Journal of Convex Analysis
Volume 7 (2000), No. 1, 115–128

# Regular Maximal Monotone Operators and the Sum Theorem

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Received July 31, 1998
Revised manuscript received July 26, 1999

In this note, which is a continuation of [17], we study two classes of maximal monotone operators on general Banach spaces which we call $\mathcal{C}_0$ (resp. $\mathcal{C}_1$)-regular. All maximal monotone operators on a reflexive Banach space, all subdifferential operators, and all maximal monotone operators with domain the whole space are $\mathcal{C}_1$-regular and all linear maximal monotone operators are $\mathcal{C}_0$-regular. We prove that the sum of a $\mathcal{C}_0$ (or $\mathcal{C}_1$)-regular maximal monotone operator with a maximal monotone operator which is locally inf bounded and whose domain is closed and convex is again maximal monotone provided that they satisfy a certain "dom–dom" condition. From this result one can obtain most of the known sum theorem type results in general Banach spaces. We also prove a local boundedness type result for pairs of monotone operators.

## 1. Introduction

Let $X$ be a Banach space and $X^*$ be its dual, endowed with the dual norm. We shall denote by $B$ (resp. $B^*$) the unit ball of $X$ (resp. $X^*$). Let also $\langle \cdot, \cdot \rangle : X^* \times X \to R$ denote the usual evaluation map, i.e. $\langle x^*, x \rangle = x^*(x)$.

Recall that a multivalued map $T : X \rightrightarrows X^*$ is called a *monotone operator* if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $x, y \in X$, $x^* \in T(x)$, and $y^* \in T(y)$. The set of all $x \in X$ such that $T(x) \neq \emptyset$ is called the *domain* of $T$ and is denoted $D_T$. If $T$ is monotone and $A$ is a subset of $X$, a pair $(y, y^*) \in A \times X^*$ is called *monotonically related to* $T$ on $A$ if $\langle x^* - y^*, x - y \rangle \geq 0$ whenever $x \in A$ and $x^* \in T(x)$. The monotone operator $T$ is called *maximal on* $A$ if $y^* \in T(y)$ whenever the pair $(y, y^*)$ is monotonically related to $T$ on $A$. (If $A = X$, we shall omit $X$ from the above terminology.)

Let $T : X \rightrightarrows X^*$ be a monotone operator, $C$ be a closed convex subset of $X$, $x \in C$, and $x^* \in X^*$. Consider the following extended numbers (extended means that $+\infty$ is a possibility)

$$L_C(x, x^*, T) = 0 \vee \sup \left\{ \frac{\langle z^* - x^*, x - z \rangle}{\|x - z\|}; \quad z \in C,\ z \neq x,\ z^* \in T(z) \right\}$$

ISSN 0944-6532 / \$ 2.50 

$$M_C(x,x^*,T) = 0 \vee \inf_{u^*\in T(x)} \sup \left\{ \frac{\langle u^* - x^*, x - z\rangle}{\|x - z\|};\quad z \in C,\ z \neq x \right\}$$

(here $a \vee b = \max\{a,b\}$; we use the usual conventions that $\inf\emptyset = +\infty$ and $\sup\emptyset = -\infty$). When $C = X$ we shall denote $L_C(x,x^*,T)$ by $L(x,x^*,T)$ and $M_C(x,x^*,T)$ by $M(x,x^*,T)$.

In the particular case when $T$ is the subdifferential of a lower semicontinuous convex function and $C = X$, these numbers were introduced by Simons [11] who proved that they are equal to each other. In [17] we proved that these numbers are also equal when $\mathrm{co}(D_T) - C$ absorbs $\overline{\mathrm{lin}(D_T - C)}$ and either $X$ is reflexive and $T$ is maximal monotone or $T$ is the subdifferential of a proper, convex, lower semicontinuous function on $X$. (Here "co" stands for "convex hull of" and "lin" stands for "linear span of".) As a matter of fact, we proved their equality in a more general setting. Before stating this result we need to introduce some notation. For $\lambda \geq 0$, let $g_{\lambda,x} : X \to R$ be defined by $g_{\lambda,x}(u) = \lambda\|u - x\|$. Let also $I_C$ denote the indicator function of $C$ (i.e. $I_C(z) = 0$ if $z \in C$ and $I_C(z) = +\infty$ otherwise). The following theorem was proved in [17] (see Theorem 1):

**Theorem 1.1.** *Let $C \subseteq X$ be nonempty, closed, convex, $T : X \rightrightarrows X^*$ be monotone, and $x \in X$. Assume that $T + \partial g_{\lambda,x} + \partial I_C$ is maximal monotone for any $\lambda \geq 0$. Then $L(x,x^*,T + \partial I_C) = M(x,x^*,T + \partial I_C)$. If $x \in C$, then $L_C(x,x^*,T) = M_C(x,x^*,T)$.*

For any monotone operator $T : X \rightrightarrows X^*$ define

$$\mathcal{C}_0(T) = \{C \subseteq X; C \text{ is closed, convex, and } D_T \cap \mathrm{int}(C) \neq \emptyset\}$$
$$\mathcal{C}_1(T) = \Big\{C \subseteq X; C \text{ is closed, convex, and } \bigcup_{\lambda>0} \lambda(\mathrm{co}(D_T) - C) = \overline{\mathrm{lin}(D_T - C)}\Big\}$$

Clearly $\mathcal{C}_0(T) \subseteq \mathcal{C}_1(T)$.

**Definition 1.2.**

(1) A maximal monotone operator $T : X \rightrightarrows X^*$ is called *$X$-regular* if $L(x,x^*,T) = M(x,x^*,T)$ for any $x \in X$ and $x^* \in X^*$.

(2) Let $i \in \{0,1\}$. A maximal monotone operator $T : X \rightrightarrows X^*$ is called *$\mathcal{C}_i$-regular* if $L(x,x^*,T + \partial I_C) = M(x,x^*,T + \partial I_C)$ for any $x \in X$, $x^* \in X^*$ and any $C \in \mathcal{C}_i(T)$.

**Remark 1.3.** (1) Of course the statement "*$T$ is $\mathcal{C}_1$-regular*" implies that "*$T$ is $\mathcal{C}_0$-regular*" which in turn implies that "*$T$ is $X$-regular*".

(2) If $T$ is $\mathcal{C}_1$-regular and $C \in \mathcal{C}_1(T)$ then $D_T \cap C \neq \emptyset$. Indeed, if $D_T \cap C = \emptyset$, then $L_X(x,x^*,T + \partial I_C) = 0 \vee \sup\emptyset = 0 \vee \{-\infty\} = 0$ and thus $M_X(x,x^*,T + \partial I_C) = 0$, implying that $x \in D_T \cap C$ for any $(x,x^*) \in X \times X^*$, which is not true.

