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Optimality Conditions for Constrained Convex Parabolic Control Problems via Duality

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The optimality system is given for a phase-constrained control problem governed by a linear parabolic equation. The control may act on the boundary and the cost function may depend on the boundary values of the final state. The optimality system follows from general results for convex state-constrained control problems by means of the theory of convex duality, using a suitable perturbation.

1. Introduction

In this paper we consider an abstract scheme for the optimal control problem of linear parabolic systems with phase constraints following the ideas proposed in [6] and [1] for the optimal control of elliptic problems and similar to those used in [19] for the case of systems governed by ordinary differential equations. We obtain optimality conditions using the convex duality methods. To this end, we introduce a natural and versatile perturbation, acting on the right member of the state equation, of a control problem governed by a linear parabolic equation. Under some qualification assumption the duality scheme is shown to be stable which yields immediately the optimality system by computing the partial directional derivatives of the associated Lagrangian. This method provides an alternative to methods using penalization of the state equation as in [4], although the results obtained have the same form. However, we use a weaker constraint qualification than in [4], [3] or [18], and our setting covers the results obtained in these papers. The general setting considered leads us to use duality results in the framework of Fréchet spaces (see [13], [22], and [23]). Applications are given to the case of boundary control and/or observation (pointwise or zone), pointwise phase constraints and boundary final cost functional. Some results of the present paper have been announced in [2]. Other results about state constrained distributed systems can be found in [12], [11], [14], [7], [10], [9].

2. The optimal control problem

2.1. The abstract framework

Let V, H be Hilbert spaces such that V is continuously and densely embedded in H. Identifying H to its dual we have

$$V \subset H \subset V^*$$

(each inclusion being continuous and dense). We shall denote by $\langle \cdot, \cdot \rangle$ the duality product between V^* and V as well as the scalar product on H (because they coincide on $H \times V$).

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We will consider also Banach spaces U (the control space), E_1 (the space of the final states) with E_1 continuously and densely embedded in H, the families $A(t) \in \mathcal{L}(V, V^*)$, $B(t) \in \mathcal{L}(U, V^*)$, $t \in [0,T]$, and the convex functionals $L : [0,T] \times V \times U \to \mathbb{R} \cup \{+\infty\}$, $l : H \times E_1 \to \mathbb{R} \cup \{+\infty\}$.

We deal with the following (constrained) convex optimal control problem:

$$\inf J(y,u) = \int_0^T L(t, y(t), u(t))\, dt + l(y(0), y(T)) \tag{P}$$

subject to

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + A(t)y(t) = B(t)u(t) + f(t) \text{ a.e. in }]0,T[\tag{2.1}$$

$$(y,u) \in \mathcal{M} \subset E(0,T) \times L^p(0,T;U), \tag{2.2}$$

where $\mathcal{M}$ is a closed convex set and $W(0,T)$ is the space

$$W(0,T) = \{y \in L^2(0,T;V) : \frac{\mathrm{d}y}{\mathrm{d}t} \in L^2(0,T;V^*)\}$$

which is a Hilbert space endowed with the norm

$$\|y\|_{W(0,T)} = \left(\int_0^T (\|y(t)\|_V^2 + \|\frac{\mathrm{d}y}{\mathrm{d}t}(t)\|_{V^*}^2)\, dt\right)^{1/2}.$$

We consider the space $C(]0,T];E_1)$ which is a Fréchet space with the topology of uniform convergence on compact subsets of $]0,T]$ and we endow the space $C(]0,T];E_1) \cap W(0,T)$ with the coarsest topology such that the injections of this space into $C(]0,T];E_1)$ and into $W(0,T)$ are continuous.

- The space $E(0,T)$ is a Fréchet space continuously and densely embedded in $C(]0,T];E_1) \cap W(0,T)$, and in $W(0,T)$. Note that since $C([0,T];E_1)$ is continuosly and densely embedded in $C(]0,T];E_1)$ and in $C([0,T];H)$, and $W(0,T)$ is continuously and densely embedded in $C([0,T];H)$.

We denote by $F(0,T)$ the linear space of the functions of the form $t \mapsto \dfrac{\mathrm{d}y}{\mathrm{d}t}(t) + A(t)y(t)$ with $y \in E(0,T)$. Note that $F(0,T) \subset L^2(0,T;V^*)$. We will consider $F(0,T)$ endowed with the finest topology which makes continuous the mapping $y(\cdot) \mapsto \dfrac{\mathrm{d}y}{\mathrm{d}t}(\cdot) + A(\cdot)y(\cdot)$ from $E(0,T)$ to $F(0,T)$. Thus $F(0,T)$ is a Fréchet space continuously and densely embedded in $L^2(0,T;V^*)$.

- For all y, $z \in V$, the function $t \mapsto \langle A(t)y, z\rangle$ is measurable on $]0,T[$ and

$$\sup_{t \in]0,T[} \|A(t)\|_{\mathcal{L}(V,V^*)} < +\infty.$$

- There exists $\alpha \in \mathbb{R}$, $\omega > 0$ such that for all $y \in V$ and $t \in]0, T[$,

$$\langle A(t)y, y\rangle + \alpha\|y\|_H^2 \geq \omega\|y\|_V^2. \tag{2.3}$$

- the map $u(\cdot) \mapsto B(\cdot)u(\cdot)$ is continuous from $L^2(0, T; U)$ to $L^2(0, T; V^*)$ and from $L^p(0, T; U)$ to $F(0, T)$ where $p \in [2, +\infty]$ is fixed (in some applications we may have $p = +\infty$ and $f \in F(0, T)$)
- The function l is proper lower semicontinuous and L is a normal convex integrand (see [21]). Thus J is convex proper and l.s.c. on $E(0, T) \times L^p(0, T; U)$. We shall assume that J is finite over $\mathcal{M}$.

Note that the initial condition

$$y(0) = y_0 \in H \tag{2.4}$$

(or, more generally, $y(0) \in H_0$ with H_0 closed convex subset of H) or the possible final restrictions might be included in the definition of $\mathcal{M}$. In this way we may consider also the case of the systems with unsufficient data (see [16], [17]). Note also that the hypothesis J finite over the feasible set is not restrictive since the constraints are not included into the cost via the indicator function, but are given separately. According to [15] the abstract problem (2.1), (2.4) has a unique solution in $W(0, T)$ for each $u \in L^2(0, T; U)$ and $y_0 \in H$. As we have $W(0, T) \subset C([0, T]; H)$ continuously (see [15, page 116, Th 1.1 and Th 1.2]), our problem is well formulated. Finally we will assume that

- for each $u \in L^p(0, T; U)$ the solution to (2.1), (2.4) lies in the space $E(0, T)$,
- for each $y_0 \in H$ the solution to the homogenous equation (2.1), (2.4) with $u = 0$, $f = 0$ belongs to $E(0, T)$.

The abstract problem (P) covers a large class of parabolic systems as we shall see later. Notice that, taking $p = 2$, $E_1 = H$, and $E(0, T) = W(0, T)$, we obtain the "classical" case (which is considered in [15], [3] or [4]). However, considering different spaces E_1 and $E(0, T)$ we are able to handle the case of pointwise phase constraints and boundary final cost criterium. The last case has been partially studied in [18] (for a particular parabolic problem with the restrictive hypothesis $y_0 = 0$).

