\mu := \{(x, z) \ : \ x \in \mathbb{R}^N \ , \ z \in T_\mu(x)\} \ .$$

When $\mu = \mathcal{L}^N \llcorner \Omega$, where Ω is an open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, we have $T_\mu(x) = \mathbb{R}^N$ for μ-a.e. x, while for $\mu = \mathcal{H}^k \llcorner M$, where M is a Lipschitz k-manifold, T_μ coincides μ-a.e. with the usual tangent space to M. For further properties of T_μ we refer to [14].

For $\psi \in \mathcal{D}$ and μ-a.e. $x \in \mathbb{R}^N$, the symbol $\nabla_\mu\psi(x)$ stands for $P_\mu(x)[\nabla\psi(x)]$, where $P_\mu(x)[\,\cdot\,]$ is the orthogonal projection of $\mathbb{R}^N$ onto $T_\mu(x)$.

A Sobolev-type space $W^{1,1}_\mu$ can be defined, following [5], as the completion of $\mathcal{D}$ with respect to the norm $\|\psi\|_{L^1_\mu} + \|\nabla_\mu\psi\|_{(L^1_\mu)^N}$. Thus any function $u \in W^{1,1}_\mu$ admits a *tangential gradient* $\nabla_\mu u \in (L^1_\mu)^N$, and the following integration by parts formula holds:

$$-\langle \mathrm{div}(\sigma\mu), u\rangle = \int \sigma \cdot \nabla_\mu u \, d\mu \ , \qquad \sigma \in X_\mu \ , \ u \in W^{1,1}_\mu \ . \tag{2.3}$$

For the proof of (2.3), which is the same as in the case $p > 1$, we refer to [5].

If $(\eta_i)_{i\in I}$ is an arbitrary family of μ-measurable closed-valued multifunctions from $\mathbb{R}^N$ to $\mathbb{R}^N$, we denote by $\mu - \mathrm{ess}\sup_{i\in I} \eta_i$ the multifunction η from $\mathbb{R}^N$ to $\mathbb{R}^N$ characterized (up to μ-negligible sets) by the two properties (see [23]):

(i) $\eta_i(x) \subseteq \eta(x)$ for μ-a.e. x , $\forall i \in I$;

(ii) η is minimal with respecto to inclusion μ-a.e. (i.e. for any other μ-measurable and closed-valued multifunction θ such that, for every $i \in I$, $\eta_i(x) \subseteq \theta(x)$ for μ-a.e. x, there holds $\eta(x) \subseteq \theta(x)$ for μ-a.e. x).

We say that a function $\varphi : T\mu \to [0, +\infty)$ is a metric integrand, and we write $\varphi \in \mathcal{M}$, if the following conditions are satisfied:

$$\text{for any } z \in \mathbb{R}^N \ , \text{ the map } x \mapsto \varphi(x, P_\mu(x)[z]) \text{ is } \mu\text{-measurable on } \mathbb{R}^N \ ; \tag{2.4}$$

$$\text{for } \mu\text{-a.e. } x \in \mathbb{R}^N \ , \text{ the map } z \mapsto \varphi(x, z) \text{ is convex on } T_\mu(x) \ ; \tag{2.5}$$

$$\text{there exists } C > 0 \text{ such that } \varphi(x, z) \leq C|z| \text{ for } (x, z) \in T\mu \ ; \tag{2.6}$$

$$\varphi(x, tz) = t\varphi(x, z) \text{ for } (x, z) \in T\mu \, , \ t > 0 \ . \tag{2.7}$$

We associate with φ the dual metric φ^o defined on the dual bundle $T^*\mu$ of $T\mu$ as

$$\varphi^o(x, z^*) := \sup\{z \cdot z^* \ : \ z \in T_\mu(x) \, , \ \varphi(x, z) \leq 1\} \ .$$

Notice that $z \cdot z^* \leq \varphi(x,z)\,\varphi^o(x,z^*)$ for any $z \in T_\mu(x)$, $z^* \in T^*_\mu(x)$, and that $\varphi^{oo} = \varphi$.

Furthermore, the metric φ^o still enjoys properties (2.4), (2.5), and (2.7) on $T^*\mu$ (see for instance [10]).

Finally, we recall that, if J is a proper functional defined on a Banach space X with values in $\mathbb{R} \cup \{+\infty\}$, the relaxed functional $\overline{J}$ of J is defined as the greatest lower semicontinuous functional less than or equal to J on X (see [9]), while the Fenchel conjugate functional J^* of J is defined on the dual space of X (see [19]) by $J^*(x^*) := \sup\{\langle x\,,\, x^*\rangle - J(x) \;:\; x \in X\}$.

3. The (μ, φ)-total variation

Some generalized notions of bounded variation functions have been proposed in the literature: for instance the theory of perimeters on a smooth manifold has been studied in [17], while more recently the class of BV functions over rectifiable currents has been introduced in [3]. Our definition reads as follows.

Let $\mu \in \mathcal{R}_+$ and $\varphi \in \mathcal{M}$. For every function $u \in L^1_\mu$ we define the (μ,φ)-total variation $|D_\mu u|_\varphi$ of u as

$$|D_\mu u|_\varphi := \sup\{-\langle \operatorname{div}(\sigma\mu), u\rangle \;: \sigma \in X_\mu\,,\ \varphi^o(x,\sigma(x)) \leq 1 \text{ for } \mu\text{-a.e. } x\} \tag{3.1}$$

and we set

$$BV_{\mu,\varphi} := \{u \in L^1_\mu \;:\; |D_\mu u|_\varphi < +\infty\}\,.$$

In particular, if E is a μ-measurable subset of $\mathbb{R}^N$ such that $\chi_E \in L^1_\mu$, we define the (μ,φ)-perimeter $P_{\mu,\varphi}(E)$ of E as $P_{\mu,\varphi}(E) := |D_\mu \chi_E|_\varphi$.

It immediately follows that the functionals $u \mapsto |D_\mu u|_\varphi$ and $E \mapsto P_{\mu,\varphi}(E)$ are lower semicontinuous with respect to the convergence in L^1_μ. Notice also that, whenever φ satisfies a coercivity-type condition of the form

$$\varphi(x,z) \geq C^{-1}|z|\,, \qquad (x,z) \in T\mu\,, \tag{3.2}$$

the space $BV_{\mu,\varphi}$ is independent of φ; in particular, when $\varphi(x,z) = |z|$, $BV_{\mu,\varphi}$ will be denoted by BV_μ.

Let us now show that, if $\mu = \mathcal{L}^N \llcorner \Omega$ and $\varphi(x,z) = |z|$, then $|D_\mu u|_\varphi$ and $BV_{\mu,\varphi}$ reduce to the usual notions $|Du|(\Omega)$ and $BV(\Omega)$, respectively.

To this aim, we recall (see [2], Theorem 1.9 with $p = N$ and $q = N/(N-1)$) that, if $\Omega \subseteq \mathbb{R}^N$ is a bounded open set with Lipschitz boundary, $u \in BV(\Omega)$, and $\sigma \in (L^\infty(\Omega))^N$ with $\operatorname{div}\sigma \in L^N(\Omega)$, then it is possible to define, in a natural way, a real valued measure (σ, Du) satisfying $\int_\Omega (\sigma, Du) \leq \|\sigma\|_{L^\infty(\Omega)}|Du|(\Omega)$, and a trace $[\sigma\cdot\nu] \in L^\infty(\partial\Omega)$ on $\partial\Omega$ of the normal component of σ, such that the following Gauss-Green formula holds:

$$\int_\Omega u \operatorname{div}\sigma\, dx + \int_\Omega (\sigma, Du) = \int_{\partial\Omega} [\sigma\cdot\nu] u \, d\mathcal{H}^{N-1} \tag{3.3}$$

(here ν is the outward unit normal to $\partial\Omega$).

