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Piecewise Affine Selections for Piecewise Polyhedral Multifunctions and Metric Projections

M. Finzel
Mathematisches Institut, Universität Erlangen-Nürnberg,
Bismarckstrasse 1 1/2, 91054 Erlangen, Germany.
e-mail: finzel@mi.uni-erlangen.de

W. Li
Department of Mathematics and Statistics, Old Dominion University,
Norfolk, VA 23529-0077, USA.
e-mail: wuli@math.odu.edu

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Piecewise polyhedral multifunctions are the set-valued version of piecewise affine functions. We investigate selections of piecewise polyhedral multifunctions, in particular, the least norm selection and continuous extremal point selections.

A special class of piecewise polyhedral multifunctions is the collection of metric projections $\Pi_{K,P}$ from $\mathbb{R}^n$ (endowed with a polyhedral norm $\|\cdot\|_P$) to a polyhedral subset K of $\mathbb{R}^n$. As a consequence, the two types of selections are piecewise affine selections for $\Pi_{K,P}$. Moreover, if $\Pi_{K,\infty}$ and $\Pi_{K,1}$ are the metric projection onto K in $\mathbb{R}^n$ endowed with the ℓ_∞-norm and the ℓ_1-norm, respectively, we prove that $\Pi_{K,1}$ has a piecewise affine and quasi-linear extremal point selection when K is a subspace, and that the strict best approximation $\mathbf{sba}_K(x)$ of x in K is a piecewise affine selection for $\Pi_{K,\infty}$.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional vector space (of column vectors). Piecewise affine functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ are useful in many applications. Early interest in this topic came from study of resistive networks [1, 27] and numerical solutions of nonlinear equations [6]. For recent work and references, see [5, 19, 30, 12, 13, 14]. Piecewise polyhedral multifunctions are the set-valued version of piecewise affine functions. One special class of such multifunctions is the one of metric projections $\Pi_{K,P}$ from $\mathbb{R}^n$ (endowed with a polyhedral norm $\|\cdot\|_P$) to a polyhedral subset K of $\mathbb{R}^n$. In many cases continuous selections of metric projections $\Pi_{K,P}$ inherit the polyhedral structure of the projections, in particular, they are piecewise affine. That is why we first study piecewise affine selections for piecewise polyhedral multifunctions. Our research was motivated by Mangasarian's least norm solution of a linear programming problem [23] as well as by special piecewise affine selections constructed for the metric projection $\Pi_{K,1}$ in $\mathbb{R}^n$ with the ℓ_1-norm, [8].

We will focus our attention to two types of piecewise affine selections of a piecewise polyhedral multifunction: the least norm solution introduced by Mangasarian [23] and continuous extremal point selections. The tools of Convex Analysis allow one to construct a continuous and piecewise affine selection of the extremal point mapping of a piecewise polyhedral mapping. Since the extremal point mapping of a piecewise polyhedral mapping has discrete image which is not convex at all, the construction of such a selection is not easy. Our construction leads to a linear programming problem. Moreover, the general

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theory on piecewise affine selections allows us to prove that the strict best approximation $\mathbf{sba}_K(x)$ of x in K is a piecewise affine selection for $\Pi_{K,\infty}$, extending the main results in [8] and [7], respectively.

In order to describe our results more precisely, we first give the necessary notation. Recall that a subset K of $\mathbb{R}^n$ is called a *polyhedral subset* of $\mathbb{R}^n$ if K is the intersection of finitely many closed halfspaces of $\mathbb{R}^n$. In other words, K is a polyhedral subset of $\mathbb{R}^n$ if and only if there exist a real $l \times n$ matrix A and a vector $b \in \mathbb{R}^l$ such that $K = \{x \in \mathbb{R}^n : Ax \geq b\}$, where $y \geq z$ means that each component of y is greater than or equal to the corresponding component of z. The interior, the boundary, the closure, and the extremal points of a subset K of $\mathbb{R}^n$ will be denoted by $\operatorname{int} K$, $\operatorname{bd} K$, $\operatorname{cl} K$, and $\operatorname{ext} K$, respectively. A subset K of a closed set X in $\mathbb{R}^n$ is said to be nowhere dense in X if $\operatorname{cl}(X \setminus \operatorname{cl} K) = X$ (or K has no relative interior point of X) [18, p. 4]. The nowhere dense sets we will use here are the boundary of a closed set and hence nowhere dense in $\mathbb{R}^n$ [18, p. 4]. These sets are also boundary sets in the sense of Kuratowski [20, p. 138]: A subset Y of a closed set X is said to be a boundary set with respect to X if its complement is a dense set in X, *i.e.*, if $\operatorname{cl}(X \setminus Y) = X$. However, in general, a boundary set in $\mathbb{R}^n$ is not necessarily the boundary of another set!

For a polyhedral subset D of $\mathbb{R}^n$ a collection of polyhedral subsets $\{D_1, \dots, D_k\}$ is called a *polyhedral subdivision* of D if the subsets D_i satisfy the following three conditions:

$$
\begin{aligned}
&D = \bigcup_{i=1}^{k} D_i, \\
&\dim(D_i) = \dim D \quad \text{for each } i, \quad \text{and} \\
&\dim(D_i \cap D_j) < \dim D \quad \text{for } i \neq j,
\end{aligned}
$$

where the dimension $\dim D$ is the dimension of the affine hull of D.

A function $f : D \to \mathbb{R}^m$ is called *piecewise affine* on D if there exists a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D such that f is affine on each D_i (cf. [5]).

In applications, it is not practical to verify that a function is piecewise affine on D by constructing a polyhedral subdivision of D. Scholtes [30] proved that a continuous function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ is piecewise affine if and only if there exist affine functions $f_1, \dots, f_r$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$
f(x) \in \{f_i(x) : 1 \leq i \leq r\} \quad \text{for } x \in \mathbb{R}^n. \tag{1.1}
$$

An analogue characterization holds if we replace $\mathbb{R}^n$ by a polyhedral subset D of $\mathbb{R}^n$. That is, any continuous "patching" of affine functions on D gives a piecewise affine function on D. As a consequence, f is a continuous and piecewise affine function on D if and only if there exist a polyhedral decomposition $D_1, \dots, D_k$ of D such that f is continuous on D and affine on each D_i. Moreover, a piecewise affine function is Lipschitz continuous.

Set-valued functions which correspond to point-valued affine functions are so-called polyhedral multifunctions. A mapping $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a multifunction if it maps each point x in $\mathbb{R}^n$ to a subset $F(x)$ of $\mathbb{R}^m$. Here $2^{\mathbb{R}^m}$ denotes the power set of $\mathbb{R}^m$, the collection of all subsets of $\mathbb{R}^m$. In particular, $F(x)$ might be empty.

A multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is said to be *Hausdorff continuous* on a subset D of $\mathbb{R}^n$, if

$$\lim_{y \to x, y \in D} \mathrm{H}(F(x), F(y)) = 0 \quad \text{for } x \in D,$$

where $\mathrm{H}(S,T)$ denotes the *Hausdorff distance* of two subsets S and T of $\mathbb{R}^m$:

$$\mathrm{H}(S,T) = \max\left\{ \sup_{x\in S} \inf_{y \in T} \|x-y\|, \sup_{y\in T} \inf_{x\in S} \|x-y\| \right\},$$

and $\|\cdot\|$ is the Euclidean 2-norm on $\mathbb{R}^m$. Here we assume $\mathrm{H}(S,\emptyset) = \mathrm{H}(\emptyset,T) = 0$.

A multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called a *polyhedral multifunction* if its graph $\{(x,y) : x \in \mathbb{R}^n, y \in F(x)\}$ is a polyhedral subset of $\mathbb{R}^n \times \mathbb{R}^m$. In particular, for each $x \in \mathbb{R}^n$, the image $F(x)$ is either empty or polyhedral. A polyhedral multifunction F is *Lipschitz continuous* on its *domain* $D := \{x : F(x) \neq \emptyset\}$ [33] (cf. also [16]), *i.e.*, there exists a positive constant λ (depending on F) such that

$$\mathrm{H}(F(x), F(y)) \le \lambda \cdot ||x-y|| \quad \text{for} \quad x, y \in D.$$

See [24] and [22] for explicit estimates of λ. Note, it is an immediate consequence of the definition of polyhedral multifunctions that its domain is a polyhedral subset of $\mathbb{R}^n$. Similar to piecewise affine functions, we define piecewise polyhedral multifunctions in terms of polyhedral multifunctions. A multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is called *piecewise polyhedral* if its domain $D := \{x : F(x) \neq \emptyset\}$ has a finite polyhedral subdivision $\{D_1, \ldots, D_k\}$ and F is a polyhedral multifunction on each D_i. If the domain D of a piecewise polyhedral multifunction F is a polyhedral set, then F is Lipschitz continuous on D (cf. the remark following Lemma 3.5).