2.2. The constraint qualification

Let us consider the linear continuous operator

$$\mathcal{T} : W(0, T) \times L^2(0, T; U) \longrightarrow L^2(0, T; V^*)$$

defined by

$$\mathcal{T}(y, u)(t) = \frac{\mathrm{d}y}{\mathrm{d}t}(t) + A(t)y(t) - B(t)u(t) \text{ a.e. in }]0, T[.$$

We need the following constraint qualification assumption:

There exists a Fréchet space $Z(0, T)$ continuously and densely embedded in $F(0, T)$ such that

$$Z(0, T) \subset \mathbb{R}_+(\mathcal{T}(\mathcal{M}) - f). \tag{QA}$$

Let us compare the condition (QA) with the condition $(\mathcal{H}')$ used in [4, section 3] namely:

There exists a Banach space $\hat{Z}(0,T) \subset L^2(0,T;V^*)$ with continuous and dense embedding such that there exists $\mathcal{M}_1 \subset \mathcal{M}$ bounded in $C^0(0,T;H) \times L^2(0,T;U)$ with

$$0 \in \mathrm{Int}(\mathcal{T}(\mathcal{M}_1) - f) \text{ in the } \hat{Z}(0,T) \text{ topology.} \tag{$\mathcal{H}'$}$$

In this case choosing $p = 2$, $E_1 = H$, $E(0,T) = W(0,T)$ and $Z(0,T) = \hat{Z}(0,T)$ it is obvious that $(\mathcal{H}')$ implies (QA). This shows that (QA) is weaker than $(\mathcal{H}')$. Also, according to [4, proposition 1.1 and remark 1.1], we obtain that (QA) is strictly weaker than classical Slater condition.

2.3. An example of a final boundary cost criterium with boundary control and pointwise phase constraints system

Here we will slightly generalize the case studied in [18] and we will show how to use our general framework for this particular problem. First we state the particular parabolic problem. Let us consider a bounded open set $\Omega \subset \mathbb{R}^n$ which is locally on one side of its boundary Γ. We assume Γ is a C^∞ manifold of dimension $n-1$. Denote $Q =]0,T[\times\Omega$, $\Sigma =]0,T[\times\Gamma$. For $i,\ j = 1,\dots,n$ let functions $a_0,\ a_{ij} \in \mathcal{D}(\bar{\Omega})$ and real number $c > 0$ be given such that $a_{ij} = a_{ji}$ and that for all $\xi \in \mathbb{R}^n$, $x \in \Omega$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c\sum_{i=1}^{n} \xi_i^2.$$

With $\alpha \geq 0$ and $y_0 \in L^2(\Omega)$, we consider the *state equation* given by

$$\frac{\partial y}{\partial t}(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}(t,x)\right) - a_0(x)y(x,t) \text{ in } Q, \tag{2.5}$$

$$\frac{\partial y}{\partial \nu_A}(t,\xi) + \alpha y(t,\xi) = u(t,\xi) \text{ in } \Sigma,$$

$$y(0,x) = y_0(x) \text{ in } \Omega. \tag{2.6}$$

The *phase constraints* are of the form

$$y(t,x) \in C(t,x) \text{ for all } (t,x) \in]0,T] \times \overline{\Omega}, \tag{2.7}$$

$$u \in U_{ad} \subset L^\infty(\Sigma). \tag{2.8}$$

where $C(\cdot,\cdot)$ is a multifunction from $]0,T] \times \bar{\Omega}$ into $\mathbb{R}$ with closed convex values, and U_{ad} is closed and convex. Finally, the *cost functional* is given by

$$l_1(y(T)) + \int_0^T \int_\Gamma L_1(t,y(t,\xi),u(t,\xi))\, d\Gamma\, dt \tag{2.9}$$

with $l_1 : C(\bar{\Omega}) \to \mathbb{R}\cup\{+\infty\}$ lower semicontinuous convex proper and $L_1 : \mathbb{R}^3 \to \mathbb{R}\cup\{+\infty\}$ normal convex integrand. Note that in [18] is considered the particular case with $y_0 = 0$,

$U_{ad} = L^\infty(\Sigma)$ and L_1 does not depend on y. To see how this particular problem of minimizing (2.9) subject to (2.5)–(2.8), fits in our general framework, we shall consider $p = \infty$ and the following spaces: $H = L^2(\Omega)$, $V = H^1(\Omega)$, $E_1 = C(\bar\Omega)$, $U = L^\infty(\Gamma)$, $E(0,T) = C(]0,T]; C(\bar\Omega)) \cap W(0,T)$. Define $A(t) \equiv A \in \mathcal{L}(V,V^*)$ and $B(t) \equiv B \in \mathcal{L}(U,V^*)$ by

$$\langle Ay, z\rangle = \int_\Omega \Big(\sum_{i,j} a_{ij}\frac{\partial y}{\partial x_j}\frac{\partial z}{\partial x_i} + a_0 yz\Big)\,dx + \alpha\int_\Gamma y(\xi)z(\xi)\,d\Gamma$$

for all y, $z \in H^1(\Omega)$,

$$\langle Bu, z\rangle = \int_\Gamma u(\xi)z(\xi)\,d\Gamma.$$

Using Green's formula we have that (2.5)–(2.6) can be equivalently written as (2.1), (2.4). On the other hand $-A$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t) \in \mathcal{L}(V^*) : t \in [0,+\infty[\}$ in V^* given by

$$S(t)y = \sum_{k=1}^{\infty} e^{-\lambda_k t}\langle y, v_k\rangle v_k \text{ for all } y \in V^*$$

where λ_k (resp. v_k) denote the eigenvalues (resp. eigenfunctions) of the operator A and $v_k \in C^\infty(\bar\Omega) \subset V$. Moreover, the restrictions of $S(t)$ to V (resp. to H) are also strongly continuous semigroups in V (resp. in H). For a given $u \in L^\infty(\Sigma)$, the solution of (2.5)–(2.6) is

$$y(t) = S(t)y_0 + \int_0^t S(t-s)Bu(s)ds$$

and the last integral is a continuous function on $\bar Q$ (see [18]), and $t \mapsto S(t)y_0$ belongs to $C(]0,T], C(\overline{\Omega}))$ (see [5]). Hence $y \in E(0,T)$.