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with Lipschitz boundary, let $\mu := \mathcal{L}^N \llcorner \Omega$ and $\varphi(x,z) := |z|$. Then $|D_\mu u|_\varphi = |Du|(\Omega)$ for every $u \in L^1_\mu = L^1(\Omega)$, hence $BV_{\mu,\varphi} = BV(\Omega)$.*

Proof. Given $\sigma \in (L^\infty(\Omega))^N$ with $\operatorname{div} \sigma \in L^\infty(\Omega)$, and $\psi \in \mathcal{D}$, (3.3) yields:

$$\langle \operatorname{div}(\sigma\mu), \psi \rangle_{(\mathcal{D}',\mathcal{D})} = -\int_\Omega \sigma \cdot \nabla\psi \, dx = \int_\Omega \psi \operatorname{div} \sigma \, dx - \int_{\partial\Omega} [\sigma \cdot \nu] \psi \, d\mathcal{H}^{N-1} .$$

In order to have $\sigma \in X_\mu$, the right hand side member needs to be controlled by the norm of ψ in $L^1(\Omega)$. This yields $[\sigma \cdot \nu] = 0$ $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$. Therefore

$$X_\mu = \left\{ \sigma \in (L^\infty(\Omega))^N \ : \ \operatorname{div} \sigma \in L^\infty(\Omega) \ , \ [\sigma \cdot \nu] = 0 \ \ \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega \right\} . \tag{3.4}$$

The inclusion $(\mathcal{C}^1_c(\Omega))^N \subseteq X_\mu$ and the fact that $\varphi^o(x,z) = |z|$ imply that $|D_\mu u|_\varphi \geq |Du|(\Omega)$, hence $BV_{\mu,\varphi} \subseteq BV(\Omega)$. It remains to prove that $|D_\mu u|_\varphi \leq |Du|(\Omega)$. For any $u \in BV(\Omega)$ and $\sigma \in X_\mu$, by (3.3) and (3.4) we have

$$-\langle \operatorname{div}(\sigma\mu), u \rangle = -\int_\Omega u \operatorname{div} \sigma \, dx = \int_\Omega (\sigma, Du) .$$

Hence, if $\|\sigma\|_{(L^\infty(\Omega))^N} \leq 1$, we get $-\langle \operatorname{div}(\sigma\mu), u \rangle \leq |Du|(\Omega)$. Passing to the supremum over σ, we deduce $|D_\mu u|_\varphi \leq |Du|(\Omega)$. As a consequence, $BV_{\mu,\varphi} = BV(\Omega)$. □

Example 3.2. A natural case to be considered is when $\mu = a\mathcal{H}^k \llcorner M$, where M is a smooth connected k-manifold and φ is a continuous metric integrand on the tangent bundle to M. If the density a is a positive function in $L^\infty_\mu \cap \mathcal{C}^1$ with $\nabla \log a \in (L^\infty_\mu)^N$, and σ is a $\mathcal{C}^1$ tangent field to M, one can easily check, using the divergence theorem on a smooth manifold (see for instance [22]), that $\sigma \in X_\mu$, as

$$\operatorname{div}(\sigma\mu) = (\operatorname{div}_\mu \sigma + \sigma \cdot \nabla \log a)\mu \tag{3.5}$$

where $\operatorname{div}_\mu \sigma := \sum_{i=1}^N (\nabla_\mu \sigma^i)_i$. Moreover, essentially due to the smoothness of $\operatorname{spt} \mu$, we will show in Section 6 that, as in the classical case $\mu = \mathcal{L}^N \llcorner \Omega$, there exists a vector valued measure $D_\mu u$ which allows to give an integral representation of the (μ,φ)-total variation on the whole space $BV_{\mu,\varphi}$.

Example 3.3. We point out that the choice

$$\mu := \left[\pi^{-\frac{N}{2}} (\lambda_1 \cdot \ldots \cdot \lambda_N)^{-\frac{1}{2}} \exp\left(-\sum_{i=1}^N x_i^2/\lambda_i \right) \right] \mathcal{L}^N , \quad \varphi(z) := \left(\sum_{i=1}^N \eta_i z_i^2 \right)^{1/2} ,$$

where $\lambda_i, \eta_i, i = 1, \ldots, N$, are suitable positive real numbers, could relate, when $N \to +\infty$, the space $BV_{\mu,\varphi}$to a theory of perimeters in infinite-dimensional Hilbert spaces (see [11] and [16]).

4. Coarea formula

Several generalizations of the classical coarea–formula for BV functions [12, Section 5.5] have been proposed in the literature (see for instance [13, 3.2.22], [3, Proposition 2.13], [15, Theorem 2.3.5], [21, Theorem 17.1]). We now present a coarea-type formula, which holds on L^1_μ for any $\mu \in \mathcal{R}_+$ and any $\varphi \in \mathcal{M}$, provided one adopts the distributional definition of (μ,φ)-total variation introduced in the previous section. As a consequence we obtain the stability of $BV_{\mu,\varphi}$ under composition (see Corollary 4.2 below). We let $\{u > t\} := \{x \in \mathbb{R}^N \ : u(x) > t\}$.

Theorem 4.1. *Let $\mu \in \mathcal{R}_+$ and $\varphi \in \mathcal{M}$. Then for any $u \in L^1_\mu$, the map $t \mapsto P_{\mu,\varphi}(\{u > t\})$ is Lebesgue-measurable and the following coarea-type formula holds:*

$$|D_\mu u|_\varphi = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ . \tag{4.1}$$

Proof. Define

$$\mathcal{K} := \{\sigma \in X_\mu \ : \ \varphi^o(x, \sigma(x)) \leq 1 \ \text{ for } \mu\text{-a.e. } x\} \ . \tag{4.2}$$

Let us fix $u \in L^1_\mu$ and define for every $\sigma \in \mathcal{K}$

$$f_\sigma(t) \ := \ -\langle \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle \ , \qquad t \in \mathbb{R} \ .$$

The function $t \mapsto \mu\big(\{u > t\}\big)$ is bounded and non-increasing on $\mathbb{R}$, hence continuous at all $t \in \mathbb{R} \setminus D$, with D at most countable. Then f_σ is also continuous on $\mathbb{R} \setminus D$ for any $\sigma \in X_\mu$. Applying the Lindelöf's Theorem to the family of continuous functions f_σ on $\mathbb{R} \setminus D$, there exists a countable sequence $\{\sigma_n\} \subset \mathcal{K}$ such that

$$\sup\{f_\sigma(t) : \sigma \in \mathcal{K}\} = \mathcal{L}^1 - \text{ess sup}\{f_\sigma(t) : \sigma \in \mathcal{K}\} = \sup_n f_{\sigma_n}(t) \ , \qquad \forall t \in \mathbb{R} \setminus D \ . \tag{4.3}$$

Therefore

$$P_{\mu,\varphi}(\{u > t\}) = \sup_n f_{\sigma_n}(t) \ , \qquad \mathcal{L}^1\text{- a.e. } \ t \in \mathbb{R} \ , \tag{4.4}$$

which entails the measurability statement of the theorem.

Notice now that, for any $\sigma \in X_\mu$, there holds

$$\begin{aligned} -\langle u, \operatorname{div}(\sigma\mu)\rangle &= -\langle u^+, \operatorname{div}(\sigma\mu)\rangle + \langle u^-, \operatorname{div}(\sigma\mu)\rangle \\ &= -\int_0^{+\infty} \langle \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle\, dt + \int_{-\infty}^0 \langle 1 - \chi_{\{u>t\}}, \operatorname{div}(\sigma\mu)\rangle\, dt \\ &= \int_{\mathbb{R}} f_\sigma(t)\, dt \ , \end{aligned} \tag{4.5}$$

where we used Fubini's theorem and the fact that $\langle 1, \operatorname{div}(\sigma\mu)\rangle = 0$ for any $\sigma \in X_\mu$.