One special class of piecewise polyhedral multifunctions is the collection of metric projections in $(\mathbb{R}^n, \|\cdot\|_P)$, where $\|\cdot\|_P$ denotes a polyhedral norm. Recall that a norm $\|\cdot\|_P$ on $\mathbb{R}^n$ is called *polyhedral* if the corresponding unit ball $\{x \in \mathbb{R}^n : \|x\|_P \le 1\}$ is polyhedral. A norm $\|\cdot\|_P$ on $\mathbb{R}^n$ is polyhedral if and only if there exist vectors $E_1, \ldots, E_r$ in $\mathbb{R}^n$ such that

$$\|x\|_P = \max_{1 \le i \le r} \langle E_i, x\rangle \quad \text{for } x \in \mathbb{R}^n. \tag{1.2}$$

The *metric projection* from $(\mathbb{R}^n, \|\cdot\|_P)$ to a polyhedral subset K in $\mathbb{R}^n$ is the multifunction defined by

$$\Pi_{K,P}(x) = \{y \in K : \|y-x\|_P = \mathrm{dist}(x,K)\} \quad \text{for } x \in \mathbb{R}^n,$$

where

$$\mathrm{dist}(x,K) := \min_{y \in K} \|y-x\|_P \tag{1.3}$$

is the distance of a point x in $(\mathbb{R}^n, \|\cdot\|_P)$ to K. The metric projection $\Pi_{K,P}$ maps each x in $\mathbb{R}^n$ to a bounded and nonempty polyhedral subset $\Pi_{K,P}(x)$ of K. Moreover, it is Lipschitz continuous [21].

Special polyhedral norms on $\mathbb{R}^n$ are the ℓ_1-norm, $\|z\|_1 := |z_1| + \cdots + |z_n|$, and the ℓ_∞-norm $\|z\|_\infty := \max\{|z_1|, \ldots, |z_n|\}$, where z_i denotes the i-th component of a vector z in $\mathbb{R}^n$. The corresponding metric projections are denoted by $\Pi_{K,1}$ and $\Pi_{K,\infty}$, respectively.

Next let us introduce some selections of a multifunction $F : D \subset \mathbb{R}^n \to 2^{\mathbb{R}^m}$. A *selection* of a multifunction F is a mapping $s : D \to \mathbb{R}^m$ such that $s(x) \in F(x)$ for all $x \in D$. See [2, 4] for surveys on various selections of multifunctions and metric projections. If F is a multifunction such that $F(x)$ is a nonempty closed convex subset of $\mathbb{R}^m$ for every x in D, then the *least norm selection* for F, denoted by **lns**, is defined by

$$\mathbf{lns}(x) := \arg\min_{y\in F(x)} \|y\| = \arg\min_{y\in F(x)} \frac{1}{2}\|y\|^2 \quad \text{for } x \in \mathbb{R}^n. \tag{1.4}$$

Since $F(x)$ is a closed convex subset of $\mathbb{R}^n$, there exists a unique solution of the strictly convex quadratic programming problem (1.4). Thus, $\mathbf{lns}(x)$ is well-defined.

Another type of selections of F is given by the *extremal point selections* s of F, which satisfy $s(x) \in \operatorname{ext} F(x)$ for $x \in D$. In contrast to the least norm selection, there exists a Lipschitz continuous multifunction F such that F has no continuous extremal point selection (cf. Example 3.1). One central result in this paper is the existence of a continuous extremal point selection for a piecewise polyhedral multifunction. Moreover, each continuous extremal point selection of a piecewise polyhedral multifunction is piecewise affine. This leads to piecewise affine and continuous selections, and to linear selections [3]. The later exist only if the metric projection satisfies very strong additional conditions [2, 4].

The paper is organized as follows. Section 2 contains some characterizations of piecewise affine functions. The central result of Section 3 is that the least norm selection of a piecewise polyhedral multifunction F is a piecewise affine selection of F. Section 4 is devoted to the study of continuous extremal point selections of a piecewise polyhedral multifunction F. Any continuous extremal point selection of F is a piecewise affine selection of F. Moreover, for any $\bar{y} \in \operatorname{ext} F(\bar{x})$, there is a continuous extremal point selection s of F such that $s(\bar{x}) = \bar{y}$. The main result of Section 5 is that the metric projection $\Pi_{K,P}$ from $(\mathbb{R}^n, \|\cdot\|_P)$ to a polyhedral subset K of $\mathbb{R}^n$ is a piecewise polyhedral multifunction, and the corresponding distance function is piecewise affine. In the last section, we apply the results of piecewise affine selections of piecewise polyhedral multifunctions to the metric projection $\Pi_{K,P}$ and obtain piecewise affine selections thereof. By the characterization of piecewise affine functions, the strict best approximation is a piecewise affine selection of $\Pi_{K,\infty}$. Moreover, there exists a continuous, piecewise affine, and quasi-linear extremal point selection of $\Pi_{G,1}$, where G is a subspace of $\mathbb{R}^n$. However, for the general metric projection $\Pi_{K,P}$, one cannot expect a continuous, piecewise affine, and quasi-linear extremal point selection for $\Pi_{K,P}$. We give a simple counterexample in $(\mathbb{R}^3, \|\cdot\|_\infty)$.

To conclude the introduction, we give some matrix and vector notations used in this paper. For a matrix A (or a vector x) A^T (or x^T) denotes the transpose of A (or x). For an index set J, let A_J (or x_J) be the submatrix (or subvector) of A (or x) consisting of the i-th rows (or the i-th components) of A where $i \in J$. If $J = \{i\}$, we also write A_i (or x_i) instead of A_J (or x_J).

2. Piecewise Affine Functions

In this section, we provide and establish several characterizations of piecewise affine functions (cf. Corollary 2.4). As an application, we recover a result of Sun [32] on the structure of differentiable piecewise quadratic functions.

First we prove that for each continuous function f satisfying (2.1) a polyhedral subdivision of D exists such that f is affine on each of these polyhedral subsets of D. Moreover, for given affine functions $f_1, \dots, f_r$ the polyhedral subdivision can be chosen independently of the particular function f.

Lemma 2.1. *Let D be a polyhedral subset of $\mathbb{R}^n$ and let $f_1, \dots, f_r$ be affine functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. There exists a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D such that any continuous function f from D to $\mathbb{R}^m$ satisfying*

$$f(x) \in \{f_i(x) : 1 \le i \le r\} \quad \textit{for } x \in D, \tag{2.1}$$

is an affine function on each D_j, $1 \le j \le k$.

Proof. First, we may assume $\dim D = n$. Otherwise an affine isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ exists such that $T(D) \subset \{x \in \mathbb{R}^n : x_i = 0,\ \dim D + 1 \le i \le n\}$, and we can reduce the problem to $\mathbb{R}^{\dim D}$.

Next we assume $f_i \not\equiv f_j$ for $1 \le i < j \le r$. Otherwise we may delete functions until this condition holds true.

Let $f_{i,1}(x), \dots, f_{i,m}(x)$ be the m components of $f_i(x)$. For $1 \le i < j \le r$ and $1 \le l \le m$, we divide D into two closed polyhedral subsets:

$$D^+_{ijl} := \{x \in D : f_{i,l}(x) - f_{j,l}(x) \ge 0\} \text{ and } D^-_{ijl} := \{x \in D : f_{i,l}(x) - f_{j,l}(x) \le 0\}.$$

Note that

$$f_i(x) = f_j(x) \quad \text{if and only if} \quad x \in \bigcap_{1 \le l \le m} (D^+_{ijl} \cap D^-_{ijl}). \tag{2.2}$$

Since $f_i \not\equiv f_j$, the set $\bigcap_{1 \le l \le m}(D^+_{ijl} \cap D^-_{ijl})$ is polyhedral and of dimension less than n. For each sign-mapping $\sigma : \{(i,j,l) : 1 \le i < j \le r, 1 \le l \le m\} \to \{-1, 1\}$, the set

$$D(\sigma) := \bigcap_{1 \le i < j \le k, 1 \le l \le m} D^{\sigma(i,j,l)}_{ijl}$$

is either empty or polyhedral. For each $x \in D$, we have $x \in D(\sigma)$ with $\sigma(i,j,l) = 1$ if $f_{i,l}(x) \ge f_{j,l}(x)$ and $\sigma(i,j,l) = -1$ otherwise. Thus,

$$D = \bigcup_{\sigma} D(\sigma). \tag{2.3}$$

Since D is a polyhedral set with nonempty interior, $D(\sigma)$ is nowhere dense in D whenever $\dim D(\sigma) < n$. By Baire's Theorem [18, p. 28] the complement of a countable union of nowhere dense sets is dense. Hence,

$$\operatorname{cl}\left[D \setminus \bigcup_{\dim D(\sigma) < n} D(\sigma)\right] = D.$$

Thus

$$D = \operatorname{cl}\left[D \setminus \bigcup_{\dim D(\sigma) < n} D(\sigma)\right] \subset \operatorname{cl}\left[\bigcup_{\dim D(\sigma) = n} D(\sigma)\right] = \bigcup_{\dim D(\sigma) = n} D(\sigma) = D,$$

where the second equality holds true because all the polyhedral sets $D(\sigma)$ are closed. So we obtain a decomposition of D:

$$D = \bigcup_{\dim D(\sigma)=n} D(\sigma).$$

Without loss of generality, we may assume that all sets $D(\sigma)$ above are pairwise distinct, since we can delete some redundant subsets $D(\sigma)$ until the remaining subsets are pairwise distinct. Now let $D(\sigma)$ and $D(\sigma')$ be two distinct polyhedral sets such that $\dim D(\sigma) = \dim D(\sigma') = n$. Since they are distinct, there exists a triple (i_0, j_0, l_0) of indices such that $D_{i_0 j_0 l_0}^{\sigma(i_0,j_0,l_0)} \neq D_{i_0 j_0 l_0}^{\sigma'(i_0,j_0,l_0)}$. This implies that $\sigma(i_0, j_0, l_0) = -\sigma'(i_0, j_0, l_0)$ and $f_{i_0,l_0} \neq f_{j_0,l_0}$. Therefore,

$$D_{i_0 j_0 l_0}^{\sigma(i_0,j_0,l_0)} \cap D_{i_0 j_0 l_0}^{\sigma'(i_0,j_0,l_0)} \subset \{x \in \mathbb{R}^n : f_{i_0,l_0}(x) = f_{j_0,l_0}(x)\},$$

which is a hyperplane. Therefore,

$$\dim(D(\sigma) \cap D(\sigma')) \leq \dim\left(D_{i_0 j_0 l_0}^{\sigma(i_0,j_0,l_0)} \cap D_{i_0 j_0 l_0}^{\sigma'(i_0,j_0,l_0)}\right) \leq n - 1.$$

Thus, $\{D(\sigma) : \dim(D(\sigma)) = n\}$ is a polyhedral subdivision of D.