3. Duality results

Here we will briefly recall some basic results on convex duality and some recent results taken from [13] and [23] on duality in Fréchet spaces. Let X be a Fréchet space and let X^* be its (topological) dual space. The duality scalar product is denoted by $\langle\cdot,\cdot\rangle$ (or, if necessary, $\langle\cdot,\cdot\rangle_{XX^*}$). Let $\Gamma_0(X)$ be the set of all convex, proper (i.e with the range in $\mathbb{R}\cup\{+\infty\}$ not identically $+\infty$) and lower semicontinuous. The conjugate of a function $f : X \to \bar{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$ is the function $f^* : X^* \to \bar{\mathbb{R}}$ given by

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x)\}.$$

The domain of a function $f : X \to \overline{\mathbb{R}}$ is the set

$$\operatorname{dom} f = \{x \in X : \ f(x) < +\infty\}.$$

The indicator function of a subset S of X is given by $i_S(x) = 0$ on S and $i_S(x) = +\infty$ on $X \setminus S$. Consider now Fréchet spaces X, Y and $F \in \Gamma_0(X \times Y)$. The primal problem associated to the the perturbation function F is $\inf_{x\in X} F(x,0)$. The dual problem is given by $\sup_{y^*\in Y^*}(-F^*(0,y^*))$. The marginal function $\varphi : Y \to \bar{\mathbb{R}}$, $\varphi(y) = \inf_{x\in X} F(x,\, y)$

verifies $\varphi(0) = \inf_{x\in X} F(x,0)$, $\varphi^*(y^*) = F^*(0,y^*)$ and $\varphi^{**}(0) = \sup_{y^*\in Y^*}(-F^*(0,y^*)) = \sup_{y^*\in Y^*}(-\varphi^*(y^*))$. The convex-concave Lagrangian associated to the perturbation function F is the function $L : X\times Y^* \to \bar{\mathbb{R}}$ given by $L(x,y^*) = \inf_{y\in Y}(F(x,y) - \langle y, y^*\rangle)$. If the marginal function φ is lower semicontinuous at 0 with $\varphi(0)$ finite, then $\inf_{x\in X} F(x,0) = \sup_{y^*\in Y^*}(-F^*(0,y^*))$. On the other hand the existence of solutions to primal and dual problem with the same optimal value can be expressed via the Lagrangian as in the next theorem.

Theorem 3.1. *The following conditions are equivalent:*

(i) $(x_0, y_0^*) \in X \times Y^*$ *solves*

$$F(x_0,0) = \inf_{x\in X} F(x,0) \text{ and } F^*(0,y_0^*) = \sup_{y^*\in Y^*}(-F^*(0,y^*))$$

with $F(x_0,0) = -F^*(0,y_0^*)$*;*

(ii) (x_0, y_0^*) *is a saddle point of the Lagrangian, i.e.* $L(x_0,y^*) \le L(x_0,y_0^*) \le L(x,y_0^*)$ *for all* $(x,y^*) \in X\times Y^*$.

If we modify (i) *(respectively* (ii)*) adding the requirement* $F(x_0,0)$ *is finite (respectively* $L(x_0,y_0^*)$ *is finite), then another equivalent condition is*

(iii) $L(x_0,y_0^*)$ *is finite and* $L'_x((x_0,y_0^*);(x-x_0)) \ge 0$, $L'_y((x_0,y_0^*);(y^*-y_0^*)) \le 0$ *for all* $(x,y^*) \in X\times Y^*$.

Notice that the directional derivatives in (iii) exist by the fact that L is convex-concave. The next result will be very useful in the sequel (see [13]) or [23]. Observing that $\operatorname{dom}\varphi$ is the projection of $\operatorname{dom} F$ on Y, then Proposition 3.1 of [13] takes the following form.

Theorem 3.2. *Assume that* $\mathbb{R}_+(\operatorname{dom}\varphi)$ *is a closed subspace and that* $\varphi(0)$ *is finite. Then* $\inf_{x\in X} F(x,0) = \max_{y^*\in Y^*}(-F^*(0,y^*))$.

4. Optimality conditions

We go back to problem (P) stated in section 2. According to [15, page 116, Th 1.1 and Th 1.2], we have that the linear continuous operator $\mathcal{A} : W(0,T) \to L^2(0,T;V^*)\times H$ defined by

$$(\mathcal{A}y)(t) = \left(\frac{\mathrm{d}y}{\mathrm{d}t}(t) + A(t)y(t), y(0)\right)$$

is bijective (hence it is an isomorphism between topological vector spaces). Using our hypotheses we obtain that the restriction of $\mathcal{A}$ to $E(0,T)$ (which will be denoted by $\mathcal{A}$ too) establishes also a linear topological isomorphism from $E(0,T)$ to $F(0,T)\times H$. Let us consider also $\hat{f} = (f,0) \in L^2(0,T;\, V^*)\times H$ and the linear continuous operators $\mathcal{B} : L^2(0,T;\, U) \to L^2(0,T;\, V^*)\times H$ and $\mathcal{I} : W(0,T) \to L^2(0,T;\, V^*)\times H$ defined by $\mathcal{B}u = (B(\cdot)u(\cdot),0)$ and $\mathcal{I}(y) = (0,y(0))$. Hence the system (2.1) can be written equivalently as $(\mathcal{A}-\mathcal{I})y(\cdot) = \mathcal{B}u + \hat{f}$. Note that the restriction of $\mathcal{B}$ to $L^p(0,T;U)$ (denoted also by $\mathcal{B}$) is a linear continuous operator from $L^p(0,T;U)$ to $F(0,T)\times H$. Note also that $(\mathcal{A}-\mathcal{I})(y) - \mathcal{B}(u) = (\mathcal{T}(y,u),0)$ for all $(y,u) \in W(0,T)\times L^2(0,T;U)$. Let us consider the perturbation function $F : (E(0,T)\times L^p(0,T;U))\times Z(0,T) \to \mathbb{R}\cup\{+\infty\}$ defined by

$$F((y,u),g) = J(y,u) + i_{\mathcal{C}}((y,u),g),$$

where

$$\mathcal{C} = \{((y,u),g) \in \mathcal{M} \times Z(0,T) : \ \mathcal{T}(y,u) - f = g\}.$$

Note that $\mathcal{C}$ is a closed convex subset in $E(0,T) \times L^p(0,T;U) \times Z(0,T)$, hence $F \in \Gamma_0((E(0,T) \times L^p(0,T;U)) \times Z(0,T))$. The control problem (P) may be written equivalently as

$$\inf_{(y,u)\in E(0,T)\times L^p(0,T;U)} F((y,u),0)$$

and the associated Lagrangian $\mathcal{L} : (E(0,T) \times L^p(0,T;U)) \times Z(0,T)^* \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\mathcal{L}((y,u),q) = \begin{cases} J(y,u) - \langle \mathcal{T}(y,u) - f, q \rangle_{Z(0,T)Z(0,T)^*} & \text{if } (y,u) \in \mathcal{D} \\ +\infty & \text{if } (y,u) \notin \mathcal{D} \end{cases}$$

where $\mathcal{D} = \{(y,u) \in \mathcal{M} : \ \mathcal{T}(y,u) - f \in Z(0,T)\}$ is the projection of the set $\mathcal{C}$ on $E(0,T) \times L^p(0,T;U)$. Note that the dual space $L^2(0,T;X)^* = L^2(0,T;X^*)$ where X is a reflexive Banach space and for $x^*(\cdot) \in L^2(0,T;X^*)$, $x(\cdot) \in L^2(0,T;X)$ the duality product is given by

$$\langle x^*(\cdot), x(\cdot) \rangle = \int_0^T \langle x^*(t), x(t) \rangle_{X^*X} dt$$