In [17], as a consequence of Theorem 1.1, we proved that *maximal monotone operators on reflexive Banach spaces as well as subdifferential operators (on any Banach space) are $\mathcal{C}_1$-regular.*

Here are other results that we proved in [17] and will be used in this paper:

(i) *If $C \subseteq X$ is closed and convex and $x \in C$ then $L_C(x,x^*,T) = L(x,x^*,T + \partial I_C)$ and $M_C(x,x^*,T) = M(x,x^*,T + \partial I_C)$.*

(ii) *$M(x, x^*, T) < +\infty$ if and only if $x \in D_T$.*
(iii) *$M(x, x^*, T) = 0$ if and only if $x^* \in \overline{T(x)}$.*
(iv) *If $T$ is $X$-regular then $\overline{D_T}$ is convex.*
(v) *If $T$ is $X$-regular and $x \in \overline{D_T}$ then $T$ is locally bounded at $x$ if and only if $x \in \mathrm{int}(D_T)$.*
(vi) *A $\mathcal{C}_0$-regular maximal monotone operator is maximal monotone locally (i.e. it is maximal monotone on any open convex subset of $X$ which intersects its domain).*

One of the important open problems in convex analysis is to find conditions under which the sum $T+S$ of two maximal monotone operators $T$ and $S$ is maximal monotone. When $X$ is reflexive, Rockafellar [10] proved that a sufficient condition for the "sum theorem" to be true is that $\mathrm{int}(D_T) \cap D_S$ be nonempty. This condition was relaxed by several authors (see [1], [3], [13]), the apparently least restrictive one being due to Simons (Theorem 26 in [13]) who proved that the sum theorem is true whenever $S$ and $T$ satisfy a certain *$\chi$ constraint qualification* (for example if $\mathrm{co}(D_T) - \mathrm{co}(D_S)$ absorbs $\overline{\mathrm{lin}(D_T - D_S)}$. Later, in [14], Simons proved that these less restrictive conditions (his and the other ones) are equivalent to each other. The proofs of all these results rely heavily on the assumption that $X$ is a reflexive Banach space. When $X$ is a Banach space, not necessarily reflexive, the sum theorem is known to be true only in a few particular cases (see Chapter IX in [15]). Here are some of them:

(a) $D_T = X = D_S$ (due to M. Heisler; see Section 3 in [5] or Theorem 40.4 in [15]).
(b) both $T$ and $S$ are linear and $D_S = X$ (see Theorem 7.2 in [7] or Theorem 37.1 in [15]).
(c) $T$ is the subdifferential of a proper lower semicontinuous convex function on $X$ and $S$ is linear with $D_S = X$ (due to H. Bauschke; see Theorem 42.2 in [15]).

It is our aim in this paper to prove the sum theorem (in any Banach space $X$) in the case when $T$ is $\mathcal{C}_i$-regular ($i \in \{0, 1\}$), $S$ is locally inf bounded (see the definition in the next section), $D_S \in \mathcal{C}_i(T)$, and $D_T$ and $D_S$ satisfy an additional condition. Since we shall also prove that a maximal monotone operator whose domain is $X$ (or which is linear or which is the subdifferential of a proper lower semicontinuous convex function) is $\mathcal{C}_0$-regular, and since a maximal monotone operator whose domain is $X$ is locally inf bounded, our result is a generalization of (a), (b), and (c).

Finally we would like to mention that one can also introduce dual numbers

$$L_V^*(x, x^*, T) = 0 \vee \sup\left\{\frac{\langle z^* - x^*, x - z\rangle}{\|x^* - z^*\|};\ \ z^* \in V,\ z^* \neq x^*,\ z^* \in T(z)\right\}$$

$$M_V^*(x, x^*, T) = 0 \vee \inf_{(z, x^*) \in \mathcal{G}(T)} \sup\left\{\frac{\langle u^* - x^*, x - z\rangle}{\|x^* - z^*\|};\ \ z^* \in V,\ z^* \neq x^*\right\}$$

for any $(x, x^*) \in X \times X^*$, $V \subseteq X^*$. We shall study properties of these numbers in another paper.

## 2. A sum theorem

We begin with a construction and a lemma that will be useful in reducing statements about monotone operators on $X$ to statements about monotone operators on a closed subspace.

Given $T : X \rightrightarrows X^*$ and $Y \subseteq X$ a closed subspace, define $T|Y : Y \rightrightarrows Y^*$ by

$$(T|Y)(y) = \{y^* \in Y^*;\ \text{there exists } x^* \in T(y) \text{ such that } y^* = x^*|Y\}.$$

If $C$ is a closed convex subset of $Y$, in addition to $I_C : X \to R \cup \{+\infty\}$ we shall also consider $I_{C,Y} : Y \to R \cup \{+\infty\}$, the indicator function of $C$ in $Y$, and its subdifferential $\partial I_{C,Y} : Y \rightrightarrows Y^*$. It is easy to verify that $\partial I_{C,Y} = (\partial I_C)|Y$.

**Lemma 2.1.** *Let $T : X \rightrightarrows X^*$ be monotone, $Y \subseteq X$ be a closed subspace such that $D_T \subseteq Y$. Let also $C$ be a closed convex subset of $Y$ and $x \in Y$. Then*

(a) *$T$ is maximal monotone $\Longleftrightarrow T = T + \partial I_Y$ and $T|Y : Y \rightrightarrows Y^*$ is maximal monotone.*
(b) *$L_X(x, x^*, T + \partial I_C) = L_Y(x, x^*|Y, T|Y + \partial I_{C,Y})$.*
(c) *$M_X(x, x^*, T + \partial I_C) = M_Y(x, x^*|Y, T|Y + \partial I_{C,Y})$.*

**Proof.** (a) See Lemma 25 in [13]. To prove (b) notice that, since $C \subseteq Y$, $\partial I_C = \partial I_C + \partial I_Y$ and therefore

$$\begin{aligned} L_X(x, x^*, T + \partial I_C) &= L_X(x, x^*, T + \partial I_C + \partial I_Y) \\ &= L_Y(x, x^*, T + \partial I_C) = L_Y(x, x^*|Y, (T + \partial I_C)|Y) \\ &= L_Y(x, x^*|Y, T|Y + (\partial I_C)|Y) = L_Y(x, x^*|Y, T|Y + \partial I_{C,Y}) \end{aligned}$$

the second equality following from Lemma 3 in [17] and the third one from the definitions. The proof of (c) is similar. □

We recall now a construction and a related result due to Simons [12]. Given a monotone operator $T : X \rightrightarrows X^*$ define $\psi_T : X \to R \cup \{+\infty\}$ by

$$\psi_T(u) = \sup\left\{\frac{\langle z^*, u - z\rangle}{1 + \|z\|};\ z^* \in T(z)\right\}.$$