If we pass to the supremum over $\sigma \in \mathcal{K}$ in (4.5), using (4.3) and (4.4), we get

$$|D_\mu u|_\varphi \leq \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ , \qquad u \in L^1_\mu \ .$$

It remains to prove the difficult part of (4.1), namely

$$|D_\mu u|_\varphi \geq \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ , \qquad u \in L^1_\mu \ . \tag{4.6}$$

First, we observe that to prove (4.6) it is enough to verify that

$$\int \varphi(x, \nabla_\mu u)\, d\mu \geq \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt \ , \qquad u \in \mathcal{D} \ . \tag{4.7}$$

Indeed, by Theorem 5.1 below, given $u \in L^1_\mu$ there exists a sequence $\{u_n\} \subseteq \mathcal{D}$ converging to u in L^1_μ such that

$$|D_\mu u|_\varphi = \lim_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu \ .$$

Possibly passing to a subsequence (still denoted by $\{u_n\}$), we have $\chi_{\{u_n>t\}} \to \chi_{\{u>t\}}$ in L^1_μ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Then applying (4.7) to every u_n and passing to the limit as $n \to +\infty$, by the L^1_μ-lower semicontinuity of the (μ, φ)-perimeter and Fatou's Lemma we have

$$\begin{aligned}\int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})\, dt &\le \int_{\mathbb{R}} \liminf_{n\to+\infty} P_{\mu,\varphi}(\{u_n > t\})\, dt \le \liminf_{n\to+\infty} \int_{\mathbb{R}} P_{\mu,\varphi}(\{u_n > t\})\, dt \\ &\le \lim_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu = |D_\mu u|_\varphi \ ,\end{aligned}$$

that is (4.6).

We are thus reduced to prove (4.7). Let $u \in \mathcal{D}$. We introduce the following subset $\mathcal{C}(u)$ of $L^1(\mathbb{R})$:

$$\mathcal{C}(u) := \left\{ \sum_{i=1}^m \alpha_i\, f_{\sigma_i} \ , \ \sigma_i \in \mathcal{K}\, , \ \alpha_i \in \mathcal{D}(\mathbb{R}; [0,1])\, , \ \sum_{i=1}^m \alpha_i = 1\, , \ m \in \mathbb{N} \right\} \ .$$

We claim that

$$\int \varphi(x, \nabla_\mu u)\, d\mu \ge \sup_{g\in\mathcal{C}(u)} \int_{\mathbb{R}} g\, dt \ . \tag{4.8}$$

Indeed, let $g = \sum_{i=1}^m \alpha_i f_{\sigma_i} \in \mathcal{C}(u)$, where $\sigma_i \in \mathcal{K}$, $\alpha_i \in \mathcal{D}(\mathbb{R}; [0,1])$, and $\sum_{i=1}^m \alpha_i = 1$. Define $A_i(t) := \int_{-\infty}^t \alpha_i(s)\, ds$ for $t \in \mathbb{R}$, $i = 1, \ldots, m$. Then we have the chain rule identity $\nabla_\mu(A_i \circ u) = \alpha_i(u) \nabla_\mu u$. Using Fubini's theorem and noticing that $\varphi(x, \nabla_\mu u(x)) \ge \nabla_\mu u(x) \cdot \sigma(x)$ for μ-a.e. x whenever σ belongs to $\mathcal{K}$, we get

$$\begin{aligned}\int_{\mathbb{R}} g(t)\, dt &= \int_{\mathbb{R}} -\langle \chi_{\{u>t\}}, \sum_{i=1}^m \alpha_i(t)\, \mathrm{div}(\sigma_i \mu)\rangle\, dt = \sum_{i=1}^m -\langle A_i \circ u, \mathrm{div}(\sigma_i\mu)\rangle \\ &= \sum_{i=1}^m \int \nabla_\mu (A_i \circ u) \cdot \sigma_i\, d\mu = \sum_{i=1}^m \int \alpha_i(u)\, \nabla_\mu u \cdot \sigma_i\, d\mu \\ &\le \sum_{i=1}^m \int \alpha_i(u)\, \varphi(x, \nabla_\mu u)\, d\mu = \int \varphi(x, \nabla_\mu u)\, d\mu \ ,\end{aligned}$$

which proves the claim (4.8).

By construction, the family $\mathcal{C}(u)$ enjoys the following property (stability by partitions of unity): for $l \in \mathbb{N}$, if $g_1, \ldots, g_l$ belong to $\mathcal{C}(u)$ and $\beta_1, \ldots, \beta_l$ are functions in $\mathcal{D}(\mathbb{R}; [0,1])$ with $\sum_{j=1}^l \beta_j = 1$, then there holds $\sum_{j=1}^l \beta_j g_j \in \mathcal{C}(u)$. This enables us to apply an argument about commutation between supremum and integral (see [7, Theorem 1]) which entails

$$\sup_{g\in\mathcal{C}(u)} \int_{\mathbb{R}} g\, dt \ = \ \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\,sup}_{g\in\mathcal{C}(u)} g \right)\, dt \ .$$

Since $\mathcal{C}(u)$ contains $\{f_\sigma : \sigma \in \mathcal{K}\}$, by (4.8) and (4.4) we deduce

$$\begin{aligned}\int \varphi(x, \nabla_\mu u)\, d\mu &\geq \sup_{g\in\mathcal{C}(u)} \int_{\mathbb{R}} g\, dt = \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\ sup}_{g\in\mathcal{C}(u)} g\right) dt \\ &\geq \int_{\mathbb{R}} \left(\mathcal{L}^1 - \operatorname*{ess\ sup}_{\sigma\in\mathcal{K}} f_\sigma\right) dt\ .\end{aligned} \tag{4.9}$$

Finally, one can check directly by the definition of μ-essential supremum that the choice of the family $\{\sigma_n\} \subset \mathcal{K}$ in (4.3) entails

$$P_{\mu,\varphi}(\{u > t\}) = \mathcal{L}^1 - \operatorname*{ess\ sup}_{\sigma\in\mathcal{K}} f_\sigma\ . \tag{4.10}$$

From (4.9) and (4.10) we get (4.7), and this completes the proof of (4.1). □

Corollary 4.2. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then for every $u \in BV_{\mu,\varphi}$, it holds $A \circ u \in BV_{\mu,\varphi}$. Moreover if A is monotone non decreasing, we can write*

$$|D(A\circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} a(t)\, P_{\mu,\varphi}(\{u > t\})\, dt\ , \tag{4.11}$$

where $a(t)$ denotes the a.e. defined derivative of A.