(see [8]). Since $V^{**} = V$ and $Z(0,T) \subset F(0,T) \subset L^2(0,T;V^*)$, continuously and densely we have $L^2(0,T;V) \subset F(0,T)^* \subset Z(0,T)^*$ continuously. Thus the duality scalar product between $Z(0,T)$ and $Z(0,T)^*$ coincides with duality product between $F(0,T)$ and $F(0,T)^*$ on $Z(0,T)\times F(0,T)^*$, and it coincides with the duality scalar product between $L^2(0,T;V^*)$ and $L^2(0,T;V)$ on $(Z(0,T) \times (L^2(0,T;V))$ too. So we can denote all these scalar products by $\langle \cdot, \cdot \rangle$, and sometimes we will write formally such products $\langle x, x^* \rangle$ by $\int_0^T \langle x(t), x^*(t) \rangle \, dt$. Note also that $L^2(0,T;V) \subset L^2(0,T;H) \subset L^2(0,T;V^*)$ continuously and densely because we identify the Hilbert space $L^2(0,T;H)$ to its dual space. We will consider the adjoint operators $\mathcal{A}^* \in \mathcal{L}(F(0,T)^* \times H, \ E(0,T)^*)$ and $\mathcal{B}^* \in \mathcal{L}(F(0,T)^* \times H, \ L^p(0,T;U)^*)$.

Theorem 4.1.

(i) *A pair $(\bar{y},\bar{u}) \in \mathcal{M}$ is optimal for the problem (P) iff there exists $\bar{q} \in Z(0,T)^*$ such that*

$$\mathcal{T}(\bar{y},\bar{u}) = f, \tag{4.1}$$

$$J'((\bar{y},\bar{u});(y-\bar{y},u-\bar{u})) - \langle \mathcal{T}(y-\bar{y},u-\bar{u}), \bar{q} \rangle \geq 0 \text{ for all } (y,u) \in \mathcal{D}. \tag{4.2}$$

(ii) *If J is Gâteaux differentiable with respect to y then* (4.2) *is equivalent to the existence of some $\bar{p}_1 \in F(0;T)^* \times H$ such that*

$$\mathcal{A}^* \bar{p}_1 = J'_y(\bar{y},\bar{u}) \tag{4.3}$$

$$\langle \mathcal{A}(y-\bar{y}), \bar{p}_1 \rangle - \langle \mathcal{T}(y-\bar{y},u-\bar{u}), \bar{q} \rangle + J'_u((\bar{y},\bar{u});u-\bar{u}) \geq 0 \tag{4.4}$$

for all $(y,u) \in \mathcal{D}$, in which we denote by $J'_u((\bar{y},\bar{u});u-\bar{u})$ the directional derivative with respect to u and by $J'_y(\bar{y},\bar{u})$ the Gâteaux derivative with respect to y.

Moreover, if $Z(0,T) = F(0,T)$, or $\bar{q} \in F(0,T)^*$, or $Z(0,T) = L^2(0,T;H)$ and $B(t) \in \mathcal{L}(U,H)$ a.e., inequality (4.4) becomes decoupled in

$$\langle \mathcal{A}(y-\bar{y}), \bar{p}_1 - (\bar{q},0)\rangle \geq 0 \text{ and } J'_u((\bar{y},\bar{u}); u-\bar{u}) + \langle \mathcal{B}^*(\bar{q},0), u-\bar{u}\rangle \geq 0$$

for all $(y,u) \in \mathcal{D}$.

Proof. Let $g \in Z(0,T)$. Assumption (QA) implies that there is $(y,u) \in \mathcal{M}$ and $\lambda > 0$ such that

$$\lambda^{-1} g = \mathcal{T}(y,u) - f.$$

Thus $F((y,u),\lambda^{-1}g) = J(y,u) < \infty$, hence $\lambda^{-1}g \in \operatorname{dom}\varphi$, since the marginal function $\varphi : Z(0,T) \longrightarrow \mathbb{R} \cup \{+\infty\}$ is given, for all $g \in Z(0,T)$ by

$$\varphi(g) = \inf_{(y,u)\in E(0,T)\times L^p(0,T;U)} F((y,u),g)$$

with $\operatorname{dom}\varphi = \mathcal{T}(\mathcal{M}) - f$. Observe that $\varphi(g) = \inf_{(y,u)\in\mathcal{M}} J(y,u)$ for all $g \in \operatorname{dom}\varphi$ since J is finite over $\mathcal{M}$. This proves that $Z(0,T) = \mathbb{R}_+\big(\operatorname{dom}\varphi\big)$. By Theorem 3.2 we get that

$$\inf_{(y,u)\in E(0,T)\times L^p(0,T;U)} F((y,u),0) = \max_{q\in Z(0,T)^*} -F^*((0,0),q).$$

Hence, if $(\bar{y},\bar{u}) \in \mathcal{M}$ is optimal for the problem (P), then there exists $\bar{q} \in Z(0,T)^*$ which solves the dual problem (with a finite optimal value). Now using Theorem 3.1, we obtain $((\bar{y},\bar{u}),\bar{q}) \in (E(0,T) \times L^p(0,T;U)) \times Z(0,T)^*$ solves the primal and dual problems with a finite optimal value iff $((\bar{y},\bar{u}),\bar{q}) \in \mathcal{D} \times Z(0,T)^*$, (4.2) is fulfiled and

$$\langle \mathcal{T}(\bar{y},\bar{u}) - f, q - \bar{q}\rangle \geq 0 \text{ for all } q \in Z(0,T)^*.$$

The last inequality is equivalent to (4.1). Since the adjoint of a linear topological isomorphism is bijective, hence $\mathcal{A}^*$ is bijective, let $\bar{p}_1 \in F(0,T)^* \times H$ be the unique solution of (4.3). Then (4.2) is equivalent to (4.4). It is easy to see that under the supplementary conditions about $Z(0,T)$ or $\bar{q}$ we can decouple (4.4). □

By the surjectivity of $\mathcal{A}$ it is easy to see that, for a fixed $y_0 \in H$, we have $(\mathcal{A} - \mathcal{I})\{y \in E(0,T) : \ y(0) = y_0\} = F(0,T) \times \{0\}$. Thus, using also the fact that $\mathcal{A}$ is an open map, the following remarks are straightforward.

Remark 4.2. If there exists an admissible pair $(\hat{y},\hat{u}) \in \mathcal{M}$ (i.e., $(\hat{y},\hat{u})$ verifies (2.1)) such that $\hat{y} \in \operatorname{int}_{E(0,T)}\mathcal{M}^{-1}(u)$ (Slater's condition) then (QA) holds with $Z(0,T) = F(0,T)$.