It remains to show that any continuous function $f : D \to \mathbb{R}^m$ satisfying (2.1) is affine on each $D(\sigma)$ with $\dim D(\sigma) = n$. Choose $D^* = D(\sigma)$ with $\dim D^* = n$, *i.e.*, $\operatorname{int} D^* \neq \emptyset$. Next we claim that $f_i(x) \neq f_j(x)$ for $1 \leq i < j \leq r$ and $x \in \operatorname{int} D^*$. Assume there exists an $x^* \in \operatorname{int} D^*$ such that $f_i(x^*) = f_j(x^*)$ for two indices $i \neq j$. But by (2.2) and by the construction of the sets $D(\sigma)$, $\dim D(\sigma) = n$, the point x^* belongs to the boundary of D^*, which stands in contradiction to $x^* \in \operatorname{int} D^*$.

By (2.1) and by the continuity of f there exists exactly one index $i^* \in \{1, \ldots, r\}$ such that $f(x) = f_{i^*}(x)$ for all $x \in \operatorname{int} D^*$. Using the continuity of f once more we have $f(x) = f_{i^*}(x)$ for $x \in D^*$. Consequently, f is affine on each $D(\sigma)$ with $\dim D(\sigma) = n$. ☐

Note that if $f_1, \ldots, f_r$ are linear functions (instead of affine functions) and D is a polyhedral cone (instead of a polyhedral set), then each D_σ in the above proof is a polyhedral cone. As a consequence, we have the following variation of Lemma 2.1.

Corollary 2.2. *Let C be a polyhedral cone in $\mathbb{R}^n$, let $f : C \to \mathbb{R}^m$ be a continuous function, and let $f_1, \ldots, f_r$ be linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f(x) \in \{f_i(x) : 1 \leq i \leq r\}$ for $x \in C$. Then there exist polyhedral cones $C_1, \ldots, C_k$ such that*

(i) $C = \bigcup_{i=1}^k C_i$;
(ii) $\dim(C_i \cap C_j) < \dim D$ *for* $i \neq j$;
(iii) $\dim C_i = \dim C$ *for each* i;
(iv) f *is a linear function on each* C_i.

Lemma 2.1 can also be rephrased in terms of continuous selections.

Theorem 2.3. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a multifunction such that $D := \{x : F(x) \neq \emptyset\}$ is a polyhedral set and $F(x) \subset \{f_i(x) : 1 \leq i \leq r\}$ for $x \in D$, where $f_i : \mathbb{R}^n \to \mathbb{R}^m$ are affine functions. Then there exist at most finitely many continuous selections for F on D and each continuous selection of F is a piecewise affine function.*

Proof. By Lemma 2.1, there exists a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D such that each continuous function $f : D \to \mathbb{R}^m$ satisfying (2.1) is affine on each D_j. As a consequence, each continuous selection of F is piecewise affine.

Let σ be a mapping from $\{1, \dots, k\}$ to $\{1, \dots, r\}$ and define

$$f_\sigma(x) := f_{\sigma(j)}(x) \quad \text{for } x \in D_j,\ j = 1, \dots, k.$$

If $f : D \to \mathbb{R}^m$ is a continuous selection for F, then it is affine on each D_j. Since $f(x) \in \{f_i(x) : i = 1, \dots, r\}$, there is an index $i(j)$ (depending on f) such that $f(x) = f_{i(j)}(x)$ for $x \in D_j$. Defining $\sigma(j) := i(j)$ for $j = 1, \dots, k$ implies $f = f_\sigma$. Since there are at most r^k different mappings σ, F has at most r^k continuous selections. □

Finally, we give a list of characterizations of piecewise affine functions.

Corollary 2.4. *Let D be a polyhedral subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}^m$ be a continuous function. Then the following statements are equivalent.*

(i) *f is a piecewise affine function on D.*
(ii) *There exist polyhedral subsets $D_1, \dots, D_k$ of D such that $D = \bigcup_{i=1}^k D_i$ and f is an affine function on each D_i.*
(iii) *There exist affine functions $f_1, \dots, f_r$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f(x) \in \{f_i(x) : 1 \leq i \leq r\}$ for $x \in D$.*
(iv) *There exist piecewise affine functions $g_1, \dots, g_r$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f(x) \in \{g_i(x) : 1 \leq i \leq r\}$ for $x \in D$.*
(v) *There exist polyhedral subsets $D_1, \dots, D_k$ of D such that $D = \bigcup_{i=1}^k D_i$ and f is a piecewise affine function on each D_i.*

Proof. It is obvious that (i) implies (ii). Assume that (ii) holds. Let f_i be an affine function such that $f = f_i$ on D_i. Thus, (iii) holds (with $r = k$). This proves that (ii) implies (iii). It is trivial to see that (iii) implies (iv). Now we prove that (iv) implies (i). By the definition of piecewise affine functions, for each fixed i, there exist affine functions $g_{i,1}, \dots, g_{i,r_i}$ such that

$$g_i(x) \in \{g_{i,j}(x) : 1 \leq j \leq r_i\} \quad \text{for } x \in D.$$

Thus,

$$f(x) \in \{g_i(x) : 1 \leq i \leq r\} \subset \{g_{i,j}(x) : 1 \leq j \leq r_i, 1 \leq i \leq r\} \quad \text{for } x \in D.$$

By Lemma 2.1, f is a piecewise affine function on D.

Finally, we prove the equivalence of (i) and (v). Obviously, (i) implies (v). Now assume that (v) holds. By the definition of piecewise affine functions, for each fixed i, f satisfies (iv) for $D = D_i$, and hence for $D = \bigcup_{i=1}^k D_i$. Thus, (iv) holds. By the implication (iv)⇒(i), f is a piecewise affine function on D. □

Related to piecewise affine functions are piecewise quadratic functions. By using (2.1), one can give the following definition of piecewise quadratic functions [31].

Definition 2.5. Let D be a polyhedral subset of $\mathbb{R}^n$ and $g : D \to \mathbb{R}$ be a continuous function. We say that g is a piecewise quadratic function on D if there exist quadratic functions $g_1, \dots, g_r$ from D to $\mathbb{R}$ such that

$$g(x) \in \{g_i(x) : 1 \le i \le r\} \quad \text{for } x \in D. \tag{2.4}$$

In general, one cannot get a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D such that g is a quadratic function on each D_i. However, for continuously differentiable piecewise quadratic functions, there is a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D such that g is a quadratic function on each D_i [32]. Here we recover the following theorem by Sun [32] on the structure of differentiable piecewise quadratic functions as an application of Lemma 2.1.

Corollary 2.6. *Let D be a polyhedral subset of $\mathbb{R}^n$ and let $g : D \to \mathbb{R}$ be a continuously differentiable function on D. Then g is a piecewise quadratic function on D if and only if there exists a polyhedral subdivision $D_1, \dots, D_k$ of D such that g is a quadratic function on each D_i.*

Proof. We assume $\dim D = n$. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$, *i.e.*, e_i is the vector whose i-th component is 1 and other components are zero. Let $g_1, \dots, g_r$ be quadratic functions such that (2.4) holds. Fixed $x \in D$. By continuity of g and g_i's, $g(x) \ne g_i(x)$ implies $g(x + te_i) \ne g_i(x + te_i)$ for t near 0. By (2.4), for t near 0, we have

$$g(x + te_i) \in \{g_j(x + te_i) : g_j(x) = g(x)\},$$

which implies

$$\frac{g(x + te_i) - g(x)}{t} \in \left\{ \frac{g_j(x + te_i) - g_j(x)}{t} : g_j(x) = g(x) \right\}. \tag{2.5}$$

By differentiability of g and g_i's, letting $t \to 0$ in (2.5), we obtain

$$\frac{\partial g(x)}{\partial x_i} \in \left\{ \frac{\partial g_1(x)}{\partial x_i}, \dots, \frac{\partial g_r(x)}{\partial x_i} \right\}. \tag{2.6}$$

