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Euler-Lagrange Inclusions and Existence of Minimizers for a Class of Non-Coercive Variational Problems

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We are concerned with integral functionals of the form

$$J(v) \doteq \int_{B_R^n} [f(|x|, |\nabla v(x)|) + h(|x|, v(x))] \, dx,$$

defined on $W_0^{1,1}(B_R^n, \mathbb{R}^m)$, where B_R^n is the ball of $\mathbb{R}^n$ centered at the origin and with radius $R > 0$. We assume that the functional J is convex, but the compactness of the sublevels of J is not required. We prove that, under suitable assumptions on f and h, there exists a radially symmetric minimizer $v \in W_0^{1,1}(B_R, \mathbb{R}^m)$ for J. Moreover, we associate to the functional J a system of differential inclusions of the Euler-Lagrange type, and we prove that the solvability of these inclusions is a necessary and sufficient condition for the existence of a radially symmetric minimizer for J.

Keywords: Calculus of variations, existence, Euler-Lagrange inclusions, radially symmetric solutions, non-coercive problems

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1. Introduction

In this paper we deal with integral functionals of Calculus of Variations of the form

$$J(v) \doteq \int_{B_R^n} [f(|x|, |\nabla v(x)|) + h(|x|, v(x))] \, dx \,, \tag{1.1}$$

defined on $W_0^{1,1}(B_R^n, \mathbb{R}^m)$, where B_R^n is the ball of $\mathbb{R}^n$ centered at 0 and with radius $R > 0$. We consider convex functionals, but no assumptions are made concerning the compactness of the sublevels of J. Our interest in this problem arises from its applications to physical models for which the energy functional has the form (1.1) with Lagrangian not necessarily superlinear with respect to ∇v (see, e.g., the Appendix of [11]).

Since, under our assumptions, the functional J need not be coercive, the existence of a minimizer in $W_0^{1,1}(B_R^n, \mathbb{R}^m)$ cannot be obtained by means of direct methods. However,

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thanks to the radial symmetry of the problem, it is possible to make a drastic simplification: the vectorial minimization problem involving J can be reduced to a one–dimensional problem, in the sense that J has a minimizer in $W_0^{1,1}(B_R^n, \mathbb{R}^m)$ if and only if there exists a solution to

$$\min_{u \in W} \int_0^R t^{n-1} \left[f(t, |u'(t)|) + h(t, u(t)) \right] dt\,, \tag{1.2}$$

where

$$W \doteq \left\{ u \in AC_{loc}(]0,R], \mathbb{R}^m);\ u(R) = 0,\ t \mapsto t^{n-1} |u'(t)| \in L^1(0,R) \right\} \tag{1.3}$$

(see Lemma 3.8 below). Nevertheless, problem (1.2) presents new nontrivial difficulties. The first one is due to the fact that the set W is not a subset of $AC([0,R], \mathbb{R}^m)$, which is the classical framework for this kind of minimization problem. In addition, (1.2) is not a Dirichlet problem, because no boundary conditions are given at $t = 0$. Finally, our regularity assumptions on f and h are very mild, hence it is not possible to adapt to (1.2) the indirect methods developed in [2], [5], [6], and [9].

Minimization problems of the form (1.2), defined on scalar functions, were studied in [7] and [8]. In particular, in [8] the existence of a minimizer is obtained by means of a fixed point technique applied to a suitable multifunction. In this paper we adapt this technique to functionals defined on vector–valued functions, obtaining that there exists at least one solution to (1.2), and then a radially symmetric minimizer of J in $W_0^{1,1}(B_R^n, \mathbb{R}^m)$. More precisely, we find two functions $u \in W$ and $p \in W^*$, where

$$W^* \doteq \left\{ p \in AC([0,R], \mathbb{R}^m) \mid p(0) = 0,\ t^{1-n} |p'(t)| \in L^1(0,R) \right\}\,, \tag{1.4}$$

such that the pair (u,p) satisfies a system of differential inclusions of the Euler–Lagrange type (see (3.8) and (3.9) below), and we show that this implies that u is a solution to (1.2), as one has to expect due to the convexity of the functional.

The last part of the paper is devoted to the study of the Euler–Lagrange inclusions. Since a minimizer $u \in W$ of (1.2) need not be an absolutely continuous function, we cannot apply the classical necessary conditions (see [4], [14], [15], and the references therein). Nevertheless, we show that for every solution u of (1.2) there exists $p \in W^*$ such that the pair (u,p) solves (3.8) and (3.9). These necessary and sufficient conditions for the solvability of (1.2) are the basic tools for the study of non-convex problems, as we show in the forthcoming paper [10].