Remark 4.3. If there are no state constraints, i.e., if $\mathcal{M} = \{(y,u) \in E(0,T) \times L^p(0,T;U) : y(0) \in K, u \in \mathcal{U}_{ad}\}$ where $K \subset H$ and $\mathcal{U}_{ad} \subset L^p(0,T;U)$ are closed and convex, then (QA) holds with $Z(0,T) = F(0,T)$. Moreover, if we put $\bar{p}_1 = (\bar{p},\bar{h}) \in F(0,T)^* \times H$ then (5.13) is equivalent to $\bar{q} = \bar{p}$ and

$$\langle \bar{h}, y(0) - \bar{y}(0)\rangle \geq 0.$$

5. Applications

First we will write some parts of Theorem 4.1 in a more explicit form. Assume that J, L and l are Gâteaux differentiable with respect to $y \in E(0,T)$, $y \in V$ and all the variables respectively, and for all y, $w \in E(0,T)$, u, $u_1 \in L^p(0,T;U)$,

$$\langle J'_y(y,u), w\rangle_{E(0,T)^*E(0,T)} = \int_0^T \langle L'_y(t,y(t),u(t)), w(t)\rangle_{V^*V} dt + \\ \langle l'_1(y(0),y(T)), w(0)\rangle_H + \langle l'_2(y(0),y(T)), w(T)\rangle_{E_1^*E_1}$$

(l'_i denotes the Gâteaux derivative with respect to the i-component, $i = 1, 2$). Of course, we have assumed that for all $y \in E(0,T)$, $u, u_1 \in L^p(0,T;U)$, the map $t \mapsto L'_y(t,y(t),u(t))$ belongs to $L^2(0,T;V^*)$, and the map $t \mapsto \langle L'_u(t,y(t),u(t)), u_1(t)\rangle_{U^*U}$ belongs to $L^1(0,T)$. Let us put $\bar{p}_1 = (\bar{p},\bar{h})$. Suppose $\bar{p} \in W(0,T)$ (recall that $W(0,T) \subset L^2(0,T;V) \subset F^*(0,T)$). Then the equation (4.3) can be written as

$$-\frac{\mathrm{d}\bar{p}}{\mathrm{d}t}(t) + A^*(t)\bar{p}(t) = L'_y(t,\bar{y}(t),\bar{u}(t)) \text{ a.e. in }]0,T[, \tag{5.1}$$

$$\bar{p}(T) = l'_2(\bar{y}(0),\bar{y}(T)), \tag{5.2}$$

and

$$\bar{h} = \bar{p}(0) + l'_1(\bar{y}(0),\bar{y}(T)). \tag{5.3}$$

Note that if $l'_2(\bar{y}(0),\bar{y}(T)) \in H$, (for example if l is defined and Gâteaux differentiable on $H \times H$) then (5.1), (5.2) has a unique solution $\bar{p}(\cdot) \in W(0,T)$ (see [15, Theorem 1.1, page 116] and change t in $T-t$).

However, in general we have $l'_2(\bar{y}(0),\bar{y}(T)) \in E_1^*$ (and $H \subset E_1^*$), so we will *define* the *generalized solution* $\bar{p}$ to (5.1), (5.2) as being the projection on $F(0,T)^*$ of the solution $\bar{p}_1$ to (4.3). In general $F(0,T)^*$ is not a subspace of the space $\mathcal{D}^*(]0,T[;V)$ of the distributions from $]0,T[$ to V, so the derivative $\dfrac{\mathrm{d}\bar{p}}{\mathrm{d}t}$ has no meaning in the sense of distributions. But, in the particular case when $E(0,T) = W(0,T)$ we have $F(0,T) = L^2(0,T;V^*)$, hence $F(0,T)^* = L^2(0,T;V) \subset \mathcal{D}^*(]0,T[;V)$ and the derivative $\dfrac{\mathrm{d}\bar{p}}{\mathrm{d}t}$ can be considered in the sense of distributions.

Also, each time we will consider the adjoint system in the form (5.1), (5.2) we will assume that its solution $\bar{p} \in C([0,T[\,;H)$. Thus (5.3) has a precise meaning. In the examples considered later this assumption is always fulfilled. Moreover, in the next subsection we will consider an important example (final boundary observation and boundary control) with $E(0,T) = C(]0,T];V) \cap W(0,T)$, when the solution to the adjoint system has a precise meaning and is continuously differentiable. Also, we can write *formally* (4.4) as

$$\int_0^T \langle \frac{\mathrm{d}(y-\bar{y})}{\mathrm{d}t}(t) + A(t)(y-\bar{y})(t), \bar{p}(t)\rangle \, dt - \\ \int_0^T \langle \frac{\mathrm{d}(y-\bar{y})}{\mathrm{d}t}(t) + A(t)(y-\bar{y})(t) - B(t)(u-\bar{u})(t), \bar{q}(t)\rangle \, dt + \\ + J'_u((\bar{y},\bar{u}); u-\bar{u}) + \langle y(0)-\bar{y}(0), \bar{p}(0) + l'_1(\bar{y}(0), \bar{y}(T))\rangle \geq 0$$

for all $(y,u) \in \mathcal{D}$.

5.1. A particular case with applications to a final boundary observation and boundary control

Let us consider the following particular problem. The operator A is time invariant ($A(t) \equiv A$) and verifies the coercitivity condition (2.3) with $\alpha = 0$. Also A is symmetric i.e.

$$\langle Av, w\rangle = \langle Aw, v\rangle$$

for all v, $w \in V$. Since $V \subset H \subset V^*$ we can consider A as un unbounded operator from V^* to itself defined on a dense subspace, hence an eigenvector is defined as usually. We denote by $\{v_n : n \in \mathbb{N}\} \subset V$ an orthonormal basis constituted of eigenvectors associated to an increasing unbounded sequence eigenvalues $\{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{R}_+$. Next, according to (2.3), we will renorm equivalently the space V (using the same notation for the new norm) by

$$\|v\|_V^2 = \langle Av, v\rangle = \sum_{n\in\mathbb{N}} \lambda_n \langle v, v_n\rangle^2$$

for all $v \in V$ The corresponding norm in V^* is given by

$$\|v^*\|_{V^*}^2 = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n}\langle v^*, v_n\rangle^2.$$

for all $v^* \in V^*$. Note that $V = \{v \in V^* : \sum_{n\in\mathbb{N}} \lambda_n\langle v, v_n\rangle^2 < +\infty\}$ and $H = \{v \in V^* : \sum_{n\in\mathbb{N}}\langle v, v_n\rangle^2 < +\infty\}$. Let us put for every $t \geq 0$ and $v \in V^*$, $S(t)v = \sum_{n\in\mathbb{N}} e^{-\lambda_n t}\langle v, v_n\rangle v_n$. It is easy to see that the family $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup of linear bounded operators in V^* (and in H or V as well) having as infinitesimal generator $-A$ (and the restrictions of $-A$ to $A^{-1}H$ or $A^{-1}V$ respectively).

Lemma 5.1.