Being the supremum of affine functions, $\psi_T$ is convex and lower semicontinuous. If $u \in D_T$ and $u^* \in T(u)$, then for any $z \in D_T$ and any $z^* \in T(z)$ we have

$$\frac{\langle z^*, u - z\rangle}{1 + \|z\|} = \frac{\langle z^* - u^*, u - z\rangle}{1 + \|z\|} + \frac{\langle u^*, u - z\rangle}{1 + \|z\|} \le 0 + \|u^*\| \cdot \frac{\|u - z\|}{1 + \|z\|} \le \|u^*\|(1 + \|u\|)$$

which shows that $\psi(u) < +\infty$. Thus

$$D_T \subseteq \operatorname{dom}(\psi_T). \tag{2.1}$$

In [4] Coodey and Simons used a generalization of $\psi_T$ to strengthen earlier results of Rockafellar [8] and Borwein and Fitzpatrick [2]. Among other results they proved that $T$ is locally bounded at each surrounded point of $\operatorname{co}(D_T)$ ($x \in X$ is a *surrounded point* of $A$ if $X \setminus \{0\} = \bigcup_{\lambda > 0} \lambda(A - x)$). A variant of this result for a pair of monotone operators is presented next.

**Proposition 2.2.** *Let $T, S : X \rightrightarrows X^*$ be monotone operators such that $\operatorname{co}(D_T) - \operatorname{co}(D_S)$ is absorbing. Then there exist $r_0 > 0$ and $c > 0$ such that*

$$\|t^*\|,\ \|s^*\| \le c(r_0 + \|z\|)(r_0 + \|t^* + s^*\|),\ \textit{whenever } z \in D_T \cap D_S, t^* \in T(z),\ \textit{and } s^* \in S(z).$$

**Proof.** Let $\psi_T$ and $\psi_S$ be defined as above. Since $\operatorname{dom}(\psi_T)$ and $\operatorname{dom}(\psi_S)$ are convex, it follows from (2.1) that $\operatorname{co}(D_T) \subseteq \operatorname{dom}(\psi_T)$ and $\operatorname{co}(D_S) \subseteq \operatorname{dom}(\psi_S)$. Thus $\operatorname{co}(D_T) - \operatorname{co}(D_S) \subseteq \operatorname{dom}(\psi_T) - \operatorname{dom}(\psi_S)$ and our assumption implies that $\operatorname{dom}(\psi_T) - \operatorname{dom}(\psi_S)$ is absorbing. From Corollary 4 in [13] it follows that there exist $\varepsilon > 0$ and $r \geq 1$ such that

$$\varepsilon B \subseteq \{x \in X; \psi_T(x) \leq r, \|x\| \leq r\} - \{x \in X; \psi_S(x) \leq r, \|x\| \leq r\}.$$

Let $x \in \varepsilon B$, $z \in D_T \cap D_S$, $t^* \in T(z)$, and $s^* \in S(z)$. Then $x = a - b$ with $\psi_T(a) \leq r$, $\|a\| \leq r$, $\psi_S(b) \leq r$, and $\|b\| \leq r$. We have

$$\begin{aligned}\langle t^*, x\rangle &= \langle t^*, a - z\rangle + \langle s^*, b - z\rangle + \langle t^* + s^*, z - b\rangle \\ &\leq \psi_T(a)(1 + \|z\|) + \psi_S(b)(1 + \|z\|) + \|t^* + s^*\|(\|z\| + r) \\ &\leq (r + \|z\|)(2r + \|t^* + s^*\|)\end{aligned}$$

from which it follows that

$$\|t^*\| \leq \frac{(r + \|z\|)(2r + \|t^* + s^*\|)}{\varepsilon}.$$

A similar estimate can be obtained for $\|s^*\|$. □

**Corollary 2.3.** *Let $T, S : X \rightrightarrows X^*$ be maximal monotone operators such that $\operatorname{co}(D_T) - \operatorname{co}(D_S)$ is absorbing. Then $T(z) + S(z)$ is a $w^*$-closed subset of $X^*$ for any $z \in D_T \cap D_S$.*

**Proof.** Since $T$ and $S$ are maximal monotone, $T(z)$ and $S(z)$ are convex and therefore $T(z) + S(z)$ is also convex. In view of the Krein-Šmulian theorem it is enough to prove that $T(z) + S(z)$ is bw*-closed, that is every bounded w*-convergent net in $T(z) + S(z)$ has its limit in $T(z) + S(z)$.

Let $\{t_i^*\} \subseteq T(z)$ and $\{s_i^*\} \subseteq S(z)$, be nets such that the net $\{t_i^* + s_i^*\}$ is bounded and w*-convergent to $z^*$. By the previous proposition, the nets $\{t_i^*\}$ and $\{s_i^*\}$ are also bounded, so they are relatively w*-compact. By replacing them with subnets we may assume that w*-$\lim t_i^* = t^*$ and w*-$\lim s_i^* = s^*$. Since $T$ and $S$ are maximal monotone, $T(z)$ and $S(z)$ are w*-closed and therefore $t^* \in T(z)$ and $s^* \in S(z)$. Then $z^* = t^* + s^* \in T(z) + S(z)$. □

**Corollary 2.4.** *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator, $C \in \mathcal{C}_1(T)$, and assume that $L_X(x, x^*, T + \partial I_C) = M_X(x, x^*, T + \partial I_C)$ for any $(x, x^*) \in X \times X^*$. Then $T + \partial I_C$ is maximal monotone. In particular, if $T : X \rightrightarrows X^*$ is $\mathcal{C}_0$ (resp. $\mathcal{C}_1$)-regular maximal monotone operator and $C \in \mathcal{C}_0(T)$ (resp. $C \in \mathcal{C}_1(T)$, then $T + \partial I_C$ is maximal monotone.*

**Proof.** Assume first that $\overline{\operatorname{lin}(D_T - C)} = X$, i.e. $\operatorname{co}(D_T) - C$ is absorbing. Let $(x, x^*) \in X \times X^*$ be monotonically related to $T + \partial I_C$. Then $L_X(x, x^*, T + \partial I_C) = 0$ and therefore $M_X(x, x^*, T + \partial I_C) = 0$ too. This means that $x^* \in \overline{(T + \partial I_C)(x)}$. By Corollary 2.3, $(T + \partial I_C)(x)$ is w*-closed and therefore norm-closed too. Thus $x^* \in (T + \partial I_C)(x)$, implying that $T + \partial I_C$ is maximal monotone.