Let σ be a mapping from $\{1, \dots, n\}$ to $\{1, \dots, r\}$ and let $f_\sigma(x)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose i-th component is $\partial g_{\sigma(i)}(x)/\partial x_i$. By (2.6), for each fixed x and i, there is an index $j(i)$ such that

$$\frac{\partial g(x)}{\partial x_i} = \frac{\partial g_{j(i)}(x)}{\partial x_i}.$$

Let $\sigma(i) = j(i)$ for $i = 1, \dots, n$. Then $\nabla g(x) = f_\sigma(x)$, where $\nabla g(x)$ denotes the gradient of g at x and f_σ is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose i-th component is $\frac{\partial g_{j(i)}(x)}{\partial x_i}$. Thus,

$$\nabla g(x) \in \bigcup_\sigma \{f_\sigma(x)\}.$$

Since ∇g is continuous and f_σ are affine, by Lemma 2.1, ∇g is a piecewise affine function. Hence a polyhedral subdivision $\{D_1, \dots, D_k\}$ of D exists such that ∇g is an affine function on each D_i. As a consequence, g is a quadratic function on each D_i. The converse is trivially true. □

3. Extremal Point Selections

3.1. Remarks on Continuous Extremal Point Selections

In this section we study extremal point selections of piecewise polyhedral multifunctions. One reason to do so is that any continuous extremal point selection of F is a piecewise affine function. In studying Lipschitz continuity of polyhedral multifunctions, Walkup and Wets proved Lipschitz continuity of the extremal points of polyhedral multifunctions (even though they did not explicitly state it) [33]. Since $\operatorname{ext} F(x)$ is a non-convex and finite point set, there is no general theory about the existence of a continuous selection for $\operatorname{ext} F$. In fact, even a Lipschitz continuous multifunction $F : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$, where $F(x)$ is a line segment in $\mathbb{R}^2$ for each x and $\operatorname{ext} F$ is Lipschitz continuous on $\mathbb{R}^2$, does not need to have a continuous extremal point selection.

However, for a piecewise polyhedral multifunction F with $\operatorname{ext} F(x) \neq \emptyset$ for all x, we shall construct a vector which determines for each x a supporting hyperplane for $F(x)$ such that it intersects with $F(x)$ in a unique point $s(x) \in F(x)$, and $s(x)$ continuously moves along the graph of F. Therefore, $s(x)$ is an extremal point of $F(x)$ and s is a continuous function of x. This allows us to construct a continuous extremal point selection for a piecewise polyhedral multifunction. That is, we obtain one type of piecewise affine selections of a piecewise polyhedral multifunction F by constructing continuous extremal point selections of F.

Let us start with the counterexample mentioned above.

Example 3.1. Let $[x, y] := \{\theta x + (1 - \theta)y : 0 \leq \theta \leq 1\}$ for $x, y \in \mathbb{R}^2$ and let $\mathbf{0}$ be the origin of $\mathbb{R}^2$. We denote the projections to the coordinate axis by

$$P_1(x) := \begin{pmatrix} x_1 \\ 0 \end{pmatrix},$$
$$P_2(x) := \begin{pmatrix} 0 \\ x_2 \end{pmatrix}.$$

For $x \in \mathbb{R}^2$, define

$$F(x) := \begin{cases} [\mathbf{0}, x] & \text{if } x \geq \mathbf{0} \\ [P_1(x), P_2(x)] & \text{otherwise.} \end{cases}$$

Then the image $F(x)$ is a line segment in $\mathbb{R}^2$ for each x in $\mathbb{R}^2$, and F and $\operatorname{ext} F$ are Lipschitz continuous on $\mathbb{R}^2$, but F does not have any continuous extremal point selection.

Proof. If x is on the boundary of the first quadrant, then either $P_1(x) = \mathbf{0}$ or $P_2(x) = \mathbf{0}$. If $P_1(x) = \mathbf{0}$, then $P_2(x) = x$; if $P_2(x) = \mathbf{0}$, then $P_1(x) = x$. Note that

$$\operatorname{ext} F(x) := \begin{cases} \{\mathbf{0}, x\} & \text{if } x \geq \mathbf{0} \\ \{P_1(x), P_2(x)\} & \text{otherwise.} \end{cases}$$

From this it is not difficult to prove that F and $\operatorname{ext} F$ are Lipschitz continuous. We leave the details to interested readers.

However, F does not have any continuous extremal point selection. Assume to the contrary, there is such a selection, say s. Then either $s(x) = \mathbf{0}$ or $s(x) = x$ for x in the first quadrant. If $s(x) = \mathbf{0}$ for x in the first quadrant, it follows from the continuity of s that $s(x) = P_1(x)$ in the second quadrant, which implies $s(x) = P_1(x)$ in the remaining quadrants. By $s(x) = \mathbf{0}$ in the first quadrant and $s(x) = P_1(x)$ in the fourth quadrant, we conclude that s is not continuous. Similarly, we will get a contradiction if $s(x) = x$ in the first quadrant. □

Remark 3.2. This example demonstrates that in general continuous extremal point selections do not exist.

One should compare this example with Michael's selection theorem [25]: if Q is a lower semicontinuous multifunction and $Q(x)$ is a closed convex set for each x, then Q has a continuous selection. The above example shows the importance of convexity of $Q(x)$ in Michael's selection theorem.

3.2. A Continuous Extremal Point Selection

In this section we study extremal point selections of piecewise polyhedral multifunctions. Let us first state the central result.

Theorem 3.3. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a piecewise polyhedral multifunction such that $D := \{x : F(x) \neq \emptyset\}$ is a polyhedral subset of $\mathbb{R}^n$. For any $\bar{x} \in D$ and any $\bar{y} \in \operatorname{ext} F(x)$, there exists a continuous selection s of $\operatorname{ext} F$ on D with $s(\bar{x}) = \bar{y}$.*

In order to prove this theorem, we study polyhedral multifunctions more closely. Let $F : D \to 2^{\mathbb{R}^m}$ be a polyhedral multifunction, where $D \subset \mathbb{R}^n$ with $F(x) \neq \emptyset$ for $x \in D$. Clearly, the graph of F is a polyhedral subset of $\mathbb{R}^{n+m}$. In particular, there exist matrices $A \in \mathbb{R}^{l\times n}$, $Q \in \mathbb{R}^{l\times m}$, and a vector $b \in \mathbb{R}^l$, such that

$$\operatorname{graph}(F) = \{(x, y) : x \in D,\ y \in F(x)\} = \{(x, y) : Ax + Qy \leq b\}.$$

Consequently, the set $D \subset \mathbb{R}^n$ is the orthogonal projection of $\operatorname{graph}(F)$ to $\mathbb{R}^n$, and hence D is a polyhedral subset of $\mathbb{R}^n$. The representation of $\operatorname{graph}(F)$ implies for $x \in D$:

$$F(x) = \{y \in \mathbb{R}^m : Qy \leq b - Ax\};$$

i.e., the image $F(x)$ is a polyhedral subset of $\mathbb{R}^m$.

Similarly, if $F : D \to 2^{\mathbb{R}^m}$, is a piecewise polyhedral multifunction, where D is a polyhedral subset of $\mathbb{R}^n$, by its definition finitely many polyhedral subsets $D_1, \ldots, D_r$ of D exist such that $D = \bigcup_{i=1}^r D_i$, and the restriction of F to D_i is a polyhedral function, $1 \leq i \leq r$. In particular, there exist matrices $A^i \in \mathbb{R}^{l_i\times n}$, $Q^i \in \mathbb{R}^{l_i\times m}$, and vectors $b^i \in \mathbb{R}^{l_i}$, such that for $1 \leq i \leq r$

$$\operatorname{graph}(F|_{D_i}) = \{(x, y) : x \in D_i,\ y \in F(x)\} = \{(x, y) : A^i x + Q^i y \leq b^i,\ x \in D_i\}.$$

Next we give some characterizations of the existence of extremal points of $F(x)$:

Proposition 3.4. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a polyhedral multifunction with polyhedral domain $D \subset \mathbb{R}^n$. Under the notation given above we have*

$$\operatorname{ext} F(x) \neq \emptyset \quad \textit{for} \quad x \in D \quad \Longleftrightarrow \quad \operatorname{rank} Q = m.$$

If $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ *is piecewise polyhedral with domain* D*, we have*

$$\operatorname{ext} F(x) \neq \emptyset \quad \textit{for} \quad x \in D \quad \Longleftrightarrow \quad \operatorname{rank} Q^i = m \textit{ for an index } i \in \{1, \ldots, r\}.$$

Proof. First let F be polyhedral. By Grünbaum [11, p. 162] the extremal points of $F(x)$ are exactly the zero-dimensional faces of $F(x) = \{y \in \mathbb{R}^m : Qy \leq b - Ax\}$, $x \in D$, which are the intersection sets of m of the hyperplanes $\{y \in \mathbb{R}^m : Q_i y = b_i - A_i x\}, 1 \leq i \leq l$, where the corresponding normal vectors Q_i are linearly independent. So we have $\operatorname{ext} F(x) \neq \emptyset$ for $x \in D$ if and only if rank $Q = m$.