(i) *For every $t > 0$ we have $S(t) \in \mathcal{L}(V^*, V)$ and $\frac{d}{dt}S(t)v = -AS(t)$ for all $v \in V^*$ for the strong topology of V.*

(ii) *For each $v \in V^*$, the mapping $t \mapsto S(t)v$ is indefinitely differentiable from $]0, +\infty[$ to V.*

Proof. The first part of this lemma results from

$$\sum_n \lambda_n\langle S(t)v, v_n\rangle^2 = \sum_n \lambda_n e^{-2\lambda_n t}\langle v, v_n\rangle^2 \leq C(t)\|v\|_{V^*}^2$$

with $C(t) = \sup_n \lambda_n^2 e^{-2\lambda_n t} < +\infty$. Thus the semigroup $S(t)$ (considered in V^*) is differentiable. The second part can be proved directly or using the properties of the differentiable semigroups (see e.g. [20]). □

Recall that for $f \in L^2(0,T;V^*)$ and $y_0 \in H$, the abstract Cauchy problem:

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + Ay(t) = f(t) \text{ a.e. in }]0,T[\tag{5.4}$$

$$y(0) = y_0 \tag{5.5}$$

has a unique solution in $W(0,T)$ [15, page 116, Th 1.1 and Th 1.2]. Moreover, the solution is given (a.e. in $]0,T[$) by

$$y(t) = S(t)y_0 + \int_0^t S(t-s)f(s)\,ds = S(t)y_0 + \sum_n \left(\int_0^t e^{-\lambda_n(t-s)}\langle f(s), v_n\rangle\,ds\right) v_n$$

the last series being uniformly convergent (with respect to $t \in [0,T]$) in H and convergent in $L^2(0,T;V)$. Our aim is to prove next that, considering two particular cases for the function f (which correspond to L^∞ boundary control and to L^2 distributed control) the corresponding solution belongs to $C(]0,T];V)$. Also we will consider the case with the initial condition $y_0 \in V^*$ which will allow us to treat explicitly the adjoint system. Recall that the mild solution to (5.4),(5.5) is the function given by $t \mapsto S(t)y_0 + \int_0^T S(t-s)f(s)ds$ which coincides in $L^2(0,T;V^*)$ with the function y given by (5.7).

Theorem 5.2. *Let*

$$f(t) = \sum_{i=1}^k u_i(t)w_i, \quad \textit{a.e. in }]0,T[\tag{5.6}$$

with $u_i \in L^\infty(0,T;\mathbb{R})$ and $w_i \in V^*$, or

$$f \in L^2(0,T;H).$$

Then, for $y_0 \in H$, the (unique) solution $y \in W(0,T)$ to the Cauchy problem (5.4), (5.5) *verifies*

$$y \in C^0(]0,T];V).$$

The solution is given by

$$y(t) = S(t)y_0 + \sum_n \left(\int_0^t e^{-\lambda_n(t-s)}\langle f(s), v_n\rangle ds\right) v_n \ \textit{a.e. in }]0,T[\tag{5.7}$$

the last series being uniformly convergent (with respect to $t \in [0,T]$) in V. Moreover, considering $y_0 \in V^*$, the function y given by (5.7) *is the unique mild solution to the problem* (5.4), (5.5) *and belongs to $C^0(]0,T];V) \cap C^0([0,T];V^*) \cap L^2(0,T;H)$.*

Proof. According to Lemma 5.1, it is sufficient to prove that the function

$$\varphi(t) = \sum_n \left(\int_0^t e^{-\lambda_n(t-s)}\langle f(s), v_n\rangle\,ds\right) v_n \text{ a.e. in }]0,T[$$

belongs to $C^0([0,T];V)$. If f verifies (5.6), then taking $M = \max_{1\le i\le k}\|u_i\|_{L^\infty}$ we have

$$\begin{aligned}\sum_n \lambda_n\Big(\int_0^t e^{-\lambda_n(t-s)}\langle f(s),v_n\rangle\,ds\Big)^2 &\le M^2\Big(\sum_n\sum_{i=1}^k \frac{1}{2\lambda_n}(1-e^{-2\lambda_n t})\langle w_i,v_n\rangle^2\Big)\\ &\le \frac{M^2}{2}\sum_{i=1}^k \|w_i\|^2_{V^*}\end{aligned}$$

for all $t\in[0,T]$, hence the series is uniformly convergent (with respect to $t\in[0,T]$) in V. If $f\in L^2(0,T;H)$ the uniform convergence of the same series results from

$$\begin{aligned}\sum_n \lambda_n\Big(\int_0^t e^{-\lambda_n(t-s)}\langle f(s),v_n\rangle\,ds\Big)^2 &\le \sum_n \lambda_n\int_0^t e^{-2\lambda_n(t-s)}\,ds\int_0^t\langle f(s),v_n\rangle^2\,ds\\ &\le \frac{1}{2}\|f\|^2_{L^2(0,T;H)}.\end{aligned}$$

In the case $y_0\in V^*$ the uniqueness of the mild solution is proved in the theory of semi-groups. □

Now we can give the abstract example with applications to the final boundary observation and (distributed or boundary) control. According to the last theorem, we obtain the following.

Theorem 5.3. *Suppose the operator A as considered in this subsection. If we take $E_1 = V$ and $E(0,T) = C(]0,T];V)\cap W(0,T)$, then*

$$\{f(t)=\sum_{i=1}^k u_i(t)w_i \text{ a.e. in }]0,T[:\ u_i\in L^\infty(0,T;\mathbb{R}),\ w_i\in V^*\}\subset F(0,T),$$

$$L^2(0,T;H)\subset F(0,T).$$

Corollary 5.4. *With A as above, if the operator $B(t)\equiv B\in\mathcal{L}(U,V^*)$ is given by $U=\mathbb{R}^k$, $Bu=\sum_{i=1}^k u_iw_i$ with $w_i\in V^*$, $i=1,\dots,k$, and considering $p=\infty$ we can take $E_1=V$ and $E(0,T)=C(]0,T];V)\cap W(0,T)$. On the other hand, if $B(t)\in\mathcal{L}(U,H)$ a.e. in $]0,T[$ such that $t\mapsto B(t)u(t)$ belongs to $L^2(0,T;H)$ for every $u\in L^2(0,T;U)$ then, considering $p=2$, we can also take $E_1=V$ and $E=C(]0,T];V)\cap W(0,T)$.*

Also we can better explain the adjoint system. Thus changing t in $T-t$ and using Theorem 5.2 we have the following.

Remark 5.5. With A as above and $E_1 = V$, the adjoint system (5.1), (5.2) has its mild solution in the space $C^0([0,T[;V)\cap C^0([0,T];V^*)$. The relation (5.3) is meaningful. However we cannot gurantee that its mild solution is an element of $F^*(0,T)$, i.e., we do not know if the mild solution coincides with the generalized solution.