The general case can be reduced to the particular one considered above as follows. First we shall show that there is no loss of generality in assuming that $0 \in D_T \cap C$. To this end, let $c \in D_T \cap C$ (this is possible because of Remark 1.3(2) in the introduction). Define $\tilde{T} : X \rightrightarrows X^*$ by $\tilde{T}(x) = T(x + c)$ and let $\tilde{C} = C - c$. It is easy to see that $\tilde{T}$ is maximal monotone, $\tilde{C} \in \mathcal{C}_1(\tilde{T})$, $L_X(x, x^*, T+\partial I_C) = L_X(x-c, x^*, \tilde{T}+\partial I_{\tilde{C}})$, $M_X(x, x^*, T+\partial I_C) = M_X(x-c, x^*, \tilde{T}+\partial I_{\tilde{C}})$, and that $T+\partial I_C$ is maximal monotone if and only if $\tilde{T}+\partial I_{\tilde{C}}$ is maximal monotone. Since $0 \in D_{\tilde{T}} \cap \tilde{C}$, our assertion is proved.

Let $Y = \overline{\text{lin}(D_T - C)}$. Since $0 \in D_T \cap C$, it follows that $D_T \subseteq Y$ and $C \subseteq Y$. Then $\text{co}(D_{(T|Y)}) - C$ is absorbing (as a subset of $Y$) and, from Lemma 2.1, $T|Y$ is maximal monotone and $L_X(x, x^*, T|Y + \partial I_{C,Y}) = M_X(x, x^*, T|Y + \partial I_{C,Y})$ for any $(x, x^*) \in Y \times Y^*$. From the particular case proved at the beginning of the proof it follows that $T|Y + \partial I_{C,Y}$ is maximal monotone. Finally, since $(T + \partial I_C)|Y = T|Y + \partial I_{C,Y}$, from Lemma 2.1 we obtain that $T + \partial I_C$ is maximal monotone. □

Let $C$ be a convex subset of $X$ and let $x \in C$. Recall that the *tangent cone* to $C$ at $x$, denoted $C_x$, and the *normal cone* to $C$ at $x$, denoted $N_C(x)$, are defined as follows

$$C_x = \overline{\bigcup_{t \geq 0} t(C - x)}$$

$$N_C(x) = \{x^* \in X^*;\ \langle x^*, z - x\rangle \leq 0, \text{ for any } z \in C\}.$$

It is easily verified that

$$N_C(x) = \{x^* \in X^*;\ \langle x^*, v\rangle \leq 0, \text{ for any } v \in C_x\}$$

and (by using a separation argument) that

$$C_x = \{v \in X;\ \langle x^*, v\rangle \leq 0, \text{ for any } x^* \in N_C(x)\}.$$

As a matter of fact, $N_C(x) = \partial I_C(x)$ whenever $C$ is closed.

**Lemma 2.5.** *Let $T : X \rightrightarrows X^*$ be a maximal monotone operator, let $C$ be a convex subset of $X$ such that $D_T \subseteq C$, and let $x \in D_T$. Then $T(x) + N_C(x) = T(x)$.*

**Proof.** Let $z \in D_T$, $z^* \in T(z)$, $x^* \in T(x)$, and $v^* \in N_C(x)$. Then, from the monotonicity of $T$ and the definition of $N_C(x)$, we get

$$\langle x^* + v^* - z^*, x - z\rangle = \langle x^* - z^*, x - z\rangle + \langle v^*, x - z\rangle \geq 0$$

and, since $T$ is maximal monotone, it follows that $x^* + v^* \in T(x)$. Thus $T(x) + N_C(x) \subseteq T(x)$. Since $0 \in N_C(x)$, the other inclusion is obvious. □

Before stating our main result, we need one more definition.

**Definition 2.6.** A multivalued map $T : X \rightrightarrows X^*$ is called *locally inf bounded* if for every $z \in D_T$ there exist $\varepsilon > 0$ and $M > 0$ such that for any $u \in D_T$ with $\|u - z\| \leq \varepsilon$ there exists $u^* \in T(u)$ with $\|u^*\| \leq M$.

**Example 2.7.** (1) Any monotone operator $T$ whose domain is open is locally inf bounded (because it is locally bounded, see for example Theorem 2.28 in [6]).

(2) If $f : X \to R \cup \{+\infty\}$ is a proper convex function which is locally Lipschitzian on its domain then $D_{\partial f} = \text{dom}(f)$ and $\partial f$ is locally inf bounded (see for example [16], where locally inf bounded monotone operators where called locally efficient).

**Theorem 2.8.** *Let $T$ and $S$ be maximal monotone operators on $X$ and let $i \in \{0,1\}$. Assume that $T$ is $\mathcal{C}_i$-regular, $S$ is locally inf bounded, and $D_S \in \mathcal{C}_i(T)$. If $i=1$ assume also that $\overline{\mathrm{co}(D_T) \cap D_S} = \overline{D_T} \cap D_S$ (for example, $S$ satisfies all these conditions if $D_S = X$). Then $T+S$ is maximal monotone.*

**Proof.** Exactly as in the proof of Corollary 2.4, we can assume, without any loss of generality, that $\overline{\mathrm{lin}(D_T - D_S)} = X$. Thus the fact that $D_S \in \mathcal{C}_i(T)$ implies that

$$\mathrm{co}(D_T) - D_S \text{ is absorbing.} \tag{2.2}$$

Being maximal monotone, both $S$ and $T$ are w*-closed and convex valued. Thus, for any $z \in D_T \cap D_S$, $T(z)+S(z)$ is convex and, by (2.2) and Corollary 2.3, w*-closed. Assume that $T+S$ is not maximal monotone. Then there exists a pair $(x,y^*) \in X \times X^*$ such that

$$\langle t^* + s^* - y^*, z - x\rangle \geq 0, \text{ whenever } z \in D_T \cap D_S, t^* \in T(z), \text{ and } s^* \in S(z) \tag{2.3}$$

but $y^* \notin (T+S)(x)$.

<u>*Claim I.*</u> $x \in D_T \cap D_S$ (to be proved later).[1]

Thus $T(x)+S(x) \neq \emptyset$ and, as noticed above, $T(x)+S(x)$ is $w^*$-closed and convex. Since $y^* \notin T(x)+S(x)$, by the separation theorem there exist $u \in X$, with $\|u\|=1$, and a real number $\beta$ such that

$$\langle t^* + s^*, u\rangle < \beta < \langle y^*, u\rangle, \text{ for any } t^* \in T(x), s^* \in S(x). \tag{2.4}$$

Since $T$ is $X$-regular, $\overline{D_T}$ is a closed convex set, (Theorem 7 in [17]), so $D = \overline{D_T} \cap D_S$ is also closed and convex. It is not difficult to see that

$$D = \overline{D_T \cap \mathrm{int}(D_S)} \text{ if } i=0. \tag{2.5$_0$}$$

The last hypothesis of the theorem implies that

$$D = \overline{\mathrm{co}(D_T) \cap D_S} \text{ if } i=1. \tag{2.5$_1$}$$

Let $D_x$ be the tangent cone to $D$ at $x$.