Now let F be piecewise polyhedral. Using the notation given above, it is sufficient to show that rank $Q^i = m$ implies rank $Q^j = m$ for all $1 \leq i, j \leq r$. Assume rank $Q^1 = m$. Since D is convex, there exists an index $j \in \{2, \ldots, r\}$ such that $D_1 \cap D_j \neq \emptyset$, say $j = 2$ and $x^* \in D_1 \cap D_2$. It suffices to prove rank $Q^2 = m$, because for an arbitrary index $j \in \{2, \ldots, r\}$ by the connectedness of D always a finite sequence $j_1 = 1, j_2, \ldots, j_k = j$ of indices in $\{1, \ldots, r\}$ exists such that $D_{j_{s-1}} \cap D_{j_s} \neq \emptyset$ for $l = 2, \ldots, k$, and hence rank $(Q^{j_{s-1}}) = m$ implies rank $(Q^{j_s}) = m$, $2 \leq s \leq k$. Consider

$$F(x^*) = \{y \in \mathbb{R}^m : Q^1 y \leq b^1 - A^1 x^*\} = \{y \in \mathbb{R}^m : Q^2 y \leq b^2 - A^2 x^*\}.$$

By rank $Q^1 = m$ and the first statement of Proposition 3.4 we have $\operatorname{ext} F(x^*) \neq \emptyset$. Consequently, by the same argument, we have rank $Q^2 = m$. $\square$

The following lemma is due to Walkup and Wets [33].

Lemma 3.5. *Let* $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ *be a polyhedral multifunction and* $\operatorname{ext} F(x)$ *be the set of extreme points of* $F(x)$ *for* $x \in \mathbb{R}^n$*. Then there exists a positive constant* λ *such that*

$$\mathrm{H}(\operatorname{ext} F(x), \operatorname{ext} F(y)) \leq \lambda \|x - y\| \quad \textit{for } x, y \in \mathbb{R}^n.$$

Remark 3.6. Obviously, the statement of Lemma 3.5 holds also true for piecewise polyhedral multifunctions with polyhedral domain D. So the extremal point mapping of any piecewise polyhedral multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is Lipschitz continuous on each D_i of the polyhedral subdivision of D, and by the convexity of D, on D.

Dealing with extremal points of $F(x)$, $x \in D$, makes sense only if $\operatorname{ext} F(x) \neq \emptyset$. So under the notation given above we assume rank $Q =$ rank $Q^i = m$, $1 \leq i \leq r$.

The representation of F given above shows that the extremal point mapping of F satisfies the conditions of Theorem 2.3. Consequently, each continuous extremal point selection of F behaves quite nicely as shown in the following theorem.

Theorem 3.7. *Let* $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ *be a piecewise polyhedral multifunction with polyhedral domain* $D \subset \mathbb{R}^n$*. Then each continuous extremal point selection of* F *is piecewise affine on* D*. There are only finitely many continuous extremal point selections of* F *on* D*.*

Proof. Since the multifunction F is piecewise polyhedral, there exists a subdivision $\{D_1, \ldots, D_r\}$ of D such that the graph of F on $D_i \times \mathbb{R}^m$ is a polyhedron. The notation given above gives a representation of the multifunction $F(x) = \{y \in \mathbb{R}^m : Q^i y \leq b^i - A^i x\}$ for $x \in D_i$, $1 \leq i \leq r$, and a representation of its extremal points:

$$\operatorname{ext} F(x) = \{y \in \mathbb{R}^m : \text{there exists } J \in \mathcal{J}_i \text{ with } \\ Q^i_J y = (b^i - A^i x)_J,\ Q^i y \leq b^i - A^i x\}, \quad (3.1)$$

where $\mathcal{J}_i$ is the collection of index sets $J \subset \{1, \dots, l_i\}$ for which Q^i_J is an $m \times m$ square nonsingular matrix. For each $J \in \mathcal{J}_i$, define

$$f^i_J(x) := [Q^i_J]^{-1}(b^i - A^i x)_J.$$

Then f^i_J is an affine mapping and from (3.1) we obtain

$$\operatorname{ext} F(x) \subset \{f^i_J(x) : J \in \mathcal{J}_i,\ 1 \le i \le r\} \quad \text{for } x \in D. \tag{3.2}$$

By Theorem 2.3, any continuous selection of $\operatorname{ext} F$ is piecewise affine on D, and there exist only finitely many continuous selections of $\operatorname{ext} F$ on D. □

Now we are going to prove Theorem 3.3 in two steps. First we prove the statement for a polyhedral multifunction F:

Theorem 3.8. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a polyhedral multifunction with domain $D \subset \mathbb{R}^n$. For any $\bar{x} \in D$ and any $\bar{y} \in \operatorname{ext} F(\bar{x})$ a continuous selection $s : D \to \mathbb{R}^m$ of F exists such that $s(\bar{x}) = \bar{y}$ and $s(x) \in \operatorname{ext} F(x)$ for all $x \in D$.*

Proof. Using the notation given above, by $\bar{y} \in \operatorname{ext} F(\bar{x})$, an index set $J \subset \{1, \dots, l\}$ exists such that J contains m indices, rank $Q_J = m$, and

$$\bar{y} = Q_J^{-1}(b_J - A_J \bar{x}).$$

In particular, rank $Q = m$. The cone

$$K(x) := \{y \in \mathbb{R}^m : Q_J y \le b_J - A_J x\}$$

contains $F(x)$ and has the vertex

$$v(x) := Q_J^{-1}(b_J - A_J x).$$

The corresponding normal cone N of $K(x)$ at the point $v(x)$ is given by

$$N = \operatorname{cone}\{Q_i^T : i \in J\} = \left\{ \sum_{i \in J} a_i Q_i^T : \alpha_i \ge 0,\ i \in J \right\},$$

i.e., we have

$$z^T(y - v(x)) \le 0 \qquad \text{for } z \in N,\ y \in K(x). \tag{3.3}$$

Because of rank $Q_J = m$ we see that $\operatorname{int} N \ne \emptyset$. Now we are going to choose a vector $u \in \operatorname{int} N$ such that it is not a normal vector of any proper face of $K(x)$ of dimension at least one. In other words, any supporting hyperplane of $K(x)$ with normal vector u will support $K(x)$ in a single point. So we consider the set of index sets $\mathcal{M} := \{M \subset \{1, \dots, l\} : M \text{ contains } m-1 \text{ indices}\}$. Since $\operatorname{span}\{Q_i^T : i \in M\}$ is a subspace of dimension at most $(m-1)$, it has no interior point (i.e., it is nowhere dense). So the finite union $\bigcup_{M \in \mathcal{M}} \operatorname{span}\{Q_i^T : i \in M\}$ has no interior points [18, Baire's Theorem, p. 27]. Consequently, a vector u exists with

$$0 \ne u \in \operatorname{int} N \setminus \bigcup_{M \in \mathcal{M}} \operatorname{span}\{Q_i^T : i \in M\}.$$

Let us assume u to be a unit vector, *i.e.*, $||u|| = 1$. Note that this vector u does not depend on the particular $x \in D$ we fixed. So for each $x \in D$ we have

$$v(x) \in H_u(x) := \{y \in \mathbb{R}^m : u^T y = u^T v(x)\}$$

and by (3.3)

$$F(x) \subset K(x) \subset \{y \in \mathbb{R}^m : u^T y \leq u^T v(x)\}. \tag{3.4}$$

Now we define an extremal point selection s on D with $s(\bar{x}) = \bar{y}$. By (3.4) we have $u^T y \leq u^T v(x)$ for all $y \in F(x)$. Because of this and since $F(x)$ is a polyhedral set, $\alpha(x) \geq 0$ exists such that

$$\alpha(x) := \max_{y \in F(x)} u^T y = \max_{Qy \leq b - Ax} u^T y. \tag{3.5}$$

Let $S(x)$ denote the solution set of the above linear programming problem for any fixed x. Then $S(x)$ is not empty. Note that $S(x)$ is the intersection of $F(x)$ and its supporting hyperplane $H(x) := \{y \in \mathbb{R}^m : u^T y = \alpha(x)\}$. If $S(x)$ is a face of $F(x)$ of dimension at least one, then u is a normal vector of this face, which stands in contrast to the way u was chosen. Thus, $S(x)$ contains only one point $s(x)$. Then it is well-known that the solution set of (3.5) is Lipschitz continuous with respect to the right-hand side perturbations (for example, see [24, 22]). Thus, $s(x)$ must be an extreme point of $F(x)$ and is Lipschitz continuous. From the choice of u, we also have $s(\bar{x}) = \bar{y}$. □

Remark 3.9. Geometrically speaking, the normal vector u, introduced in the proof, determines the selection s as follows: It is the normal vector of the hyperplane $H_u(\bar{x})$ in the y-space $\mathbb{R}^m$ and it supports the cone $K(\bar{x})$ as well as the polyhedron $F(\bar{x})$ at their joint vertex $v(\bar{x})$ such that the corresponding set of intersection contains the single point $v(\bar{x})$. Embedded in $\mathbb{R}^{n+m}$, $H_u(x)$ is an $(m-1)$-dimensional affine subspace. If we move $H_u(\bar{x})$ parallel to itself along the polyhedral graph of F it intersects $F(x)$ in the single point $s(x)$ for $x \in D$ — the way u was chosen makes sure that this set of intersection contains exactly one point.