Example 5.6. Now let us consider the operator $A\in\mathcal{L}(H^1(\Omega),(H^1(\Omega))^*)$ defined in subsection 2.3 with $a_0\ge 0$ and the following boundary control parabolic problem:

$$\frac{\partial y}{\partial t}(t,x)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}(t,x)\right)-a_0(x)y(x,t) \text{ in } Q \tag{5.8}$$

$$\frac{\partial y}{\partial \nu_A}(t,\xi) + \alpha y(t,\xi) = \sum_{i=1}^{k} u_i(t) g_i(\xi) \text{ in } \Sigma$$

$$y(0,x) = y_0(x) \text{ in } \Omega \tag{5.9}$$

The functions $g_i \in L^2(\Gamma)$, $i = 1, \dots, k$ are given. We consider the (usually unknown) initial condition $y_0 \in L^2(\Omega)$ and the controls $u_i \in L^\infty(0,T;\mathbb{R})$, $i = 1, \dots, k$. We will consider the following spaces $E_1 = V = H^1(\Omega)$, $H = L^2(\Omega)$, $U = \mathbb{R}^k$, $E(0,T) = C^0(]0,T];V) \cap W(0,T)$, and $p = \infty$. Defining the operator $B \in \mathcal{L}(\mathbb{R}^k, V^*)$ by

$$\langle Bu, v\rangle = \sum_{i=1}^{k} u_i \int_\Gamma g_i(\xi) v(\xi)\, d\Gamma = \sum_{i=1}^{k} u_i w_i$$

for all $v \in V$ where $w_i \in V^*$, $\langle w_i, v\rangle = \int_\Gamma g_i v\, d\Gamma$, we can write our parabolic problem in abstract form

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + Ay(t) = Bu(t) \text{ a.e. in }]0,T[, \tag{5.10}$$

$$y(0) = y_0. \tag{5.11}$$

Since the conditions of Theorem 5.3 and its corollary are fulfiled, we obtain that the choice of the final state space E_1 and the state space $E(0,T)$ is justified and we can consider the following cost functional:

$$J(y,u) = \int_{\Gamma_1} (y(T,\xi) - z^d(\xi))^2\, d\Gamma + \|u\|_\infty^2$$

where $\Gamma_1 \subset \Gamma$ is a C^∞ manifold of dimension $(n-1)$, $z^d \in L^2(\Gamma_1)$ is given and $\|\cdot\|_\infty$ is a norm in $L^\infty(0,T;\mathbb{R}^k)$ (e.g. $\|u\|_\infty = \text{ess sup}_{t\in[0,T]}|u(t)|$ where $|\cdot|$ is a euclidean norm in $\mathbb{R}^k$). This functional corresponds to a regional boundary final observation. Assume also that the phase constrains are of the form $(y,u) \in \mathcal{M} \subset E(0,T) \times L^\infty(0,T;\mathbb{R}^k)$, with $\mathcal{M}$ closed and convex and the constraint qualification (QA) is fulfiled . So, our problem (P) is to minimize J over the elements of $\mathcal{M}$ wich are verifying the problem (5.8)–(5.9). This problem has a unique optimal solution due to the fact that the functional $u \mapsto J(y_u, u)$, (with y_u the solution of (5.10), (5.11) associated to u) is convex, continuous and coercive on $L^\infty(0,T;\mathbb{R}^k)$, hence it is weakly * lower semicontinuous on the closed balls because a convex bounded set is closed iff it is weakly * closed. Thus, taking a minimizing sequence (u_n) with $(y_{u_n}, u_n) \in \mathcal{M}$, the coercitivity implies that (u_n) is bounded, hence we can find a cluster point for the weakly * topology which is the optimal control. In this case the adjoint system (5.1), (5.2) is given by

$$-\frac{\partial \bar p}{\partial t}(t,x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \bar p}{\partial x_j}(t,x)\right) - a_0(x)\bar p(t,x) \text{ in } Q$$

$$\frac{\partial \bar p}{\partial \nu_A}(t,\xi) + \alpha \bar p(t,\xi) = 0 \text{ in } \Sigma$$

$$\bar{p}(T,\cdot) = Az(\cdot),$$

where z is the solution of the elliptic problem

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}(x)\right) - a_0(x)z(x) = 0 \quad \text{in } Q$$

$$\frac{\partial z}{\partial \nu_A}(\xi) + \alpha z(\xi) = \begin{cases} (\bar{y}(T,\xi) - z^d(\xi) & \text{in } \Gamma_1 \\ 0 & \text{in } \Gamma \setminus \Gamma_1. \end{cases}$$

To see this consider the operator $C : V \mapsto L^2(\Gamma)$ $Cv = \chi_{\Gamma_1} v$ and note that

$$l(y(0), y(T)) = \|C(y(T) - z^d)\|^2_{L^2(\Gamma)},$$

hence the right hand side of (5.2) is (changing $\bar{p}$ in $2\bar{p}$) equal to $C^*C(y(T) - z^d)$. Since A is also the canonical isomorphism between the Hilbert space V (endowed with the scalar product $(v_1, v_2) \longmapsto \langle Av_1, v_2\rangle$) and its dual V^*, a simple computation gives the above systems.

Theorem 4.1 and Theorem 5.2, along with the fact that $\langle Az, v_n\rangle = \lambda_n\langle z, v_n\rangle$ and that we have $\langle Az, v\rangle = \int_{\Gamma_1}(\bar{y}(T,\xi) - z^d(\xi))v(\xi)\,d\Gamma$ by the definition of z, imply the following.

Theorem 5.7. *A pair $(\bar{y}, \bar{u}) \in \mathcal{M}$ is optimal for the problem (P) if and only if there exist $\bar{q} \in Z(0,T)^*$ and $\bar{p} \in F(0,T)^* \cap C^0([0,T[, V) \cap C^0([0,T], V^*) \cap L^2(0,T;H)$ such that*

$$\bar{y}(t,x) = \sum_n \left(e^{-\lambda_n t}\int_\Omega y_0 v_n + \sum_{i=1}^{k}\int_0^t e^{-\lambda_n(t-s)}\bar{u}_i(s)ds \int_\Gamma g_i(\xi)v_n(\xi)\,d\Gamma \right) v_n(x),$$

$$\bar{p}(t,x) = \sum_n e^{-\lambda_n(T-t)}\int_{\Gamma_1}(\bar{y}(T,\xi) - z^d(\xi))v_n(\xi)\,d\Gamma v_n(x),$$

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(y-\bar{y})(t), \bar{p}(t)\rangle\,dt -$$

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(y-\bar{y})(t) - B(u-\bar{u})(t), \bar{q}(t)\rangle\,dt + \tag{5.12}$$

$$\Phi((\bar{u}); (u-\bar{u})) \geq 0$$

for all $(y,u) \in \mathcal{D}$. $\Phi((\bar{u}); (u-\bar{u}))$ is the directional derivative of the convex map $u \mapsto \|u\|^2_\infty$. Moreover, if $Z(0,T) = F(0,T)$ or $\bar{q} \in F(0,T)^*$ or $Z(0,T) = L^2(0,T;H)$ and $B \in \mathcal{L}(U,H)$, the inequality (5.12) becomes decoupled in

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(y-\bar{y})(t), \bar{p}(t) - \bar{q}(t)\rangle\,dt \geq 0, \tag{5.13}$$

and

$$\Phi((\bar{u}); (u-\bar{u})) + \int_0^T \langle B^*\bar{q}(t), u(t) - \bar{u}(t)\rangle\,dt \geq 0$$

for all $(y,u) \in \mathcal{D}$.