Proof. Let us assume first that A is monotone non decreasing. Set $B(t) := \sup\{s : A(s) < t\}$; then $B(t)$ is a non decreasing function in $BV_{loc}(\mathbb{R})$ such that $B \circ A(t) = t$ and $\{A \circ u > t\} = \{u > B(t)\}$ for a.e. t. Applying (4.1) to $A(u)$, we deduce

$$|D(A\circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > B(t)\})\, dt\ .$$

We deduce (4.11) by noticing that the functions $f(s) := \int_{-\infty}^{A(s)} P_{\mu,\varphi}(\{u > B(t)\})\, dt$ and $g(s) := \int_{-\infty}^{s} P_{\mu,\varphi}(\{u > t\})a(t)\, dt$ are absolutely continuous, non decreasing and have a.e. the same derivative. Therefore they coincide and have the same limit as $s \to +\infty$ that is

$$\int_{\mathbb{R}} P_{\mu,\varphi}(\{u > B(t)\})\, dt \ = \int_{\mathbb{R}} P_{\mu,\varphi}(\{u > t\})a(t)\, dt$$

Since $a(t)$ is bounded, it follows that for $u \in BV_{\mu,\varphi}$, $|D(A \circ u)|_{\mu,\varphi} < +\infty$ and so $A \circ u$ belongs to $BV_{\mu,\varphi}$. This implication can be extended to the general case, by writting $A \circ u = (A_+ \circ u) - (A_- \circ u)$ where A_+ and A_- denote primitives of the positive and of the negative parts of $a(t)$. □

Remark 4.3. In light of Section 5, the left hand side of (4.11) can be written in an integral form with respect to μ when $u \in W^{1,1}_\mu$. Indeed the chain rule $\nabla_\mu(A\circ u) = a(u)\nabla_\mu u$ applies, see Theorem 5.4. Thus, due to Theorem 5.3, when μ and φ satisfy suitable regularity assumptions (see Theorem 5.7), (4.11) becomes:

$$\int \varphi(x, \nabla_\mu u)\, a(u)\, d\mu = \int_{\mathbb{R}} a(t)\, P_{\mu,\varphi}(\{u > t\})\, dt\ .$$

5. Relaxation on $BV_{\mu,\varphi}$ and integral representation on $W^{1,1}_\mu$

Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let $J : L^1_\mu \to [0, +\infty]$ be the functional defined by

$$J(u) := \begin{cases} \int \varphi(x, \nabla_\mu u)\, d\mu & \text{if } u \in \mathcal{D} \\ +\infty & \text{if } u \in L^1_\mu \setminus \mathcal{D} . \end{cases} \tag{5.1}$$

In this section, we consider the relaxed functional of J defined by

$$\overline{J}(u) = \inf \left\{ \liminf_{n\to+\infty} J(u_n) \ : \ u_n \to u \ \text{ in } L^1_\mu \right\} .$$

The next theorem relates $\overline{J}$ to the distributional notion (3.1) of (μ, φ)-total variation.

Theorem 5.1. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let J be defined as in (5.1). Then for every $u \in L^1_\mu$ we have $\overline{J}(u) = |D_\mu u|_\varphi$; in particular $BV_{\mu,\varphi}$ is the finiteness domain of $\overline{J}$.*

Proof. It is useful to extend φ to a metric integrand Φ defined on $\mathbb{R}^N \times \mathbb{R}^N$ by setting

$$\Phi(x, z) := \varphi(x, P_\mu(x)[z]) . \tag{5.2}$$

For any $z \in \mathbb{R}^N$, the map $x \mapsto \Phi(x, z)$ is μ-measurable, and Φ satisfies (2.5), (2.6), and (2.7) on $\mathbb{R}^N \times \mathbb{R}^N$. Moreover, if we identify $T^*\mu$ with $T\mu$ through the canonical scalar product on $\mathbb{R}^N$ and we use the homogeneity of Φ, it turns out that the Fenchel conjugate $\Phi^*(x, \cdot)$ of $\Phi(x, \cdot)$ is

$$\Phi^*(x, z^*) = \begin{cases} 0 & \text{if } z^* \in T_\mu(x),\ \varphi^o(x, z^*) \le 1 \\ +\infty & \text{otherwise} \end{cases} , \qquad (x, z^*) \in \mathbb{R}^N \times \mathbb{R}^N . \tag{5.3}$$

In terms of the integrand Φ, the functional (5.1) can be written as $J(u) = \int \Phi(x, \nabla u)\, d\mu$ if $u \in \mathcal{D}$, and $J(u) = +\infty$ if $u \in L^1_\mu \setminus \mathcal{D}$. Then we can use such expression for J in order to compute $\overline{J}$: this allows to closely follow the proof of [5, Theorem 3.1], to which we refer for the details. Since J is convex, there holds $\overline{J} = J^{**}$ [20, Theorem 5], where J^{**} is defined as the double Fenchel transform of J in the duality between L^1_μ and L^∞_μ.

Let A be the densely defined linear operator from L^1_μ to $(L^1_\mu)^N$ given by $A(u) = \nabla u$ with domain $D(A) := \mathcal{D}$; denoting by A^* the adjoint of A, we have $D(A^*) = X_\mu \subseteq (L^\infty_\mu)^N$, $A^* : D(A^*) \ni \sigma \mapsto -\operatorname{div}(\sigma\mu) \in L^\infty_\mu$. Hence, by [5, Theorem 5.1] we have

$$J^*(v) = \inf \left\{ \int \Phi^*(x, \sigma)\, d\mu \ : \ \sigma \in X_\mu \ , \ A^*\sigma = v \right\} , \quad v \in L^\infty_\mu .$$

Arguing exactly as in the proof of Theorem 3.1 of [5], we get

$$J^{**}(u) = \sup \left\{ -\langle \operatorname{div}(\sigma\mu), u \rangle - \int \Phi^*(x, \sigma)\, d\mu \ : \ \sigma \in X_\mu \right\} , \quad u \in L^1_\mu .$$

Therefore taking into account (5.3), and recalling (4.2), we have for every $u \in L^1_\mu$

$$\overline{J}(u) = \sup \{ -\langle \operatorname{div}(\sigma\mu), u \rangle \ : \sigma \in \mathcal{K} \} = |D_\mu u|_\varphi .$$

$\square$

Remark 5.2. Notice that $\overline{J} = \overline{J'}$ for any functional J' with $\overline{J} \le J' \le J$ on L^1_μ. This in particular applies for

$$J'(u) := \begin{cases} \int \varphi(x, \nabla_\mu u)\, d\mu & \text{if } u \in W^{1,1}_\mu \\ +\infty & \text{if } u \in L^1_\mu \setminus W^{1,1}_\mu \ . \end{cases}$$

Indeed $J' \le J$. Moreover, since $J'(u) \ge \int \sigma \cdot \nabla_\mu u \, d\mu = -\langle \mathrm{div}(\sigma\mu), u \rangle$ for any $u \in W^{1,1}_\mu$ and $\sigma \in \mathcal{K}$, passing to the supremum over $\sigma \in \mathcal{K}$ and using Theorem 5.1 we get $J' \ge \overline{J}$.

We next give an integral representation result for the (μ, φ)-total variation on the Sobolev space $W^{1,1}_\mu$. Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$ and $\mathcal{K}$ be given by (4.2). We define a new integrand $h : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty)$ by

$$h(x, z) := \mu - \operatorname*{ess\ sup}_{\sigma \in \mathcal{K}} \{\sigma(x) \cdot z\} \ . \tag{5.4}$$

It is easy to check that the restriction of h to $T\mu$ belongs to the class $\mathcal{M}$; in view of Lemma 5.5 below, this restriction coincides with φ under suitable regularity assumptions.