Remark 3.10. The idea of using a normal vector u for determining a continuous extremal point selection of a polyhedral multifunction $F : D \to 2^{\mathbb{R}^m}$ may also work in a more general situation. The assumption that finitely many normal vectors exist such that for all $x \in D$ the set $F(x)$ is given as intersection of certain corresponding halfspaces determined by these normal vectors is not necessary.

It is sufficient to make sure that F is Hausdorff continuous and that a normal vector u exists such that a corresponding hyperplane supports $F(x)$ in a single point for all $x \in D$. For example, if F is Hausdorff continuous and if $F(x)$ is strictly convex for all $x \in D$ such a normal vector defines a continuous extremal point selection of F. However, we cannot expect the selection to be piecewise affine if F is not piecewise polyhedral.

Let us now complete the proof of Theorem 3.3. The idea of the second part of the proof is to start with the subpolyhedron D_i of D with $\bar{x} \in D_i$, where by Theorem 3.8 an extremal point selection s of F with $s(\bar{x}) = \bar{y}$ exists, and to extend s successively to neighboring subpolyhedra D_j of D via common points $x \in D_i \cap D_j$. In order to ensure that s is well-defined on the entire set D, a minor modification of the choice of the determining normal vector u is necessary.

Proof of Theorem 3.3. Using the notation given above, an index $i \in \{1, \ldots, r\}$ exists such that $\bar{x} \in D_i$, say $\bar{x} \in D_1$. By Theorem 3.8 a continuous selection $s : D_1 \to \mathbb{R}^m$ exists with $s(\bar{x}) = \bar{y}$ and $s(x) \in \operatorname{ext} F(x)$ for all $x \in D_1$. By the proof of Theorem 3.8 this selection is determined by a normal vector u of $F(\bar{x})$ at $\bar{y}$. In detail, u belongs to the interior of the normal cone N of $F(\bar{x})$ at $\bar{y}$, but not to the nowhere dense set $\bigcap_{M \in \mathcal{M}} \operatorname{span}\{[Q_i^1]^T : i \in M\}$ where $\mathcal{M}$ denotes the collection of all subsets of $\{1, \ldots, l_1\}$ which contain $m-1$ elements.

The minor modification of the choice of u mentioned above is to substitute the exceptional set given above by a bigger exceptional set which is still a nowhere dense set in $\mathbb{R}^m$. In order to do this, define index sets $\mathcal{M}_i := \{M \subset \{1, \ldots, l_i\} : M$ contains exactly $m-1$ indices$\}$, $i = 1, \ldots, r$. Again the finite union $\bigcup_{i=1}^r \bigcup_{M \in \mathcal{M}_i} \operatorname{span}\{[Q_j^i]^T : j \in M\}$ is a nowhere dense set of $\mathbb{R}^m$ (cf. [18, Baire's Theorem, p. 27]. So a vector u exists with

$$0 \neq u \in \operatorname{int} N \setminus \bigcup_{i=1}^{r} \bigcup_{M \in \mathcal{M}_i} \operatorname{span}\{[Q_j^i]^T : j \in M\}.$$

By an analogue argument used in the proof of Theorem 3.8 u cannot be a normal vector of any face of the polyhedron $F(x) = \{y \in \mathbb{R}^m : Q^i y \leq b^i - A^i x\}$, $x \in D_i$, $1 \leq i \leq r$.

Next we extend s to D and determine so a continuous extremal point selection on D with $s(\bar{x}) = \bar{y}$. Since D is convex, an index $i \in \{2, \ldots, r\}$ exists such that $D_1 \cap D_i \neq \emptyset$, say $x^1 \in D_1 \cap D_2$. By $x^1 \in D_1$, $s(x^1)$ is an extremal point of $F(x^1)$, and by the construction of s on D_1, u is a normal vector of $F(x^1)$ at $s(x^1)$. By $x^1 \in D_2$ and by Theorem 3.8, u determines a continuous selection $\tilde{s} : D_2 \to \mathbb{R}^m$ of F with $\tilde{s}(x) \in \operatorname{ext} F(x)$ for all $x \in D_2$ and $\tilde{s}(x^1) = s(x^1)$.

Since both selections are determined by the same normal vector u, we have $\tilde{s}(x) = s(x)$ for all $x \in D_1 \cap D_2$. In other words, s continuously has been extended to $D_1 \cup D_2$, u is a normal vector of $F(x)$ at $s(x)$ and the corresponding supporting hyperplane intersects with $F(x)$ in the single point $s(x)$ for all $x \in D_1 \cup D_2$. Repeating this process iteratively extends s to the entire set D. □

4. Least Norm Selections

It is well-known that for a Hausdorff continuous multifunction F the least norm selection is continuous, cf. [10, Theorem 1].

Lemma 4.1. *Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a Hausdorff continuous multifunction with closed domain $D \subset \mathbb{R}^n$ such that $F(x)$ is a closed convex subset of $\mathbb{R}^m$ for every $x \in D$. Then the least norm selection for F is continuous on D.*

When F is a piecewise polyhedral multifunction, the least norm selection is actually a piecewise affine and Lipschitz continuous function (cf. Theorem 4.4 below). In order to prove this we need the following well-known Karush-Kuhn-Tucker characterization of an optimal solution of a convex minimization problem with linear constraints.

Lemma 4.2 ([29, Corollary 28.3.1]). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function and consider the following convex minimization problem:*

$$\inf_z \; g(z) \text{ subject to } Az \geq b, \tag{4.1}$$

where A is an $l \times n$ matrix and $b \in \mathbb{R}^l$. Then $\bar{z}$ is a solution of (4.1) if and only if there exist an index set J and $w \in \mathbb{R}^l$ such that

$$A\bar{z} \geq b, \ \nabla g(\bar{z}) = A_J^T w_J, \ w_J \geq 0, \ A_J \bar{z} - b_J = 0. \tag{4.2}$$

Remark 4.3. The index set J is called the active set (of constraints) and w is the Lagrange multiplier corresponding to the inequality constraints $Az \geq b$.

Theorem 4.4. *Suppose that $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a piecewise polyhedral multifunction and its domain $D := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}$ is polyhedral. Then the least norm selection of F is a piecewise affine (and Lipschitz continuous) function on D.*

Proof. First assume that F is a polyhedral multifunction on D. So there exist an $l \times m$ matrix A, an $l \times n$ matrix Q, and a vector b in $\mathbb{R}^l$ such that

$$F(x) = \{y \in \mathbb{R}^m : Ay \geq b - Qx\} \quad \text{for } x \in \mathbb{R}^n. \tag{4.3}$$

Note that $F(x) = \emptyset$ for $x \notin D$. Fix $x \in D$. By the characterization of the least norm solution (cf. Lemma 4.2), we know that there exist an index set J and a vector $w \in \mathbb{R}^l$ such that

$$\begin{aligned} &\mathbf{lns}(x) = A_J^T w_J, \quad A_J\, \mathbf{lns}(x) - Q_J x - b_J = 0, \\ &w_J \geq 0, \quad \text{and} \ \ A\, \mathbf{lns}(x) - Qx - b \geq 0. \end{aligned}$$

Equivalently, we have $\mathbf{lns}(x) = A_J^T w_J$, where w_J satisfies the following conditions:

$$w_J \geq 0, \quad AA_J^T w_J - Qx - b \geq 0, \quad \text{and } A_J A_J^T w_J - Q_J x - b_J = 0. \tag{4.4}$$

For each index set J, let

$$\bar{D}_J = \{(x, z, w_J) : x \in D, z = A_J^T w_J \text{ and (4.4) holds}\},$$

$$D_J = \{x \in D : \text{ there exists } w_J \text{ such that (4.4) holds}\}.$$

By its definition and (4.4), $\bar{D}_J$ is a polyhedral set. Since D_J is the linear projection of $\bar{D}_J$ onto $\mathbb{R}^n$, D_J is also a polyhedral set. Moreover, (4.4) is a characterization of the least norm solution $\mathbf{lns}(x)$ and, for any x in D, there exist J and w_J such that (4.4) holds. Therefore,

$$D = \bigcup_J D_J.$$

For $J \in \mathcal{J}$ and $x \in D_J$, define

$$F_J(x) := \{z \in \mathbb{R}^m : \text{ there exists } w_J \text{ such that } (x, z, w_J) \in \bar{D}_J\}.$$

Note that the graph of F_J has the following form:

$$\text{graph}(F_J) = \{(x, z) : x \in D_J \text{ and there exists } w_J \text{ such that } (x, z, w_J) \in \bar{D}_J\}.$$

So it is the linear projection of the polyhedral set $\bar{D}_J$ onto $\mathbb{R}^n \times \mathbb{R}^m$ and, in particular, it is a polyhedral set. Thus, F_J is a polyhedral multifunction on D_J. Moreover, by the characterization (4.4) for $\mathbf{lns}(x) = A_J^T w_J$, we obtain

$$F_J(x) = \{\mathbf{lns}(x)\} \quad \text{for } x \in D_J.$$

Since $F_J(x)$ is a singleton, we have $F_J(x) = \operatorname{ext} F_J(x)$. Since **lns** is continuous (cf. Lemma 4.1) and $\mathbf{lns}(x) \in F_J(x) = \operatorname{ext} F_J(x)$, $\mathbf{lns}(x)$ is a continuous extremal point selection for the polyhedral multifunction F_J on D_J. By Theorem 3.7, **lns** is a piecewise affine function on D_J.