2. Preliminaries

2.1. Notation

In what follows, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ will denote the standard scalar product and the Euclidean norm in $\mathbb{R}^d$, $d \geq 1$, while $B_r^d \subset \mathbb{R}^d$ will denote the open ball centered at the origin and with radius $r > 0$. We write $\mathbb{R} \doteq \mathbb{R}^1$, and $\overline{\mathbb{R}} \doteq]-\infty, +\infty]$. The norm of a matrix $Q = (q_{ij})_{\substack{i=1..n \\ j=1..m}}$ is defined by $|Q| \doteq \left(\sum_{ij} q_{ij}^2 \right)^{1/2}$.

We shall denote by $\overline{A}$ and int$A$ respectively the closure and the interior of a set A. We recall that the relative interior ri$A$ of a convex set A is defined as the interior of A regarded as a subset of its affine hull.

As is customary, $L^p(\Omega, \mathbb{R}^d)$ and $W^{1,p}(\Omega, \mathbb{R}^d)$, $1 \leq p \leq +\infty$, will denote the Lebesgue and Sobolev spaces of functions defined in an open set Ω and with values in $\mathbb{R}^d$. If $d = 1$ we write $L^p(\Omega) \doteq L^p(\Omega, \mathbb{R})$ and $W^{1,p}(\Omega) \doteq W^{1,p}(\Omega, \mathbb{R})$. The usual norm in $L^p(\Omega, \mathbb{R}^d)$ will be denoted by $\| \cdot \|_{L^p}$, and $\mathcal{B}^p_r \subseteq L^p(\Omega, \mathbb{R}^d)$ will be the ball centered at 0 and with radius $r > 0$. If $\Omega = [0, R] \subset \mathbb{R}$, we set $AC([0, R], \mathbb{R}^d) \doteq W^{1,1}([0, R], \mathbb{R}^d)$, while $AC_{loc}(]0, R], \mathbb{R}^d)$ will denote the set of all functions u such that $u \in AC([\varepsilon, R], \mathbb{R}^d)$ for every $0 < \varepsilon < R$.

2.2. Convex functions and subgradients

Given a function $\psi\colon \mathbb{R}^d \to \overline{\mathbb{R}}$, we shall denote by $\operatorname{Dom}\psi$ its effective domain, defined as $\{\xi \in \mathbb{R}^d; \psi(\xi) \in \mathbb{R}\}$, and by ψ^* its dual function, defined by $\psi^*(p) \doteq \sup_{\xi \in \mathbb{R}^d} \{\langle p, \xi \rangle - \psi(\xi)\}$ for every $p \in \mathbb{R}^d$. If ψ is a convex function, we define its subgradient at $\xi \in \operatorname{Dom}\psi$ by

$$\partial\psi(\xi) \doteq \left\{p \in \mathbb{R}^d;\ \psi(\eta) \geq \psi(\xi) + \langle p, \eta - \xi \rangle\,,\ \text{for every } \eta \in \mathbb{R}^d \right\}. \tag{2.1}$$

By definition, we set $\partial\psi(\xi) \doteq \emptyset$ for every $\xi \notin \operatorname{Dom}\psi$. We recall that, if ψ is differentiable at ξ, then $\partial\psi(\xi) = \{\nabla\psi(\xi)\}$.

In the rest of the paper we shall deal with families of convex functions $\xi \mapsto \psi(t, \xi)$ depending on a real parameter t, and $\partial\psi(t, \xi)$ will denote the subgradient with respect to the second variable.

A function $\psi\colon [0, R] \times [0, +\infty[\to \overline{\mathbb{R}}$ is said to be a convex integrand if the map $\xi \mapsto \psi(t, |\xi|)$, $\xi \in \mathbb{R}$, is convex and lower semicontinuous for a.e. $t \in [0, R]$, and there exists a Borel function $\hat{\psi}\colon [0, R] \times [0, +\infty[\to \overline{\mathbb{R}}$ such that $\hat{\psi}(t, \cdot) = \psi(t, \cdot)$ for a.e. $t \in [0, R]$.

In the following proposition we collect some well known properties of the subgradient (see [13]).