Theorem 5.3. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let h be defined by* (5.4). *Then the following representation formula of the (μ, φ)-total variation holds:*

$$|D_\mu u|_\varphi = \int h(x, \nabla_\mu u)\, d\mu \ , \quad u \in W^{1,1}_\mu \ . \tag{5.5}$$

Proof. By applying Theorem 5.1 and using (2.3) , we get

$$\overline{J}(u) = \sup_{\sigma \in \mathcal{K}} \{-\langle \mathrm{div}(\sigma\mu), u \rangle\} = \sup_{\sigma \in \mathcal{K}} \left\{ \int \sigma \cdot \nabla_\mu u \, d\mu \right\} \ , \qquad u \in W^{1,1}_\mu \ . \tag{5.6}$$

We notice that the subset $S(u)$ of L^1_μ defined by $S(u) := \{\sigma \cdot \nabla_\mu u \ : \ \sigma \in \mathcal{K}\}$ is stable by smooth partitions of unity. Indeed let $\sigma_1, \sigma_2, \ldots, \sigma_l$ be elements of $\mathcal{K}$ and let $\alpha_1, \alpha_2, \ldots, \alpha_l$ belong to $\mathcal{D}$, with $\alpha_i \ge 0$ and $\sum_{i=1}^l \alpha_i = 1$. By the convexity of $\varphi^o(x, \cdot)$, $\sigma := \sum_{i=1}^l \alpha_i \sigma_i$ still satisfies $\varphi^o(x, \sigma(x)) \le 1$ μ-a.e.; moreover, since $\mathrm{div}(\sigma\,\mu) = \sum_{i=1}^l [\alpha_i \,\mathrm{div}(\sigma_i \mu) + (\nabla \alpha_i \cdot \sigma_i)\mu]$, we have also $\sigma \in X_\mu$. Applying the commutation result between supremum and integral proved in [7, Theorem 1], and taking into account (5.6) and (5.4), we conclude, for $u \in W^{1,1}_\mu$,

$$\overline{J}(u) = \int \mu - \operatorname*{ess\ sup}_{\sigma \in \mathcal{K}} \{\sigma \cdot \nabla_\mu u\} \ d\mu = \int h(x, \nabla_\mu u)\, d\mu \ ,$$

that is (5.5). □

As a consequence of Theorem 5.3, we get the following chain rule formula on $W^{1,1}_\mu$.

Theorem 5.4. *Let $\mu \in \mathcal{R}_+$ and $u \in W^{1,1}_\mu$. Then for any $\mathcal{L}^1$- negligible set $N \subset \mathbb{R}$ it holds*

$$\nabla_\mu u = 0 \qquad \mu\text{-a.e. on } u^{-1}(N) \ ; \tag{5.7}$$

moreover, for any $a \in L^\infty(\mathbb{R})$, setting $A(t) := \int_0^t a(s)\,ds$, $A \circ u$ belongs to $W^{1,1}_\mu$, and the following chain rule holds:

$$\nabla_\mu(A \circ u) = a(u)\nabla_\mu u \qquad \mu\text{-a.e.}\ . \tag{5.8}$$

Proof. We first prove that $A \circ u$ belongs to $W^{1,1}_\mu$ and that (5.8) holds under the assumption $A \in \mathcal{C}^\infty(\mathbb{R})$. Let $\{u_n\} \subset \mathcal{D}$ be a sequence converging to u in $W^{1,1}_\mu$. Then $A \circ u_n \in \mathcal{D}$ for every n, and projecting onto the tangent space to μ the usual chain rule, we get $\nabla_\mu(A \circ u_n) = a(u_n)\nabla_\mu u_n$; since $a(u_n)\nabla_\mu u_n \to a(u)\nabla_\mu u$ in measure μ, and since a is bounded, by Vitali's convergence criterion we get that $\nabla_\mu(A \circ u_n) \to a(u)\nabla_\mu u$ in L^1_μ. This implies by definition that $A \circ u \in W^{1,1}_\mu$ and that $\nabla_\mu(A \circ u) = a(u)\nabla_\mu u$.

Let $\varphi(x, z) := |z|$, and let h be the function defined in (5.4) associated with φ. Then, using Theorem 5.3, we get

$$\begin{aligned}\int a(u)h(x, \nabla_\mu u)\,d\mu &= \int h(x, \nabla_\mu(A \circ u))\,d\mu \\ &= |D(A \circ u)|_{\mu,\varphi} = \int_{\mathbb{R}} a(t)P_{\mu,\varphi}(\{u > t\})\,dt\ .\end{aligned} \tag{5.9}$$

One can check that (5.9) still holds if we replace the $\mathcal{C}^\infty$ function a by the characteristic function of a Borel subset $N \subset \mathbb{R}$: indeed N can be approximated from the exterior and from the interior respectively by open and compact sets, which in turn can be approximated respectively by an increasing or a decreasing sequence of smooth functions.

In particular, let us take $a = \chi_N$ in (5.9), where $N \subset \mathbb{R}$ is $\mathcal{L}^1$-negligible. We get

$$\int_{u^{-1}(N)} h(x, \nabla_\mu u)\,d\mu = \int_N P_{\mu,\varphi}(\{u > t\})\,dt = 0\ .$$

Then, recalling the definition (5.4) of h, it follows that $\nabla_\mu u(x) \in T^\perp_\mu(x)$ for μ-a.e. $x \in u^{-1}(N)$. As by construction $\nabla_\mu u(x) \in T_\mu(x)$ for μ-a.e. x, this implies (5.7).

It remains to prove that (5.8) holds for any $a \in L^\infty(\mathbb{R})$. Let $\{a_n\}$ be a sequence of $\mathcal{C}^\infty$ functions bounded in $L^\infty(\mathbb{R})$ and converging to a $\mathcal{L}^1$-a.e. . Then $A_n \circ u \to A \circ u$ in L^1_μ. Let N be a negligible subset of $\mathbb{R}$ such that $a_n(t) \to a(t)$ for every $t \in \mathbb{R} \setminus N$. We have $a_n(u) \to a(u)$ μ-a.e. on $\mathbb{R}^N \setminus u^{-1}(N)$, which implies by (5.7) that $a_n(u)\nabla_\mu u \to a(u)\nabla_\mu u$ μ-a.e. . Since we know that $\nabla_\mu(A_n \circ u) = a_n(u)\nabla_\mu u$ for every n, it follows from the completeness of $W^{1,1}_\mu$ that the sequence $\{A \circ u_n\}$ converges in $W^{1,1}_\mu$, hence its limit is $A \circ u$ and (5.8) holds. □

Our aim now is to compare the integrands h and φ. We will adopt the notation $h_1 \preceq h_2$ (or equivalently $h_1(x, z) \preceq h_2(x, z)$) and $h_1 \equiv h_2$ to denote the following relations between metric integrands of the class $\mathcal{M}$:

$$\begin{aligned} h_1 \preceq h_2 \quad &\Longleftrightarrow \quad h_1(x, z) \le h_2(x, z) \text{ for all } x \in \mathbb{R}^N \setminus E, \text{ with } \mu(E) = 0\ ,\ z \in T_\mu(x)\ ; \\ h_1 \equiv h_2 \quad &\Longleftrightarrow \quad h_1 \preceq h_2 \text{ and } h_2 \preceq h_1\ . \end{aligned}$$

Next, we associate with φ a lower semicontinuous regularization (with respect to μ) by setting

$$\overline{\varphi}_\mu(x, z) := \sup\left\{\phi(x) \cdot z\ :\ \phi \in \mathcal{C}^N\ ,\ \phi(x) \cdot z \preceq \varphi(x, z)\ ,\ P_\mu \circ \phi \in X_\mu\right\}\ . \tag{5.10}$$

It is easy to check that $\overline{\varphi}_\mu$ belongs to the class $\mathcal{M}$ when restricted to $T\mu$ (note however that the equality $\overline{\varphi}_\mu = +\infty$ will occur mostly on the complement of $T\mu$).

The following comparison result holds.

Lemma 5.5. *Let $\mu \in \mathcal{R}_+$, $\varphi \in \mathcal{M}$, and let h, $\overline{\varphi}_\mu$ be defined by (5.4) and (5.10) respectively. Then*

$$\overline{\varphi}_\mu \ \preceq \ h \ \preceq \ \varphi \, . \tag{5.11}$$

Proof. Let $\mathcal{K}$ be defined by (4.2) and set

$$\begin{aligned}
\mathcal{C}_\mu &:= \left\{ P_\mu \circ \phi \ : \ \phi \in \mathcal{C}^N, \ P_\mu \circ \phi \in X_\mu \ , \ \phi(x) \cdot z \preceq \varphi(x,z) \right\} , \\
\mathcal{K}_\mu &:= \left\{ \sigma \in (L^\infty_\mu)^N \ : \ \sigma(x) \in T_\mu(x) \ , \ \varphi^o(x, \sigma(x)) \leq 1 \ \text{ for } \mu\text{ -a.e. } x \right\} .
\end{aligned}$$

Note that a field σ in $\mathcal{K}_\mu$ is not necessarily in X_μ, since in the definition of $\mathcal{K}_\mu$ we have skipped the (non local) condition on $\operatorname{div}(\sigma\,\mu)$.