Now assume that F is a piecewise polyhedral multifunction. Let $D_1, \dots, D_k$ be polyhedral subsets of D such that $D = \bigcup_{i=1}^k D_i$ and F is a polyhedral multifunction on each D_i. By the previous proof, **lns** is a piecewise affine function on each D_i. By Corollary 2.4, **lns** is a piecewise affine function on D. $\square$

5. Metric Projections in Polyhedral Spaces

For a linear programming problem with inequality constraints $Ax \geq b$, the optimal function value is actually a piecewise affine and convex function of b as shown by Guddat, Hollatz, and Nozicka [15].

Lemma 5.1 ([15, Theorem 6.7]). *Let A be an $n \times m$ matrix and $c \in \mathbb{R}^m$. Define*

$$\varphi(b) := \min_{Ay \geq b} \langle c, y \rangle$$

and

$$C := \{b \in \mathbb{R}^n : \text{ there exists a vector } \bar{y} \text{ such that } A\bar{y} \geq b \text{ and } \varphi(b) > -\infty\} \neq \emptyset.$$

Then there exist finitely many polyhedral cones $C_1, \cdots, C_k$ such that

(i) $C = \bigcup_{i=1}^k C_i$;
(ii) $\operatorname{int}(C_i) \cap \operatorname{int}(C_j) = \emptyset$ *for* $i \neq j$;
(iii) $\operatorname{int}(C_i) \neq \emptyset$ *for each* i;
(iv) φ *is a linear function on each* C_i.

The reader might also prove this lemma using Corollary 2.2.

Theorem 5.2. *Let K be a polyhedral subset of $\mathbb{R}^n$ and $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^n$. Then the distance function $\operatorname{dist}(\cdot, K)$ defined in (1.3) is a piecewise affine function.*

Proof. Let $K = \{y \in \mathbb{R}^n : Ay \geq b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. By (1.2), we can rewrite (1.3) as

$$\operatorname{dist}(x, K) = \min_{\beta, y} \beta \quad \text{subject to} \quad Ay \geq b, \quad \text{and}$$
$$\langle E_i, x - y \rangle \leq \beta \text{ for } i = 1, \dots, r,$$

i.e.,

$$\begin{aligned} \operatorname{dist}(x, K) = \min_{\beta, y} \beta \quad &\text{subject to} \quad Ay \geq b, \quad \text{and} \\ &\langle E_i, y \rangle + \beta \geq \langle E_i, x \rangle \ \text{ for } i = 1, \dots, r. \end{aligned} \tag{5.1}$$

By Lemma 5.1, there is a piecewise affine function φ such that

$$\operatorname{dist}(x, K) = \varphi(b, \langle E_1, x \rangle, \dots, \langle E_r, x \rangle) \quad \text{for } x \in \mathbb{R}^n.$$

Thus, $\operatorname{dist}(x, K)$ is a piecewise affine function of x in $\mathbb{R}^n$. $\square$

Using the piecewise affine behavior of the distance function $\text{dist}(\cdot, K)$, it is easy to prove that the corresponding metric projection $\Pi_{K,P}$ is a piecewise polyhedral multifunction.

Theorem 5.3. *Let K be a polyhedral subset of $\mathbb{R}^n$ and $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^n$. Then the metric projection $\Pi_{K,P}$ is a piecewise polyhedral multifunction.*

Proof. Let $K = \{y \in \mathbb{R}^n : Ay \geq b\}$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then, by (1.2), we have

$$\begin{aligned}\Pi_{K,P}(x) &= \{z : Az \geq b, \|x - z\|_P \leq \text{dist}(x, K)\} \\ &= \{z : Az \geq b, \langle E_i, x - z\rangle \leq \text{dist}(x, K) \quad \text{for } 1 \leq i \leq r\}.\end{aligned}$$

By Theorem 5.2, the distance function $\text{dist}(x, K)$ is piecewise affine. Hence there exist finitely many polyhedral subsets $D_1, \ldots, D_k$ such that $\mathbb{R}^n = \bigcup_{j=1}^k D_j$ and the restriction of the distance function to each D_j is affine, *i.e.*, there exist $u^j \in \mathbb{R}^n$ and $\beta_j \in \mathbb{R}$ such that

$$\text{dist}(x, K) = \langle u^j, x\rangle + \beta_j \quad \text{for } x \in D_j,\ 1 \leq j \leq k.$$

Consequently, for $x \in D_j$, we have

$$\begin{aligned}\Pi_{K,P}(x) &= \{z : Az \leq b,\ \langle E_i, x - z\rangle \leq \langle u^j, x\rangle + \beta_j,\ 1 \leq i \leq r\} \\ &= \{z : Az \leq b,\ \langle E_i, z\rangle \geq \langle E_i - u^j, x\rangle - \beta_j,\ 1 \leq i \leq r\}.\end{aligned}$$

That is, $\Pi_{K,P}$ is a polyhedral multifunction on each D_j. Since $\Pi_{K,P}$ is Lipschitz continuous [21], $\Pi_{K,P}$ is a piecewise polyhedral multifunction. □

6. Piecewise Affine Selections for Metric Projections

6.1. Least Norm Selections for Metric Projections

In this section we study least norm selections of the metric projection $\Pi_{K,P}$ from $(\mathbb{R}^n, \|\cdot\|_P)$ to a polyhedral subset K of $\mathbb{R}^n$. Recall

$$\mathbf{lns}(x) = \arg \min_{y \in \Pi_{K,P}(x)} \frac{1}{2}\|y\|^2 \quad \text{for} \quad x \in \mathbb{R}^n.$$

It is easy to see that $0 = \mathbf{lns}(x)$ for all $x \in \mathbb{R}^n$ with $0 \in \Pi_{K,P}(x)$. Nürnberger [26] calls a selection which selects the origin whenever possible, *a selection with null-property (Nulleigenschaft)*. He introduced this notion when he characterized lower semi-continuity of upper semi-continuous metric projections to certain subspaces of a general normed space. In particular, he proved the existence of a continuous selection with Nulleigenschaft of the metric projection to a linear subspace of $\mathbb{R}^n$ endowed with an arbitrary norm. Lemma 4.1 extends this statement to the metric projection $\Pi_G : \mathbb{R}^n \to 2^G$, where G is a closed subset of $\mathbb{R}^n$ such that Π_G is Hausdorff continuous and $\Pi_G(x)$ is convex for all $x \in \mathbb{R}^n$.

In particular, if we restrict ourselves to the polyhedral case, by Lemma 4.1 and Theorem 4.4, we can obtain a piecewise affine selection of $\Pi_{K,P}$ by using the least norm selection of $\Pi_{K,P}$.

Theorem 6.1. *Let K be a polyhedral subset of $\mathbb{R}^n$ and $\|\cdot\|_P$ a polyhedral norm on $\mathbb{R}^n$. Then the least norm selection of $\Pi_{K,P}$ is piecewise affine, hence it is Lipschitz continuous.*

The least norm selection $\mathbf{lns}(x)$ is either 0 (when $0 \in \Pi_{K,P}(x)$) or a point on the relative boundary of $\Pi_{K,P}(x)$ (when $0 \notin \Pi_{K,P}(x)$). In general, $\mathbf{lns}(x)$ is not necessarily an extremal point of $\Pi_{K,P}(x)$, and it is different from any extremal point selection of $\Pi_{K,P}$.

Remark 6.2. Let K be a polyhedral subset of $\mathbb{R}^n$ and $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^n$. Then there exists a positive constant ϵ such that the least norm selection of $\Pi_{K,P}$ is the unique solution of the following quadratic programming problem:

$$\min_{y\in K} \left(\|x-y\|_P + \frac{\epsilon}{2}\|y\|^2\right) \quad \text{for } x \in \mathbb{R}^n. \tag{6.1}$$

This follows from Mangasarian's characterization of the least norm solution of a linear program [23]: the least norm solution of the linear programming problem

$$\min_{y\in C} c^T y$$

is the solution of the following quadratic programming problem

$$\min_{y\in C} \left(c^T y + \frac{\epsilon}{2}\|y\|^2\right),$$

where C is a polyhedral set and $\epsilon > 0$ is a small positive constant. By (1.2), for each fixed x, $\Pi_{K,P}(x)$ is the set of vectors y^* which solve the following linear programming problem:

$$\min \tau \quad \text{subject to} \quad y \in K, \langle E_i, x-y\rangle \le \tau \text{ for } i = 1, \dots, r.$$

Therefore, by Mangasarian's characterization, the least norm selection of $\Pi_{K,P}$ is the vector y^* that solves the following quadratic programming problem:

$$\min \left(\tau + \frac{\epsilon}{2}\|y\|^2\right) \quad \text{subject to} \quad y \in K, \langle E_i, x-y\rangle \le \tau \text{ for } i = 1, \dots, r,$$

which is an equivalent form of (6.1).

6.2. Extremal Point Selections for Metric Projections

In this section we investigate continuous extremal point selections for the metric projection $\Pi_{K,P}$. By Theorem 5.3, $\Pi_{K,P}$ is a piecewise polyhedral multifunction. Since $\Pi_{K,P}(x)$ is bounded, we have $\operatorname{ext}\Pi_{K,P}(x) \ne \emptyset$ for all x in $\mathbb{R}^n$. By Theorems 3.3 and 3.7, $\Pi_{K,P}$ always has a continuous extremal point selection, which is a piecewise affine selection.