<u>*Claim II.*</u> $u \in D_x$ (to be proved later).

We shall now choose some constants. First, since $S$ is locally inf bounded, there exist $r>0$ and $\alpha>0$ such that

$$S(z) \cap rB^* \neq \emptyset \text{ whenever } z \in D_S \text{ and } \|z-x\| \leq \alpha. \tag{2.6}$$

Choose also $0 < \delta \leq \frac{1}{2}$ such that $r\delta \leq \varepsilon$, where $\varepsilon$ is defined next, independently of $\delta$. Let

$$\varepsilon = \frac{1}{4}(\langle y^*, u\rangle - \beta),$$

$$\mu = \sup_{x^* \in T(x)} \langle x^*, u\rangle,$$

[1]In order to make the proof easier to follow, we shall claim several results and verify them later.

$$W = \{x^* \in X^*; \langle x^*, u\rangle < \beta + \varepsilon - \mu\}.$$

(It follows from (2.4) that $\mu < +\infty$.) Then $W$ is a w*-open subset of $X^*$ and (2.4) implies that $S(x) \subseteq W$. Finally, since $S$ is maximal, we may also assume that $\alpha$ is small enough that

Claim III. $S(z) \cap rB^* \subseteq W$ whenever $z \in D_S$ and $\|z - x\| \leq \alpha$ (to be proved later).

We shall next use the fact that $T$ is $\mathcal{C}_i$-regular to derive a contradiction from (2.4). To this end consider the closed convex set

$$C = \{z \in X;\ \|z-x\| \leq \alpha,\ z = x+tv \text{ for some } t > 0 \text{ and some } v \in X \text{ with } \|v - u\| < \frac{\delta}{4}\}.$$

Claim IV. $\|\frac{1}{\|z-x\|}(z - x) - u\| \leq \delta$ for any $z \in C$ (to be proved later).

Since $x + \frac{\alpha}{2}u \in \mathrm{int}(C)$ and $u \in D_x$, from the definition of $D_x$ it follows that

$$D \cap \mathrm{int}(C) \neq \emptyset. \tag{2.7}$$

Let $K = D_S \cap C$. If $i = 0$, then (2.7) and (2.5$_0$) imply that

$$D_T \cap \mathrm{int}(K) = D_T \cap \mathrm{int}(D_S) \cap \mathrm{int}(C) \neq \emptyset$$

and therefore $K \in \mathcal{C}_0(T)$. If $i = 1$, then (2.7), and (2.5$_1$) imply that $\mathrm{co}(D_T) \cap D_S \cap \mathrm{int}(C) \neq \emptyset$. From (2.2) one can deduce now that $\mathrm{co}(D_T) - K$ is absorbing and therefore $K \in \mathcal{C}_1(T)$.

Since $T$ is $\mathcal{C}_i$-regular, $L(x, y^*, T + \partial I_K) = M(x, y^*, T + \partial I_K) < +\infty$ (the inequality because $x \in D_T \cap K$). From the definition of $M(x, y^*, T + \partial I_K)$ there exists $t_0^* \in T(x)$ such that

$$\frac{\langle y^* - t_0^*, z - x\rangle}{\|x - z\|} < L(x, y^*, T + \partial I_K) + \varepsilon, \text{ for any } z \in K.$$

Since $u \in D_x$, there exist $w \in X$ and $\gamma > 0$ such that $\langle y^* - t_0^*, u\rangle \leq \langle y^* - t_0^*, w\rangle + \varepsilon$ and $x + \gamma w \in K$. Then, from the above inequality, we obtain

$$\langle y^* - t_0^*, u\rangle \leq \langle y^* - t_0^*, w\rangle + \varepsilon = \frac{\langle y^* - t_0^*, (x + \gamma w) - x\rangle}{\|x - (x + \gamma w)\|} < L(x, y^*, T + \partial I_K) + 2\varepsilon.$$

From the definition of $L(x, y^*, T + \partial I_K)$, there exist $z \in K \cap D_T$ and $t^* \in T(z)$ such that

$$\langle y^* - t_0^*, u\rangle < \frac{\langle t^* - y^*, x - z\rangle}{\|x - z\|} + 2\varepsilon.$$

This inequality can be rewritten as

$$\langle y^* - t_0^*, u\rangle < \langle y^* - t^*, v\rangle + 2\varepsilon, \tag{2.8}$$

where $v = (1/\|z - x\|)(z - x)$; since $z \in C$, by Claim IV,

$$\|v - u\| < \delta. \tag{2.9}$$

Choose now $s^* \in S(z) \cap rB^*$ (this is possible by (2.6)). From (2.2) we obtain

$$\langle t^* + s^* - y^*, v\rangle \geq 0$$

or

$$\langle y^* - t^*, v\rangle \leq \langle s^*, v\rangle.$$

Using (2.8) and this last inequality we get

$$\begin{aligned}\langle y^*, u\rangle &< \langle t_0^*, u\rangle + \langle y^* - t^*, v\rangle + 2\varepsilon \leq \langle t_0^*, u\rangle + \langle s^*, v\rangle + 2\varepsilon \\ &= \langle t_0^*, u\rangle + \langle s^*, u\rangle + \langle s^*, v - u\rangle + 2\varepsilon \\ &\quad \text{(by Claim III)} \\ &\leq \langle t_0^*, u\rangle + \beta + \varepsilon - \mu + \|s^*\|\,\|v - u\| + 2\varepsilon \\ &\quad \text{(by (2.9) and then by the choice of } \delta) \\ &\leq \langle t_0^*, u\rangle - \mu + \beta + 3\varepsilon + r\delta \leq 0 + \beta + 4\varepsilon = \langle y^*, u\rangle\end{aligned}$$

which is impossible. Thus $T + S$ must be maximal monotone.