Since $\mathcal{C}_\mu \subseteq \mathcal{K} \subseteq \mathcal{K}_\mu$, we have, for μ-a.e. x in $\mathbb{R}^N$ and for all $z \in T_\mu(x)$:

$$\begin{aligned}
\overline{\varphi}_\mu(x,z) &= \mu - \operatorname*{ess\ sup}_{\sigma \in \mathcal{C}_\mu} \{\sigma(x) \cdot z\} \leq \mu - \operatorname*{ess\ sup}_{\sigma \in \mathcal{K}} \{\sigma(x) \cdot z\} = h(x,z) \\
&\leq \mu - \operatorname*{ess\ sup}_{\sigma \in \mathcal{K}_\mu} \{\sigma(x) \cdot z\} = \sup\{ z \cdot z^* \ : \ z^* \in T_\mu(x) \, , \ \varphi^o(x, z^*) \leq 1 \} \\
&= \varphi(x, P_\mu(x)[z]).
\end{aligned}$$

This completes the proof of (5.11). $\square$

Remark 5.6. The following examples, in which we have $\overline{\varphi}_\mu \equiv h$, show that the equivalence $h \equiv \varphi$ may not hold, due to the lack of regularity either of φ or of μ.

(i) Let $N = 1$, let $\mu = \mathcal{L}^1 \llcorner (0,1)$, let $F \subset (0,1)$ be a closed set with empty interior such that $\mathcal{L}^1(F) > 0$ and let us set

$$\varphi(x,z) := \begin{cases} |z| & \text{if } x \in F \\ 0 & \text{if } x \notin F \, . \end{cases}$$

Then one can check that $\mathcal{K}$ is reduced to $\{0\}$, hence $h \equiv 0$, while φ is by definition nonzero on a set of positive measure μ.

(ii) Let $N = 1$, let $\varphi(x,z) = |z|$, let F be as in the example i), and let $\mu = a(x)\mathcal{L}^1 \llcorner (0,1)$, where

$$a(x) := \begin{cases} 2 & \text{if } x \in F \\ 1 & \text{if } x \in (0,1) \setminus F. \end{cases}$$

For any $\sigma \in \mathcal{K}$, we have $|a\sigma| \leq 1$ on $(0,1)$: indeed $a\sigma$ is continuous, because $\frac{d}{dx}(a\sigma) \in L^\infty(0,1)$, and it holds $|a\sigma| \leq 1$ on the dense set $(0,1) \setminus F$. Hence $|\sigma| \leq a^{-1}$ μ-a.e. for any $\sigma \in \mathcal{K}$; moreover, for any constant $\lambda \leq 1$, the function $\sigma(x) := \lambda\,(a(x))^{-1}$ belongs to $\mathcal{K}$. By definition (5.4) it follows that $h(x,z) = a(x)^{-1}|z|$, and in particular that we do not have $h \equiv \varphi$.

In order to avoid the pathological-type behaviours described in Remark 5.6, we are led to introduce the following assumption on μ:

$$\mathcal{V}_\mu := \{\phi \in \mathcal{C}_0^N \ : \ P_\mu \circ \phi \in X_\mu \ \} \text{ is dense in } \mathcal{C}_0^N \ . \tag{5.12}$$

This condition will also be useful in Section 6, and it can be shown that it is related to the regularity of the mean curvature vector $H(\mu) := \mathrm{div}(P_\mu \mu)$ of μ [4].

Theorem 5.7. *Let $\mu \in \mathcal{R}_+$ satisfy (5.12), and let $\varphi \in \mathcal{M}$ be a lower semicontinuous metric on $T\mu$ such that (3.2) holds. Then $\overline{\varphi}_\mu \equiv \varphi$. In particular, it holds*

$$|D_\mu u|_\varphi = \int \varphi(x, \nabla_\mu u)\, d\mu \ , \qquad u \in W^{1,1}_\mu \ .$$

Proof. By Lemma 5.5, it is enough to show that $\varphi \preceq \overline{\varphi}_\mu$. Let $t < 1$, and let $\phi \in \mathcal{C}^N$ such that $\phi(x) \cdot z \preceq t\varphi(x, z)$. Then by (5.12) there exists a sequence $\{\phi_n\} \subseteq \mathcal{V}_\mu$ which converges uniformly to ϕ. For n large enough, we have $\|\phi_n - \phi\|_\infty \le (1-t)C^{-1}$, where C is the positive constant appearing in (3.2); thus, there exists $\bar{n}$ such that, for $n > \bar{n}$,

$$\begin{aligned}\phi_n(x) \cdot z &\preceq \|\phi_n - \phi\|_\infty |z| + |\phi(x) \cdot z| \\ &\preceq (1-t)C^{-1}|z| + t\varphi(x, z) \preceq \varphi(x, z) \ .\end{aligned}$$

By definition (5.10), we deduce

$$\phi(x) \cdot z = \lim_{n \to +\infty} \phi_n(x) \cdot z \ \preceq \ \overline{\varphi}_\mu(x, z) \ ;$$

then, using also the lower semicontinuity assumption on φ, we get (see for instance [9, Lemma 2.2.3])

$$t\varphi(x, z) \ = \ \sup \Big\{ \phi(x) \cdot z \ : \ \phi \in \mathcal{C}^N \ , \ \phi(x) \cdot z \ \le \ t\varphi(x, z) \Big\} \ \preceq \ \overline{\varphi}_\mu(x, z) \ .$$

Since $t < 1$ was arbitrary, it follows $\varphi \ \preceq \ \overline{\varphi}_\mu$. $\square$

6. Integral representation on $BV_{\mu,\varphi}$

The natural generalization of Theorem 5.3 would be an integral representation for the (μ, φ)-total variation ot the type

$$|D_\mu u|_\varphi = \int \mathrm{h}(x, D_\mu u) \ , \qquad u \in BV_{\mu,\varphi} \ , \tag{6.1}$$

for some metric integrand h and some element $D_\mu u \in \mathcal{R}^N$. Here and in the following, the notation must be intended in the usual sense of integration theory, namely

$$\int \mathrm{h}(x, D_\mu u) = \int \mathrm{h}(x, \nu^u_\mu(x))\, d|D_\mu u| \ ,$$

where $|D_\mu u|$ is the total variation measure of $D_\mu u$, and ν^u_μ is the density of $D_\mu u$ with respect to $|D_\mu u|$.

In order to have (6.1), we ask whether it is possible to find a vector measure $D_\mu u$ which yields such an integral representation. This question turns out to be quite delicate, and we will prove in Example 6.4 that it has, in general, a negative answer. In fact, as we will show, this problem is directly related to the possibility of finding a closed extension $\overline{T}$ of the (densely defined) linear operator T defined by

$$T : L^1_\mu \to \mathcal{R}^N \quad , \quad D(T) := W^{1,1}_\mu \quad , \quad Tu := (\nabla_\mu u)\, \mu \; ,$$

where $\mathcal{R}^N$ is endowed with the weak star topology. Recall that $\overline{T}$, if it exists, is unique and its graph coincides with the closure $\overline{G(T)}$ in $L^1_\mu \times \mathcal{R}^N$ of the graph $G(T)$ of T. Unfortunately, the closability of T is not satisfied for a general measure μ (see again Example 6.4 below, where the closure of the graph of T fails to be a graph). The following lemma gives a necessary and sufficient condition on μ for the closability of T.