Theorem 6.3. *Let K be a polyhedral subset of $\mathbb{R}^n$ and let $\|\cdot\|_P$ be a polyhedral norm on $\mathbb{R}^n$.*

(i) *For any $\bar{x} \in \mathbb{R}^n$ and any $\bar{y} \in \operatorname{ext}\Pi_{K,P}(\bar{x})$, there is a continuous extremal point selection s of the metric projection $\Pi_{K,P}$ on $\mathbb{R}^n$ with $s(\bar{x}) = \bar{y}$.*

(ii) *Any continuous extremal point selection for $\Pi_{K,P}$ is a piecewise affine (and Lipschitz continuous) selection.*

(iii) *There are only finitely many continuous extremal point selections of $\Pi_{K,P}$.*

Proof. By Theorem 5.3, $\Pi_{K,P}$ is a piecewise polyhedral multifunction on $\mathbb{R}^n$. Since $\Pi_{K,P}(x)$ is a bounded polyhedral set in $\mathbb{R}^n$, $\operatorname{ext}\Pi_{K,P}(x) \ne \emptyset$ for all $x \in \mathbb{R}^n$. Thus, the statements in Theorem 6.3 follow from Theorems 3.3 and 3.7. □

6.3. Quasi-linear Selections for Metric Projections

A multifunction $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is *homogeneous*, if

$$F(\alpha x) = \alpha F(x) \quad \text{for all} \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}.$$

For a linear subspace G of $\mathbb{R}^n$, the multifunction F is *quasi-additive* with respect to G, if

$$F(x+z) = F(x) + F(z) \quad \text{for all} \quad x \in \mathbb{R}^n,\ z \in G.$$

We call F *quasi-linear* with respect to G, if F is homogeneous and quasi-additive with respect to G. A mapping $s : \mathbb{R}^n \to \mathbb{R}^m$ is called a quasi-linear selection of F with respect to G if $s(x) \in F(x)$ for $x \in \mathbb{R}^n$ and if s is a quasi-linear mapping with respect to G.

Two decades ago Nürnberger [26] proved the existence of a quasi-linear selection of a quasi-linear multifunction F under some mild assumption on F.

Lemma 6.4. *Suppose that $F(x)$ is a closed convex subset of $\mathbb{R}^m$ for every $x \in \mathbb{R}^n$ and F is a quasi-linear multifunction with respect to a subspace G of $\mathbb{R}^n$ such that G is a set of fix-points of F (i.e., $F(z) = \{z\}$ for $z \in G$). Then F has a quasi-linear selection with respect to G.*

For a linear subspace G of $\mathbb{R}^n$, the metric projection Π_G from $\mathbb{R}^n$ (endowed with any norm) to G is quasi-linear with respect to G. Since $\Pi_{K,P}(z) = z$ for all $z \in G$, Nürnberger obtained the following as a corollary to Lemma 6.4.

Corollary 6.5. *The metric projection Π_G onto a linear subspace G of $\mathbb{R}^n$ endowed with any norm has a quasi-linear selection.*

For a metric projection $\Pi_{G,P}$ from $(\mathbb{R}^n, \|\cdot\|_P)$ to a linear subspace of G of $\mathbb{R}^n$, by Theorem 6.3, we know that $\Pi_{G,P}$ has a continuous and piecewise affine extremal point selection; and by Corollary 6.5, we know that $\Pi_{G,P}$ has a quasi-linear selection. Is it possible to construct a continuous, piecewise affine, and quasi-linear extremal point selection for $\Pi_{G,P}$? The next example shows that the answer is negative.

Example 6.6. Consider the linear subspace

$$G = \left\{ \theta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \theta \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

and $\|\cdot\|_P = \|\cdot\|_\infty$. Then $\Pi_{G,\infty}$ does not have a quasi-linear and continuous extremal point selection.

Proof. Note that $\operatorname{dist}(x, G) = \max\{|x_2|, |x_3|\}$. Consider the subset

$$D := \left\{ x \in \mathbb{R}^3 : x = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}, \|x\|_\infty = 1 \right\},$$

which is the boundary of the unit square in the hyperplane $\{x \in \mathbb{R}^3 : x_1 = 0\}$, and is orthogonal to G. All points x in D have the same set of best approximants in G:

$$\Pi_{G,\infty}(x) = \{\theta u : -1 \leq \theta \leq 1\}, \quad \text{where} \quad u := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

There are two choices of extremal point selections for $\Pi_{G,\infty}$ on D: u and $-u$. Let s be a continuous extremal point selection for $\Pi_{G,\infty}$. By the continuity of s, there exists a constant $\epsilon = \pm 1$ such that $s(x) = \epsilon u$ for $x \in D$. Let $v \in D$. Then $-v \in D$. But

$$s(-v) = \epsilon u \neq -\epsilon u = -s(v).$$

Hence s is not quasi-linear, since the homogeneity is not satisfied. □

However, when G is a linear subspace of $\mathbb{R}^n$ and $\|\cdot\|_P = \|\cdot\|_1$, Finzel [8] constructed a piecewise affine and quasi-linear extremal point selection of $\Pi_{G,1}$ by using a perturbation analysis.

6.4. Strict Best Approximation

The strict best approximation was introduced by Rice [28] as a selection for the metric projection onto a linear subspace of $(\mathbb{R}^n, \|\cdot\|_\infty)$ which chooses the so-called best element among the best approximants. This idea can easily be extended to selections for the metric projection to closed convex subsets of $(\mathbb{R}^n, \|\cdot\|_\infty)$. Two years ago, Huotari and Li [17] gave another characterization of the strict best approximation in terms of monotone rearrangement and lexicographic order of vectors. They proved that the strict best approximation of x in K, denoted by $\mathbf{sba}_K(x)$, is a continuous function of x if $\Pi_{K,\infty}$ is Hausdorff continuous [17]. In this case, the strict best approximation $\mathbf{sba}_K(x)$ is actually the limit of best ℓ_p-approximations $\Pi_{K,p}(x)$ as $p \to \infty$:

$$\mathbf{sba}_K(x) = \lim_{p \to \infty} \Pi_{K,p}(x) \quad \text{for } x \in \mathbb{R}^n, \tag{6.2}$$

where $\Pi_{K,p}(x)$ is the unique element in K such that

$$\|x - \Pi_{K,p}(x)\|_p = \min_{z \in K} \|x - z\|_p \tag{6.3}$$

and $\|z\|_p = (|z_1|^p + \ldots + |z_n|^p)^{\frac{1}{p}}$. Note that $\Pi_{K,P}$ is Lipschitz continuous [21] when K is a polyhedral set. As a consequence, $\mathbf{sba}_K(x)$ can also be defined by (6.2) if K is a polyhedral set. Recently, Finzel and Li [9] proved that $\mathbf{sba}_K$ is Lipschitz continuous when K is a polyhedral set. In this subsection, we prove that $\mathbf{sba}_K$ is piecewise affine, which implies the Lipschitz continuity of $\mathbf{sba}_K$, extending the following result proved by Finzel [7].

Lemma 6.7. *Let G be a subspace of $\mathbb{R}^n$. Then the strict best approximation $\mathbf{sba}_G$ is a piecewise affine function.*

Our approach is to represent $\mathbf{sba}_K$ in terms of $\mathbf{sba}_{G_i}$, where G_i denote those finitely many affine subspaces in $\mathbb{R}^n$ which are the affine hulls of the faces of K. For this purpose, we need the following lemma on the structure of $\Pi_{K,p}(x)$ [9].

Lemma 6.8. *Let K be a polyhedral subset of $\mathbb{R}^n$. Then there exist vectors $z^1, \dots, z^r$ in $\mathbb{R}^n$ and subspaces $G_1, \dots, G_r$ of $\mathbb{R}^n$ such that*

$$\Pi_{K,p}(x) \in \{z^1 + \Pi_{G_1,p}(x - z^1), \dots, z^r + \Pi_{G_r,p}(x - z^r)\} \qquad \text{for } x \in \mathbb{R}^n,\ 1 < p < \infty. \tag{6.4}$$

Theorem 6.9. *If K is a polyhedral subset of $\mathbb{R}^n$, then the strict best approximation $\mathbf{sba}_K$ is a piecewise affine selection for $\Pi_{K,\infty}$.*

Proof. By Lemma 6.8, there exist vectors $z^1, \dots, z^r$ in $\mathbb{R}^n$ and subspaces $G_1, \dots, G_r$ of $\mathbb{R}^n$ such that (6.4) holds. Since (6.2) holds for any polyhedral set, letting $p \to \infty$ in (6.4) we obtain

$$\mathbf{sba}_K(x) \in \{z^1 + \mathbf{sba}_{G_1}(x - z^1), \dots, z^r + \mathbf{sba}_{G_r}(x - z^r)\} \quad \text{for } x \in \mathbb{R}^n. \tag{6.5}$$

By Lemma 6.7, for each i, $g_i(x) := z^i + \mathbf{sba}_{G_i}(x - z^i)$ is a piecewise affine function. By Corollary 2.2, the strict best approximation $\mathbf{sba}_K$ is a piecewise affine function. $\square$

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