Proposition 2.1. *Let $\psi\colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be a convex function. Then the following properties hold:*

(i) *if ψ is bounded from above in a non–empty open set A, then ψ is locally Lipschitz continuous in A;*
(ii) *for every $\xi \in \mathbb{R}^d$, the set $\partial\psi(\xi)$ (possibly empty) is convex and closed in $\mathbb{R}^d$;*
(iii) *if $\xi \in \operatorname{int}\operatorname{Dom}\psi$, then $\partial\psi(\xi)$ is a non-empty compact set;*
(iv) *if $\psi^* \not\equiv -\infty$, then $p \in \partial\psi(\xi)$ if and only if $\xi \in \partial\psi^*(p)$.*

2.3. Set-valued mappings

Let X and Y be normed spaces. A multifunction $\Psi\colon X \to 2^Y$ is said to be upper semicontinuous if $\Psi^{-1}(A)$ is closed in X whenever $A \subseteq Y$ is closed. It can be checked that, if Ψ has compact values, then Ψ is upper semicontinuous if and only if for every sequence $(x_k)_k \subseteq X$ converging to a point $x \in X$ and every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\Psi(x_k) \subseteq \Psi(x) + B_\varepsilon(0)$ for every $k \geq k_\varepsilon$ (see [12], Proposition 1.1). We recall that, if $\psi\colon X \to \overline{\mathbb{R}}$ is a convex function, then the multifunction $\Psi(x) \doteq \partial\psi(x)$ is upper semicontinuous.

The following fixed point theorem will be fundamental to our aims (see [12], Corollary 11.3).

Proposition 2.2. *Let X be a Banach space, $D \subseteq X$ be a nonempty closed bounded convex set, and $\Psi\colon D \to 2^X$ be a upper semicontinuous multifunction with closed convex* *values. Assume that* $\Psi(D) \subseteq D$, *and* $\overline{\Psi(D)}$ *is a compact subset of* X. *Then* Ψ *admits a fixed point, that is, there exists* $x \in X$ *such that* $x \in \Psi(x)$.

3. The existence result

Let us consider the minimization problem

$$\min_{v \in W_0^{1,1}(B_R^n, \mathbb{R}^m)} \int_{B_R^n} \left[f\left(|x|, |\nabla v(x)|\right) + h(|x|, v(x)) \right] dx \doteq \min_{v \in W_0^{1,1}(B_R^n, \mathbb{R}^m)} J(v) \qquad (3.1)$$

where $n, m \geq 1$, and the maps f and h satisfy the following properties:

(H1) The map $f\colon [0,R] \times [0,+\infty[\to \overline{\mathbb{R}}$ is a convex integrand.

(H2) The map $x \mapsto f(|x|, 0)$ is integrable on B_R^n, that is $t \mapsto t^{n-1} f(t,0)$ is integrable on $(0,R)$.

(H3) $M \doteq \operatorname{ess\,inf}_{t \in [0,R]} M(t) > 0$, where $M(t) \doteq \lim_{s \to +\infty} f(t,s)/s$.

(H4) For every $\rho < M$ there exists a function c_ρ belonging to $L^1(0,R)$ such that $f(t,s) \geq \rho s - t^{1-n} c_\rho(t)$ for a.e. $t \in [0,R]$ and every $s \geq 0$.

(H5) $h\colon [0,R] \times \mathbb{R}^m \to \mathbb{R}$ is a measurable function such that $h(t,\cdot)$ is convex for every $t \in [0,R]$, and $t \mapsto t^{n-1} h(t,0) \in L^1(0,R)$.

(H6) There exists a measurable function $H\colon [0,R] \to [0,+\infty]$ such that, for every $u \in \mathbb{R}^m$, $\partial h(t,u) \subset \overline{B}^m_{H(t)}$ for a.e. $t \in [0,R]$, and $\gamma(t) \doteq t^{n-1} H(t) \in L^1(0,R)$.

(H7) $M_0 \doteq \sup_{t \in]0,R]} \left\{ t^{1-n} \int_0^t \gamma(s)\, ds \right\} < M$.

Remark 3.1. From the definition of convex integrand given in Section 2.2, it follows that, for a.e. $t \in [0,R]$, the map $s \mapsto f(t,s)$, $s \geq 0$, is monotone non-decreasing. We remark that the same property also holds for the polar function f^*.

Remark 3.2. From the convexity of $h(t,\cdot)$ and the definition of H we deduce that, for every $u \in \mathbb{R}^m$, $|h(t,u)| \leq |h(t,0)| + H(t)|u|$ for a.e. $t \in [0,R]$, hence (H5) and (H6) imply that the map $t \mapsto t^{n-1} h(t,u)$ belongs to $L^1(0,R)$.