Lemma 6.1. *Let $\mu \in \mathcal{R}_+$. Then T is closable if and only if (5.12) holds. In this case, $\overline{T}u$ is the vector-valued measure determined by*

$$\langle \overline{T}u, \phi \rangle_{(\mathcal{R}^N, \mathcal{C}_0^N)} := -\langle u, \operatorname{div}[(P_\mu \phi)\mu] \rangle \; , \qquad \phi \in \mathcal{V}_\mu \; , \tag{6.2}$$

and we have $BV_\mu \subseteq D(\overline{T})$.

Proof. Recalling that $\mathcal{R}^N$ is endowed with the weak star topology, the adjoint operator $T^* : \mathcal{C}_0^N \to L^\infty_\mu$ has domain $D(T^*) = \mathcal{V}_\mu$ and is defined by $T^*\phi := -\operatorname{div}[(P_\mu\phi)\mu]$ for all $\phi \in \mathcal{V}_\mu$. Since $\overline{G(T)} = \left[\mathcal{J}(G(T^*))\right]^\perp$, where $\mathcal{J} : L^1_\mu \times \mathcal{R}^N \ni (u,\lambda) \to (-\lambda, u) \in \mathcal{R}^N \times L^1_\mu$ [8, II.6], we have

$$\overline{G(T)} = \Big\{ (u,\lambda) \in L^1_\mu \times \mathcal{R}^N \;:\; \langle \lambda, \phi \rangle_{(\mathcal{R}^N, \mathcal{C}_0^N)} := -\langle u, \operatorname{div}[(P_\mu\phi)\mu] \rangle \; , \; \phi \in \mathcal{V}_\mu \Big\} . \tag{6.3}$$

Thus, $\overline{G(T)}$ is a graph if and only if the linear subspace of $\mathcal{R}^N$ given by $\{\lambda \in \mathcal{R}^N \;:\; (0,\lambda) \in \overline{G(T)}\} = \mathcal{V}_\mu^\perp$ reduces to $\{0\}$, that is if and only if $\mathcal{V}_\mu$ is dense in $\mathcal{C}_0^N$. In this case, $\overline{G(T)}$ coincides with the graph of the operator $\overline{T}$ given by (6.2). Whenever $u \in BV_\mu$, applying (3.1) (with $\varphi(x,z) = |z|$), we find that there exists a positive constant C such that

$$|\, \langle u, \operatorname{div}[(P_\mu\phi)\mu] \rangle \,| \leq \; C \, \| \, P_\mu \phi \, \|_{(L^\infty_\mu)^N} \; \leq C \| \, \phi \, \|_{(L^\infty_\mu)^N} \; , \qquad \phi \in \mathcal{V}_\mu \; .$$

Therefore $BV_\mu \subseteq D(\overline{T})$. □

Given $\mu \in \mathcal{R}_+$ satisfying (5.12) and $u \in BV_\mu$, we set $D_\mu u := \overline{T}u$. Notice that $D_\mu u = (\nabla_\mu u)\, \mu$ whenever $u \in W^{1,1}_\mu$. Then we can derive the following

Lemma 6.2. *Let $\mu \in \mathcal{R}_+$ satisfy (5.12) and let $\{u_n\}$ be a sequence in $W^{1,1}_\mu$ such that*

$$u_n \to u \;\; in \; L^1_\mu \quad , \quad \sup_n \int |\nabla_\mu u_n| \, d\mu \; < +\infty \; .$$

Then $u \in BV_\mu$ and $(\nabla_\mu u_n)\, \mu \to \; D_\mu u$ weakly star in $\mathcal{R}^N$.

Proof. The sequence $\{(u_n, (\nabla_\mu u_n)\,\mu)\}$ is precompact in $L^1_\mu \times \mathcal{R}^N$ and, by definition of $\overline{T}$, any of its cluster points (u, λ) belongs to $G(\overline{T})$. Therefore the whole sequence converges to $(u, \overline{T}u)$. It remains to check that u belongs to BV_μ, which follows from the lower semicontinuity of the (μ, φ)-total variation when $\varphi(x,z) = |z|$ and from Remark 5.2. We have indeed

$$|D_\mu u|_\varphi \leq \liminf_{n\to+\infty} |D_\mu u_n|_\varphi = \liminf_{n\to+\infty} \int |\nabla_\mu u_n|\, d\mu < +\infty\ .$$

□

Let $\mu \in \mathcal{R}_+$ satisfy (5.12). We are going to prove that, for any smooth and coercive $\varphi \in \mathcal{M}$, (6.1) holds with $\mathrm{h} = \varphi$ provided the following additional regularity assumption on μ is fulfilled:

$$\begin{aligned}&\sup\Big\{\langle u, \operatorname{div}\left[(P_\mu\phi)\mu\right]\rangle\ :\ \phi \in \mathcal{V}_\mu\ ,\ |\phi| \leq 1\Big\} = \\ &\sup\Big\{\langle u, \operatorname{div}(\sigma\mu)\rangle\ :\ \sigma \in X_\mu\ ,\ |\sigma| \leq 1\ \mu\text{-a.e.}\Big\}\ .\end{aligned} \tag{6.4}$$

Notice that conditions (5.12) and (6.4) are both satisfied when we take $\mu = a\mathcal{H}^k \llcorner M$ as in Example 3.2. Indeed in that case (5.12) is satisfied because $\mathcal{V}_\mu$ contains $\mathcal{C}^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$ (which happens whenever μ has mean curvature in $(L^\infty_\mu)^N$, see also Remark 5.6), while (6.4) can be checked by using local coordinates and approximation by convolution.

Theorem 6.3. *Let $\mu \in \mathcal{R}_+$ such that* (5.12) *and* (6.4) *hold. Assume that $\varphi \in \mathcal{M}$ satisfies the coercivity condition* (3.2) *and admits a continuous extension on the whole of $\mathbb{R}^N \times \mathbb{R}^N$. Then*

$$|D_\mu u|_\varphi = \int \varphi(x, D_\mu u)\ , \qquad u \in BV_\mu\ . \tag{6.5}$$

Proof. Denote by $G(u)$ the right hand side of (6.5). Let us first show that $|D_\mu u|_\varphi \geq G(u)$. We can assume that $|D_\mu u|_\varphi$ is finite. By Theorem 5.1, there exists a sequence $\{u_n\} \subset \mathcal{D}$ such that $u_n \to u \in L^1_\mu$ and $\int \varphi(x, \nabla_\mu u_n)\, d\mu \to |D_\mu u|_\varphi$. By Lemma (6.2), we deduce that $u \in BV_\mu$ and $(\nabla_\mu u_n)\,\mu \to D_\mu u$ weakly star in $\mathcal{R}^N$. By the continuity of (the extension of) φ, we can apply Reshetnyak's lower semicontinuity theorem [18, Theorem 2], which gives the claimed inequality:

$$|D_\mu u|_\varphi = \lim_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu \geq \int \varphi(x, D_\mu u) = G(u)\ .$$

In order to show the converse inequality, let us fix $u \in BV_\mu$ and divide the proof into in two steps. As a first step, we show that (6.5) holds when $\varphi(x,z) := |z|$. Indeed, by (6.4), (6.2), and (5.12), we get:

$$\begin{aligned}|D_\mu u|_{\varphi=|\cdot|} &= \sup\Big\{\langle u, \operatorname{div}(\sigma\mu)\rangle\ :\ \sigma \in X_\mu\ , |\sigma| \leq 1\ \mu\text{-a.e.}\Big\} \\ &= \sup\Big\{\langle u, \operatorname{div}\left[(P_\mu\phi)\mu\right]\rangle\ :\ \phi \in \mathcal{V}_\mu\ ,\ |\phi| \leq 1\Big\} = \int |D_\mu u|\ .\end{aligned}$$