5.2. The linear-quadratic problem with phase constraints

A particular and very important problem is the case when the cost function is quadratic

$$J(y,u) = \|C_1 y(T) - z_1^d\|^2_{Z_1} + \int_0^T \left(\|C_2(t)y(t) - z_2^d\|^2_{Z_2} + \langle N(t)u(t), u(t)\rangle\right) dt \qquad (5.14)$$

Unless otherwise stated we keep all the notations and hypotheses of the previous sections, but U is assumed a Hilbert space. Z_1, Z_2 are Hilbert spaces (output spaces) $C_1 \in \mathcal{L}(E_1, Z_1)$, $C_2(t) \in \mathcal{L}(V, Z_2)$, $N(t) \in \mathcal{L}(U,U)$, for all $t \in [0,T]$, the maps $t \mapsto \|C_2(t)\|_{\mathcal{L}(V,Z_2)}$, $t \mapsto \|N(t)\|_{\mathcal{L}(U)}$ belong to $L^\infty(0,T)$, $N(t)$ is self adjoint and positive. In this case all the hypotheses about J are fulfilled. Moreover J is Fréchet differentiable. We will denote by Λ_i (respectively Λ_U) the canonical isomorphism from Z_i to Z_i^*, $i \in \{1,2\}$ (resp. from U to U^*). Then, assuming (QA), Theorem 4.1 implies the following.

Theorem 5.8. *A pair $(\bar{y}, \bar{u}) \in \mathcal{M}$ is optimal for the problem* (5.14) *if and only if there exist $\bar{q} \in Z(0,T)^*$ and $\bar{p} \in F(0,T)^*$ such that*

$$\frac{d\bar{y}}{dt}(t) + A(t)\bar{y}(t) = B(t)\bar{u}(t) + f(t) \text{ a.e. in }]0, T[,$$

$$-\frac{dp}{dt}(t) + A^*(t)p(t) = C_2(t)^*\Lambda_2(C_2(t)(y(t)) - z_2^d(t)) \text{ a.e. in }]0,T[,$$

$$p(T) = C_1^*\Lambda_1(C_1(y(T)) - z_1^d),$$

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(t)(y-\bar{y})(t), \bar{p}(t)\rangle\, dt -$$

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(t)(y-\bar{y})(t) - B(t)(u-\bar{u})(t), \bar{q}(t)\rangle\, dt + \qquad (5.15)$$

$$+ \int_0^T \langle N(t)\bar{u}(t), u(t) - \bar{u}(t)\rangle_U\, dt + \langle y(0) - \bar{y}(0), \bar{p}(0)\rangle \geq 0$$

for all $(y,u) \in \mathcal{D}$. Moreover, if $Z(0,T) = F(0,T)$ or $\bar{q} \in F(0,T)^*$ or $Z(0,T) = L^2(0,T;H)$ and $B(t) \in \mathcal{L}(U,H)$, for all t, the inequality (5.15) *becomes decoupled in*

$$\int_0^T \langle \frac{d(y-\bar{y})}{dt}(t) + A(t)(y-\bar{y})(t), \bar{p}(t) - \bar{q}(t)\rangle\, dt + \langle y(0) - \bar{y}(0), \bar{p}(0)\rangle \geq 0,$$

and

$$\int_0^T \langle N(t)\bar{u}(t) + \Lambda_U^{-1}B^*(t)\bar{q}(t), u(t) - \bar{u}(t)\rangle_U\, dt \geq 0$$

for all $(y,u) \in \mathcal{D}$.

References

[1] D. Azé: On optimality conditions for state-constrained optimal control problems governed by partial differential equations, to appear in Ann. Ştint Univ. Al. I. Cuza Iaşi.

[2] D. Azé, S. Bolintinéanu: A boundary control problem with parabolic dynamic and final boundary observation, Proceedings of the 2nd Catalan Days on Applied Mathematics, Presses Universitaires de Perpignan (1995) 13–28.

[3] V. Barbu, Th. Precupanu: Convexity and Optimization in Banach Spaces, Mathematics and its Applications, East European Series, D. Reidel Publishing Company, 1985.

[4] M. Bergounioux, D. Tiba: General optimality conditions for constrained convex control problems, SIAM J. Control Opt. 34 (1995) 698–711.

[5] H. Brezis: Analyse Fonctionnelle, Masson, Paris, 1987.

[6] H. Dietrich: Duality theorems for an optimal control problem with a linear unbounded operator, SIAM J. Control Optimization 25 (1987) 1192–1201.

[7] E. Casas, L. A. Fernandez, J. Yong: Optimal control of quasilinear parabolic equations, J. Proc. R. Soc. Edinb. Sect. A125 3 (1994) 345–565.

[8] R. E. Edwards: Functional Analysis, Holt, Rinehart and Winston, New York, 1965.

[9] H. O. Fattorini: Optimal control problems with state constraints for semilinear distributed-parameter systems, J. Optimization Theory and Appl. 88 (1996) 25–59.

[10] H. O. Fattorini, H. Frankowska: Infinite dimensional control problems with state constraints, Proc. IFIP-IIASA Conf., Laxenburg, Austria 1989, Lecture Notes Control and Inf. Sci. 154 (1991) 52–62.

[11] Z. X. He: State constrained control problems governed by variational inequalities, SIAM J. Control and Optimization 25 (1987) 1119–1144.

[12] B. Hu, J. Yong: Pontryagin maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints, SIAM J. Control and Optimization 33 (1995) 1857–1880.

[13] V. Jeyakumar: Duality and infinite dimensional optimization, Nonlinear Analysis 15(12) (1990) 1111–1122.

[14] X. Li, J. Yong: Optimal Control Theory for Infinite Dimensional Systems, Birkhäuser, Basel, 1994.

[15] J.-L. Lions: Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles, Dunod, Gauthier-Villars, Paris, 1968.

[16] J-.L. Lions: Some Methods in the Mathematical Analysis of Systems and their Control, Science Press, Beijing China 1981, Gordon and Breach, Science Publishers, Inc., New York.

[17] J.-L. Lions: Optimal Control of Non Well-Posed Distributed Systems, Mathematical Control Theory, Banach Center Publication, Volume 14, PWN - Polish Scientific Publishers, Warsaw, 1985.

[18] U. Mackenroth: Convex parabolic control problems with pointwise state constraints, J. Math. Anal. Appl. 87 (1982) 256–277.

[19] M. Moussaoui, A. Seeger: Epsilon-maximum principle of Pontryagin's type and perturbation analysis of convex optimal control problems, SIAM J. Control Optimization 34 (1996) 407–427.

[20] Pazy: Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag 44, 1983.

[21] R. T. Rockafellar: Integrals which are convex functionals, II, Pacific J. Math. 39 (1971) 439–469.

[22] B. Rodrigues, S. Simons: Conjugate functions and subdifferentials in non normed situation, Nonlinear Analysis Theory Methods Appl. 12 (1988) 1069–1078.

[23] S. Simons: The occasional distributivity of $\bullet$ over $\underset{e}{+}$ and the change of variables for conjugate functions, Nonlinear Analysis Theory Methods Appl. 14 (1990) 1111–1120.

[24] R. Temam: Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag 68, 1988.

[25] Z. S. Wu, K. L. Teo: Optimal control problems involving second boundary value problems of parabolic type, SIAM J. Control Optimization 21 (1983) 729–757.

[26] H. Zerrik: Regional observability of distributed systems, Int. J. Systems Sci. 25 (1994) 301–313.