Remark 3.3. If h does not depend on t, then in (H6) we can choose H not depending on t, so that (H7) reduces to $H < \frac{Mn}{R}$.

Remark 3.4. If f is a superlinear function, then $M = +\infty$ and (H7) is automatically satisfied.

Remark 3.5. The following example shows that the existence of a solution to (3.1) may fail if (H7) is not verified. Let $n = m = 1$, $R = 1$, $h(t,u) \doteq \lambda\, u$ with $\lambda > 0$, and

$$f(t,s) = \tilde{f}(s) \doteq \begin{cases} s - \sqrt{s}, & \text{if } s > 1/4, \\ -1/4, & \text{if } 0 \leq s \leq 1/4. \end{cases}$$

It is easy to see that f and h satisfy (H1)–(H6), with $M = 1$ and $H(t) \equiv \lambda$, so that $M_0 = \lambda R/n$. We are going to show that, if (H7) does not hold, that is if $\lambda \geq n\, M/R = 1$, then the variational problem (3.1) has no solution. Integrating by parts the term in

u, (3.1) can be rewritten as

$$\min_{u \in W_0^{1,1}(-1,1)} \int_{-1}^{1} \left[\tilde{f}\left(|u'(x)|\right) - \lambda\, x\, u'(x)\right] dx. \tag{3.2}$$

It is easy to check that $\xi \mapsto \tilde{f}(|\xi|)$ is continuously differentiable and $\tilde{f}'(s) = 1 - s^{-1/2}/2$ if $s > 1/4$, while $\tilde{f}'(s) = 0$ if $0 \leq s \leq 1/4$. By the classical Euler-Lagrange necessary conditions, for every solution $u \in W_0^{1,1}(-1,1)$ to (3.2) there exists $p \in \mathbb{R}$ such that $\tilde{f}'(|u'(x)|)\operatorname{sign} u'(x) = p + \lambda\, x$ for a.e. $x \in [-1,1]$. On the other hand, since $|\tilde{f}'(s)| < 1$ for every s, then the Euler-Lagrange conditions cannot hold for a.e. $x \in [-1,1]$ whenever $\lambda > 1$. Moreover, if $\lambda = 1$, then it must be $p = 0$, so that $u'(x) = \operatorname{sign}(x)\left[4(1-|x|)^2\right]^{-1}$ for a.e. $x \in [-1,1]$. Since $u' \notin L^1(-1,1)$, u cannot be a solution to (3.1).

In order to simplify the notation, we introduce the function $g\colon [0,R] \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined by $g(t,\xi) \doteq f(t,|\xi|)$, so that

$$\partial g(t,\xi) = \begin{cases} \partial f(t,|\xi|)\dfrac{\xi}{|\xi|}, & \text{if } |\xi| \neq 0, \\ \overline{B}_{r_0(t)}, & \text{if } |\xi| = 0, \end{cases} \tag{3.3}$$

$$\partial g^*(t,p) = \begin{cases} \partial f^*(t,|p|)\dfrac{p}{|p|}, & \text{if } |p| \neq 0, \\ \overline{B}_{r_1(t)}, & \text{if } |p| = 0, \end{cases} \tag{3.4}$$

where $r_0(t) \doteq f'_+(t,0)$ and $r_1(t) \doteq (f^*)'_+(t,0)$.

The following lemma gives an a-priori bound on the selections of ∂g^* which will be frequently used in the rest of the paper.

Lemma 3.6. *Assume that* (H1)–(H4) *hold, and let* $M_0 \in]0, M[$ *be fixed. Then there exists a function* $U \in L^1(0,R)$ *such that for every* $q \in L^\infty([0,R],\mathbb{R}^m)$ *with* $\|q\|_{L^\infty} \leq M_0$, *and for every measurable selection* $\xi(t)$ *of the multifunction* $t \mapsto t^{n-1}\partial g^*(t,q(t))$ *one has* $|\xi(t)| \leq U(t)$ *for a.e.* $t \in [0,R]$.