Proof of Claim I. Let $x_0 \in D_T \cap \operatorname{int}(D_S)$ if $i = 0$ (resp. $x_0 \in D_T \cap D_S$ if $i = 1$) and let $[x, x_0]$ be the segment joining $x$ and $x_0$. Since $D_S$ is closed and convex there exists $x_1 \in [x, x_0] \cap D_S$ such that $[x, x_0] \cap D_S = [x_1, x_0]$. Since $S$ is locally inf bounded and $[x_1, x_0]$ is compact, there exists $m > 0$ and an open neighborhood $A$ of $[x_1, x_0]$ such that for any $z \in A \cap D_S$ there exists $s^* \in S(z)$ with $\|s^*\| \leq m$. For any $\rho > 0$ let $K_\rho = \{(1-t)x + tu; 0 \leq t \leq 1, \|u - x_0\| \leq \rho\}$. We claim that if $\rho$ is small enough, then $K_\rho \cap D_S \subseteq A$. If this was not true, then there would exist a sequence $u_n \to x_0$ in $X$ and a sequence $t_n \to t$ in $[0,1]$ such that $z_n = (1-t_n)x + tu_n \in D_S$ and $z_n \notin A$. Then $z = (1-t)x + tx_0 = \lim z_n \notin A$ (since $A$ is open), but $z \in [x, x_0] \cap D_S$ (since $D_S$ is closed). Since $[x, x_0] \cap D_S = [x_1, x_0] \subseteq A$ it follows that $z \in A$, which is a contradiction. Thus there exists $\rho > 0$ such that $K_\rho \cap D_S \subseteq A$. Let $K = K_\rho$.

Let $z \in D_T \cap K \cap D_S$, $t^* \in T(z)$, and $u^* \in \partial I_{K \cap D_S}(z) = \partial(I_K + I_{D_S})(z)$. Since $\operatorname{int}(K) \cap D_S \neq \emptyset$, from the sum formula for subdifferentials, there exist $j^* \in \partial I_K(z)$ and $i^* \in \partial I_{D_S}(z)$ such that $u^* = j^* + i^*$. From the previous discussion, we can also choose $s^* \in S(z)$ with $\|s^*\| \leq m$. Then $\langle j^*, x - z\rangle \leq 0$ and, by Lemma 2.5, $s^* + i^* \in S(z)$. From (2.3) we obtain

$$\frac{\langle t^* + u^* - y^*, x - z\rangle}{\|x - z\|} = \frac{\langle t^* + s^* + i^* - y^*, x - z\rangle}{\|x - z\|} + \frac{\langle j^*, x - z\rangle}{\|x - z\|} - \frac{\langle s^*, x - z\rangle}{\|x - z\|} \leq 0 + \|s^*\| \leq m.$$

Thus $L(x, y^*, T + \partial I_{K \cap D_S}) < +\infty$. Since $x_0 \in \operatorname{int}(K)$, then either $x_0 \in D_T \cap \operatorname{int}(D_S \cap K)$ if $i = 0$ or, using (2.2), $\operatorname{co}(D_T) - D_S \cap K$ absorbs $X$ if $i = 1$. In both cases $D_S \cap K \in \mathcal{C}_i(T)$. Since $T$ is $\mathcal{C}_i$-regular, it follows that $M(x, y^*, T + \partial I_{K \cap D_S}) = L(x, y^*, T + \partial I_{K \cap D_S}) < +\infty$, hence $x \in D_T \cap D_S \cap K$ (by (ii) in the Introduction). This proves Claim I.

Proof of Claim II. Assume that $u \notin D_x$. Then, by the separation theorem, there exists $u^* \in X^*$ such that

$$\langle u^*, z\rangle < \langle u^*, u\rangle \text{ for all } z \in D_x. \tag{2.10}$$

Since $0 \in D_x$, $\langle u^*, u\rangle > 0$; since $tz \in D_x$ for any $t > 0$ and any $z \in D_x$, it follows from (2.10) that $\langle u^*, z\rangle \leq 0$ for any $z \in D_x$, which means that $u^* \in \partial I_D(x) = \partial(I_{\overline{D_T}} + I_{D_S})(x)$. Since $D_T - D_S$ is absorbing, the sum formula for subdifferentials implies that $u^* = v^* + w^*$, $v^* \in \partial I_{\overline{D_T}}(x)$, $w^* \in \partial I_{D_S}(x)$. Choose $t^* \in T(x)$ and $s^* \in S(x)$. Then, by Lemma 2.5, $t^* + \lambda v^* \in T(x)$ and $s^* + \lambda w^* \in S(x)$ for any $\lambda \geq 0$. It follows from (2.4) that $\langle u^*, u\rangle = \langle v^*, u\rangle + \langle w^*, u\rangle \leq 0$ which contradicts our earlier finding that $\langle u^*, u\rangle > 0$. This contradiction proves Claim II.

Proof of Claim III. If the claim was not true, (2.9) would imply that there exist a sequence $\{x_n\}$ converging to $x$ and a sequence $\{w_n^*\}$ with $w_n^* \in S(x_n) \cap rB^*$ but

$$\langle w_n^*, u\rangle \geq \beta + \varepsilon - \mu. \tag{2.11}$$

If $w^*$ is a limit point of $\{w_n^*\}$ (there are such points because $\{w_n^*\} \subset rB^*$), then $\langle w^* - s^*, x - s\rangle \geq 0$ for any $s \in D_S$ and any $s^* \in S(s)$ (because $\langle w_n^* - s^*, x_n - s\rangle \geq 0$). The maximal monotonicity of $S$ implies that $w^* \in S(z)$. From (2.11) we obtain that $\langle w^*, u\rangle \geq \beta + \varepsilon - \mu$ which contradicts the fact that $S(x) \subseteq W$.

Proof of Claim IV. Given $z = x + tv \in C$, clearly $\frac{1}{\|z-x\|}(z - x) = \frac{1}{\|v\|}v$. We have

$$1 = \|u\| \leq \|v\| + \|u - v\| \leq \|v\| + \frac{\delta}{4}$$

and thus

$$\|v\| \geq 1 - \frac{\delta}{4} \geq \frac{1}{2} \quad (\text{ since } \delta \leq \frac{1}{2}).$$

Similarly

$$\|v\| \leq \|v - u\| + \|u\| \leq \frac{\delta}{4} + 1.$$

Combining the last inequalities we obtain

$$\big|1 - \|v\|\big| \leq \frac{\delta}{4} \quad \text{and} \quad \|v\| \geq \frac{1}{2}.$$

It follows that

$$\left\|\frac{1}{\|v\|}v - u\right\| = \frac{\big\|v - \|v\|u\big\|}{\|v\|} \leq \frac{\|v - u\| + \big\|u - \|v\|u\big\|}{\|v\|} \leq \frac{\frac{\delta}{4} + \big|1 - \|v\|\big|}{\|v\|} \leq \frac{\frac{\delta}{4} + \frac{\delta}{4}}{\frac{1}{2}} = \delta$$

and thus $\|\frac{1}{\|z-x\|}(z - x) - u\| \leq \delta$ for any $z \in C$. This proves Claim IV and completes the proof of the theorem. □

**Corollary 2.9.** *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator and $f$ be a proper, convex, lower semicontinuous function on $X$. Assume that one of the following conditions is satisfied*

(a) $D_S = X$.