As a second step, we pass to consider the case of a general metric $\varphi \in \mathcal{M}$ as in the assumptions. By the first step, we can choose a suitable sequence $\{u_n\} \subset \mathcal{D}$ such that $u_n \to u \in L^1_\mu$ and $\int |\nabla_\mu u_n|\, d\mu \to \int |D_\mu u|$. By the lower semicontinuity of the (μ, φ)-total variation, and by Reshetnyak's continuity theorem [18, Theorem 3], we infer

$$|D_\mu u|_\varphi \leq \liminf_{n\to+\infty} \int \varphi(x, \nabla_\mu u_n)\, d\mu = \int \varphi(x, D_\mu u) = G(u)\ .$$

□

Finally, let us give an example of measure μ for which condition (5.12) is not satisfied (and in fact the closure of the graph of the operator T is not a graph). The measure μ under consideration is the one dimensional Hausdorff measure on a Lipschitz (but not $\mathcal{C}^1$) curve S in $\mathbb{R}^2$. We find that, if E is a set whose boundary (in S) meets the singular part of S, the explicit expression for the (μ, φ)-perimeter of E (see formula (6.6)) is not compatible with an integral representation of the form $|D_\mu u|_\varphi = \int \varphi(x, \alpha_u)$, for a vector-valued measure α_u. Indeed, in order to have such an integral representation, the map $\varphi \mapsto |D_\mu u|_\varphi$ needs to be linear for any $u \in BV_{\mu,\varphi}$, while (6.6) shows that this is not the case for $u = \chi_E$.

Example 6.4. Let $N = 2$ and $\mu := \mathcal{H}^1 \llcorner S$, where

$$S := S_+ \cup S_-\ , \qquad S_+ := \{(t,t)\ :\ t \in [0,1]\}\ , \qquad S_- := \{(t,-t)\ :\ t \in [-1,0]\}\ .$$

Clearly the tangent space $T_\mu(x)$ is μ-a.e. one-dimensional and it is determined by the direction $\nu(x) := \nu_\pm$ respectively for $x \in S_\pm \setminus \{O\}$, where O is the origin of $\mathbb{R}^2$ and $\nu_\pm := (1/\sqrt{2}\,, \pm 1/\sqrt{2})$. Let $\varphi \in \mathcal{M}$ not depend explicitly on x, i.e. $\varphi(x,z) = \varphi(z)$ for $x \in S \setminus \{O\}$ and $z \in \mathbb{R}\nu(x)$.

We claim that, if we take $E := S_+$, there holds

$$P_{\mu,\varphi}(E) = \min\{\varphi(\nu_+)\,,\, \varphi(\nu_-)\}\ . \tag{6.6}$$

Let us prove (6.6). Let $\gamma : I := [-\sqrt{2}, \sqrt{2}] \to S$ be a parametrization by arc-length of $S \setminus \{O\}$ such that $\gamma(\pm\sqrt{2}) := A^\pm := (\pm 1, 1)$ and $\gamma(0) = O$. A measurable vector field σ such that $\sigma(x) \in T_\mu(x)$ for μ-a.e. x can be written as

$$\sigma(x) = \tilde{\sigma}(\gamma^{-1}(x))\, \nu(x)\ , \qquad x \in S \setminus \{O\}\ ,$$

where $\tilde{\sigma}$ is a scalar function defined $\mathcal{L}^1$-a.e. on I. One can check that $\sigma \in X_\mu$ if and only if $\tilde{\sigma}$ belongs to $W^{1,\infty}_0(I)$ and in that case $\mathrm{div}(\sigma\mu) = (\tilde{\sigma}' \circ \gamma^{-1})\, \mu$ where $\tilde{\sigma}'$ coincides with the $\mathcal{L}^1$-a.e. defined derivative of the Lipschitz function $\tilde{\sigma}$.

Let $u = \chi_E$. Recalling (3.1), letting $\tilde{u}$ such that $u = \tilde{u} \circ \gamma^{-1}$, and integrating by parts, we get

$$
\begin{aligned}
P_{\mu,\varphi}(E) &:= \sup\Big\{ -\langle \mathrm{div}(\sigma\mu), u\rangle \ :\ \sigma \in X_\mu\,,\ \varphi^o(\sigma(x)) \le 1 \text{ for } \mu\text{-a.e. } x\Big\} \\
&= \sup\Big\{ -\int_{-\sqrt{2}}^{\sqrt{2}} \tilde u(s)\,\tilde\sigma'(s)\,ds \ :\ \tilde\sigma \in W_0^{1,\infty}(I),\ \varphi^o(\tilde\sigma(s)\nu) \le 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in I\Big\} \\
&= \sup\Big\{ \tilde\sigma(0) :\ \tilde\sigma \in W_0^{1,\infty}(I)\,,\ \varphi^o(\tilde\sigma\,\nu) \le 1 \text{ for } \mathcal{L}^1\text{-a.e. } s\in I\Big\} \\
&= \sup\Big\{ \lambda \in \mathbb{R} \ :\ \max\{\varphi^o(\lambda\nu_+), \varphi^o(\lambda\nu_-)\} \le 1\Big\} \\
&= \min\Big\{ \varphi(\nu_+)\,,\ \varphi(\nu_-)\Big\}\ .
\end{aligned}
$$

We conclude the paper with three further observations concerning Example 6.4.

(i) The formula (6.6) seems rather disconcerting. The key point is that the condition $\sigma \in X_\mu$ allows the existence of tangent fields σ whose norm is continuous but whose direction jumps at the corner O. Computing the concentration of energy at O for a given u leads to individuate the optimal jump of σ.

(ii) By Theorem 5.1, we have $P_{\mu,\varphi}(E) = \overline{J}(\chi_E)$, hence there exists a sequence $\{u_n\} \subset W_\mu^{1,1}$ converging to χ_E in L^1_μ such that $P_{\mu,\varphi}(E) = \lim_{n\to+\infty} J(u_n)$. For instance we can take $u_n := v_n \circ \gamma^{-1}$, where $\{v_n\}$ is any sequence of non decreasing functions in $W^{1,1}(I)$ such that $v_n(-\sqrt{2}) = 0$, $v_n(\sqrt{2}) = 1$, $v_h \to \chi_E \circ \gamma$ $\mathcal{L}^1$-a.e. on I and $v_n(0) = 1$ if $\varphi(\nu_+) \ge \varphi(\nu_-)$ and $v_n(0) = 0$ otherwise.

(iii) The closure in $\mathcal{C}_0^2$ of the subspace $\mathcal{V}_\mu$ associated with μ defined in (5.12) is given here by

$$
\overline{\mathcal{V}}_\mu = \Big\{ \phi \in \mathcal{C}_0^2 \ :\ \phi(O) \in \mathbb{R}(1,0)\Big\}\ . \tag{6.7}
$$

This can be derived from the fact that, if we ask that $\phi \in X_\mu$ when ϕ is a tangent vector field in $(\mathcal{C}^1(S\setminus\{O\}))^2$, we find the necessary condition $\phi(O^+)\cdot\nu_+ = \phi(O^-)\cdot\nu_-$, where $\phi(O^\pm) := \lim_{S_\pm \ni x\to O} \phi(x)$. Thus, by (6.7), $\overline{\mathcal{V}}_\mu$ is a strict closed subspace of $\mathcal{C}_0^2$ and therefore the operator T is not closable, or equivalently $\overline{G(T)}$ is not a graph. In fact, one can check by using (6.3) and (6.7) that $(\chi_E, (t, 1/\sqrt{2})\,\delta_O)$ belongs to $\overline{G(T)}$ for any real t.

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