(b) *$S$ is $\mathcal{C}_0$ (resp. $\mathcal{C}_1$)-regular,* $\mathrm{dom}(f) \in \mathcal{C}_0(S)$ *(resp.* $\mathrm{dom}(f) \in \mathcal{C}_1(S)$ *and* $\overline{\mathrm{co}(D_S) \cap \mathrm{dom}(f)} = \overline{D_S} \cap \mathrm{dom}(f)$*), and $f$ is locally Lipschitz on* $\mathrm{dom}(f)$.

*Then $\partial f + S$ is maximal monotone.*

**Proof.** As mentioned in the introduction, $\partial f$ is a $\mathcal{C}_1$-regular maximal monotone operator. The maximal monotonicity of $\partial f + S$ when (a) is satisfied follows from Theorem 2.8. If (b) is satisfied, then $\partial f$ is locally inf bounded and the assertion follows again from Theorem 2.8. □

**Corollary 2.10.** *Let $S : X \rightrightarrows X^*$ be a maximal monotone operator.*

(i) *$S$ is $\mathcal{C}_0$-regular if and only if $S + \partial g_{\lambda,x} + \partial I_C$ is maximal monotone for any $\lambda \geq 0$, any $x \in X$, and any $C \in \mathcal{C}_0(S)$.*

(ii) *If $S$ is $\mathcal{C}_0$-regular and $C \in \mathcal{C}_0(S)$ then $S + \partial I_C$ is $\mathcal{C}_0$-regular.*

**Proof.** (i) The "if" part is an immediate consequence of Theorem 1.1 in the introduction, while the "only if" part follows from Corollary 2.9 because $f = \partial(g_{\lambda,x} + I_C)$ satisfies condition (b) and $\partial(g_{\lambda,x} + I_C) = \partial g_{\lambda,x} + \partial I_C$.

(ii) It is enough to show that $S + \partial g_{\lambda,x} + \partial I_C + \partial I_K$ is maximal monotone for any $\lambda \geq 0$, any $x \in X$, and any $K \in \mathcal{C}_0(S + \partial I_C)$. Since $D_{S+\partial I_C} = D_S \cap C$ and $K \in \mathcal{C}_0(S + \partial I_C)$, it follows that $D_S \cap C \cap \operatorname{int}(K) \neq \emptyset$ and therefore $S + \partial g_{\lambda,x} + \partial I_C + \partial I_K = S + \partial g_{\lambda,x} + \partial I_{C\cap K}$. If we can show that, $D_S \cap \operatorname{int}(C \cap K) \neq \emptyset$, then $C \cap K \in \mathcal{C}_0(S)$ and (ii) follows from (i).

Choose $u \in D_S \cap \operatorname{int}(C)$ (possible since $C \in \mathcal{C}_0(S)$) and $v \in D_{S+\partial I_C} \cap \operatorname{int}(K) = D_S \cap C \cap \operatorname{int}(K)$. If $u = v$, we are done. Otherwise, $[u, v) \subseteq \operatorname{int}(C)$, $[u, v] \subseteq \overline{D_S}$ (because $S$ is $X$-regular and therefore $\overline{D_S}$ is convex), and $v \in \operatorname{int}(K)$. Thus there exists $w$ on the open segment $(u, v)$ such that $w \in \overline{D_S} \cap \operatorname{int}(C) \cap \operatorname{int}(K)$. It is now obvious that $D_S \cap \operatorname{int}(C \cap K) \neq \emptyset$ and, as mentioned above, this proves (ii). □

**Corollary 2.11.**

(i) *If $S : X \rightrightarrows X^*$ is maximal monotone, locally inf bounded, and $D_S$ is closed and convex then $S$ is $\mathcal{C}_1$-regular. In particular, a maximal monotone operator $S$ with $D_S = X$ is $\mathcal{C}_1$-regular.*

(ii) *If $T, S : X \rightrightarrows X^*$ are maximal monotone, locally inf bounded, $D_T, D_S$ are closed and convex, and $\bigcup_{\lambda>0} \lambda(D_T - D_S) = \overline{\operatorname{lin}(D_T - D_S)}$, then $T + S$ is maximal monotone.*

**Proof.** (i) Let $C \in \mathcal{C}_1(S)$ and $\lambda \geq 0$. $T = \partial g_{\lambda,x} + \partial I_C = \partial(g_{\lambda,x} + I_C)$ is $\mathcal{C}_1$-regular and $D_T = C$. Since $\operatorname{co}(D_T) = C = \operatorname{co}(C)$ and $\operatorname{co}(D_S) = D_S$, we have

$$\bigcup_{\lambda>0}(\operatorname{co}(D_T) - \operatorname{co}(D_S)) = \bigcup_{\lambda>0}(C - D_S) = \overline{\operatorname{lin}(C - D_S)} = \overline{\operatorname{lin}(D_T - D_S)}$$

hence $D_S \in \mathcal{C}_1(T)$. Also, since $D_T = C$ and $D_S$ are closed,

$$\overline{D_T \cap D_S} = \overline{C \cap D_S} = C \cap D_S = \overline{D_T} \cap D_S.$$

By Theorem 2.8, $T + S = S + \partial g_{\lambda,x} + \partial I_C$ is maximal monotone. Thus, by Theorem 1.1 in the introduction, $S$ is $\mathcal{C}_1$-regular.

(ii) By (i), $T$ is $\mathcal{C}_1$-regular. Since the conditions of Theorem 2.8 are satisfied, $T + S$ is maximal monotone. □

## 3. Linear maximal monotone operators

A single valued monotone operator $T$ is called *linear* if $D_T$ is a linear subspace of $X$ and $T : D_T \to X^*$ is linear. We begin by recalling a result proved in [7] and [15] (we shall reformulate it so that we can use our notation). Assume that $T : D_T \to X^*$ is linear and monotone. Then

$$T \text{ is maximal monotone} \iff D_T \text{ is dense in } X \text{ and } D_T = \{x \in X;\ L_X(x, 0, T) < +\infty\}. \quad (3.1)$$

**Lemma 3.1.** *Let $T : D_T \to X^*$ be linear and maximal monotone. Let $C$ be a closed convex subset of $X$ such that $\operatorname{int}(C) \cap D_T \neq \emptyset$, let $x \in X$ and $x^* \in X^*$. If $L_X(x, x^*, T + \partial I_C) < +\infty$ then $x \in D_T$.*

**Proof.** Assume first that $0 \in \operatorname{int}(C) \cap D_T$. In view of (3.1) and the maximal monotonicity of $T$, it is enough to show that $L_X(x, 0, T) < +\infty$. To this end let $g : X \to R$ denote the gauge function associated to $C$, i.e. $g(z) = \inf\{t \geq 0;\ z \in tC\}$. Since $\operatorname{int}(C) \neq \emptyset$, $g$ is a continuous semi-norm on $X$ and therefore it is Lipschitzian. Let $\alpha$ denote the Lipschitz constant of $g$. Set $M = (L_X(x, x^*, T + \partial I_C) + \|x^*\|)(\alpha\|x\| + 1)$ and let $z \in D_T$.

Case I: $z \in C$. We have

$$\begin{aligned}
\frac{\langle T(z), x - z\rangle}{\|x - z\|} &= \frac{\langle T(z) - x^*, x - z\rangle}{\|x - z\|} + \frac{\langle x^*, x - z\rangle}{\|x - z\|} \\
&\leq \frac{\langle T(z) - x^*, x - z\rangle}{\|x - z\|} + \|x^*\| \\
&\leq L_X(x, x^*, T + \partial I_C) + \|x^*\| \leq M \quad (\text{since } 0 \in \partial I_C(z)).
\end{aligned}$$

Case II: $z \notin C$. Let $t = g(z)$ and $u = \frac{1}{t}z$. Since $C$ is closed, $t > 1$ and $u \in C$. We have

$$\begin{aligned}
\frac{\langle T(z), x - z\rangle}{\|x - z\|} &= \frac{\langle T(tu), x - tu\rangle}{\|x - tu\|} = \frac{t\langle T(u), x - u\rangle + t\langle T(u), (1 - t)u\rangle}{\|x - tu\|} \\
&\quad \text{(use the facts that } \langle T(u), u\rangle \geq 0 \text{ and } t > 1) \\
&\leq \frac{t\langle T(u), x - u\rangle}{\|x - tu\|} = \frac{t\langle T(u), x - u\rangle}{t\|x - u\|} \cdot \frac{\|tx - tu\|}{\|x - tu\|} \\
&= \frac{\langle T(u), x - u\rangle}{\|x - u\|} \cdot \frac{\|tx - tu\|}{\|x - tu\|} \\
&\quad \text{(use a computation similar to that in Case I)} \\
&\leq (L_X(x, x^*, T + \partial I_C) + \|x^*\|)\frac{\|tx - tu\|}{\|x - tu\|} \\
&\leq (L_X(x, x^*, T + \partial I_C) + \|x^*\|)\frac{\|tx - x\| + \|x - tu\|}{\|x - tu\|} \\
&= (L_X(x, x^*, T + \partial I_C) + \|x^*\|)\left(\frac{(t - 1)\|x\|}{\|x - tu\|} + 1\right) \\
&\leq (L_X(x, x^*, T + \partial I_C) + \|x^*\|)\left(\frac{(g(z) - g(x))\|x\|}{\|x - z\|} + 1\right) \\
&\leq (L_X(x, x^*, T + \partial I_C) + \|x^*\|)(\alpha\|x\| + 1) \leq M.
\end{aligned}$$

Thus $L_X(x,0,T)<+\infty$ and, as mentioned above, this proves the lemma in the case when $0\in D_T\cap \mathrm{int}(C)$.

To prove the general case, let $c\in \mathrm{int}(C)\cap D_T$. A fairly trivial verification shows that

$$L_X(x-c,x^*-T(c),T+\partial I_{C-c})=L_X(x,x^*,T+\partial I_C)<+\infty.$$

Since $0\in \mathrm{int}(C-c)$, it follows that $x-c\in D_T$. Since $c\in D_T$ and since $D_T$ is a linear subspace of $X$, it follows that $x\in D_T$. □

**Proposition 3.2.** *Let $T:D_T\to X^*$ be linear and maximal monotone. Then $T$ is $\mathcal{C}_0$-regular.*

**Proof.** Let $C$ be a closed convex subset of $X$ such that $\mathrm{int}(C)\cap D_T\neq\emptyset$ and let $x\in C$. Since always $0\le L_X(x,x^*,T+\partial I_C)\le M_X(x,x^*,T+\partial I_C)$ (Lemma 1 in [17]), it is enough to assume that $L_X(x,x^*,T+\partial I_C)$ is finite and that $M_X(x,x^*,T+\partial I_C)>0$ and show that $M_X(x,x^*,T+\partial I_C)\le L_X(x,x^*,T+\partial I_C)$.

First notice that, by Lemma 3.1, $x\in D_T$. Let $\varepsilon>0$ and $M<M_X(x,x^*,T+\partial I_C)$. In view of (3.1), $D_T$ is dense in $X$. Therefore we can find $z\in C\cap D_T$ such that

$$M\le\frac{\langle T(x)-x^*,x-z\rangle}{\|x-z\|}$$

(notice that, since $T$ is single valued, the "inf" from the definition of $M_X(x,x^*,T+\partial I_C)$ disappears).

Let $0<\lambda<1$ and let $z_\lambda=(1-\lambda)x+\lambda z\in C\cap D_T$. Clearly $x-z_\lambda=\lambda(x-z)$. Choose $\lambda$ so small that

$$\frac{\lambda\langle T(x-z),x-z\rangle}{\|x-z\|}\le\varepsilon.$$

We have

$$\begin{aligned}M&\le\frac{\langle T(x)-x^*,x-z\rangle}{\|x-z\|}=\frac{\langle T(x)-x^*,x-z_\lambda\rangle}{\|x-z_\lambda\|}\\&\le\frac{\langle T(x)-T(z_\lambda),x-z_\lambda\rangle}{\|x-z_\lambda\|}+\frac{\langle T(z_\lambda)-x^*,x-z_\lambda\rangle}{\|x-z_\lambda\|}\\&\le\frac{\langle T(x-z_\lambda),x-z\rangle}{\|x-z\|}+L_X(x,x^*,T+\partial I_C)\\&\le\frac{\lambda\langle T(x-z),x-z\rangle}{\|x-z\|}+L_X(x,x^*,T+\partial I_C)\le\varepsilon+L_X(x,x^*,T+\partial I_C)\,.\end{aligned}$$

Since this is true for any $\varepsilon>0$ and any $M<M_X(x,x^*,T+\partial I_C)$, it follows that $M_X(x,x^*,T+\partial I_C)\le L_X(x,x^*,T+\partial I_C)$ and the theorem is completely proved. □

Combining Proposition 3.2 with Theorem 2.8 we obtain the next corollary which was proved by different methods in both [7] and [15].

**Corollary 3.3.** *If $T$ and $S$ are linear maximal monotone operators on $X$ and $D_S=X$ then $T+S$ is (linear) maximal monotone.*

Since a $\mathcal{C}_0$-regular maximal monotone operator satisfies the conditions of Theorem 10 in [17], we obtain another proof of the following result due to Phelps and Simons [7] and Simons [15].

**Corollary 3.4.** *A linear maximal monotone operator is maximal monotone locally (i.e. it is maximal monotone on any open convex subset of $X$ which intersects its domain).*

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