Proof. Let $\delta_0 \doteq \dfrac{M - M_0}{2}$, and let us define the function

$$U(t) \doteq t^{n-1}\frac{f^*(t, M_0 + \delta_0) - f^*(t,0)}{\delta_0}\,. \tag{3.5}$$

Notice that, by the very definition of $M(t)$ in (H3), one obtains that, for a.e. $t \in [0,R]$, $\overline{\operatorname{Dom}} f^*(t,\cdot) = [-M(t), M(t)]$, and then $]-M, M[\subseteq \operatorname{Dom} f^*(t,\cdot)$. Hence the function U is well defined. In order to prove that $U \in L^1(0,R)$, we show that the map $t \mapsto t^{n-1}f^*(t,s)$ is integrable on $[0,R]$ for every $s \in]-M, M[$. Indeed, given $s \in]-M, M[$, by (H4) and the very definition of f^* we deduce that there exists $c_s \in L^1(0,R)$ such that

$$-t^{n-1}f(t,0) \leq t^{n-1}f^*(t,s) \leq c_s(t)\,, \qquad \text{a.e. } t \in [0,R]\,,$$

which implies, together with (H2), that the map $t \mapsto t^{n-1}f^*(t,s)$, and hence U, is integrable on $[0,R]$.

Let us fix $q \in L^\infty([0,R],\mathbb{R}^m)$ with $\|q\|_{L^\infty} \le M_0$, and let $\xi(t)$ be a measurable selection of the multifunction $t \mapsto t^{n-1}\partial g^*(t,q(t))$. For a.e. $t \in [0,R]$ such that $q(t) \neq 0$ we have $|\xi(t)| \in t^{n-1}\partial f^*(t,|q(t)|)$, so that, thanks to the monotonicity of $f^*(t,\cdot)$,

$$\begin{aligned} t^{n-1}\left[f^*(t,M_0+\delta_0) - f^*(t,0)\right] &\ge t^{n-1}\left[f^*(t,M_0+\delta_0) - f^*(t,|q(t)|)\right] \ge \\ &\ge |\xi(t)|\left(M_0+\delta_0-|q(t)|\right) \ge \delta_0\,|\xi(t)|\,, \end{aligned}$$

whereas, if $|q(t)| = 0$, from (3.4) we have $|\xi(t)| \le t^{n-1}(f^*)'_+(t,0)$, which directly implies that $|\xi(t)| \le U(t)$. □

As a first step, we reduce problem (3.1) to a minimization problem on the set W defined in (1.3).

Remark 3.7. Notice that, if $v \in W_0^{1,1}(B_R^n,\mathbb{R}^m)$ is a radially symmetric function, that is $v(x) = u(|x|)$ for some function $u\colon [0,R] \to \mathbb{R}^m$, then u belongs to W. Namely, it can be easily checked that u belongs to $AC_{loc}(]0,R],\mathbb{R}^m)$, and, denoting by α_n the $(n-1)$–dimensional Hausdorff measure of ∂B_1^n,

$$\alpha_n \int_0^R t^{n-1}\,|u'(t)|\,dt = \int_{B_R^n} \left|\left\langle \nabla v(x), \frac{x}{|x|}\right\rangle\right|\,dx \le \int_{B_R^n} |\nabla v(x)|\,dx\,,$$

so that $t^{n-1}|u'(t)| \in L^1(0,R)$. Finally, since $v(\omega R) = 0$ for a.e. ω with $|\omega| = 1$, we obtain $u(R) = 0$, hence $u \in W$.

The following lemma, together with Remark 3.7, shows that problem (3.1) has a solution if and only if problem

$$\min_{u\in W} \int_0^R t^{n-1}\left[f(t,|u'(t)|) + h(t,u(t))\right]\,dt \doteq \min_{u\in W} F(u) \tag{3.6}$$

has a solution. The proof follows the lines of the analogue for problems defined in $W_0^{1,1}(B_R^n)$, proved in [1].

Lemma 3.8. *For every $v \in W_0^{1,1}(B_R^n,\mathbb{R}^m)$ there exists a radially symmetric function $\tilde v \in W_0^{1,1}(B_R^n,\mathbb{R}^m)$ such that $J(\tilde v) \le J(v)$.*

Proof. Given $v \in W_0^{1,1}(B_R^n,\mathbb{R}^m)$, we define

$$\tilde v(x) \doteq \frac{1}{\alpha_n}\int_{|\omega|=1} v(\omega\,|x|)\,d\omega\,.$$

Clearly $\tilde v$ is radially symmetric, and

$$\nabla \tilde v_i(x) = \frac{1}{\alpha_n}\frac{x}{|x|}\int_{|\omega|=1} \langle \nabla v_i(\omega\,|x|),\omega\rangle\,d\omega, \qquad a.e.\ x \in B_R^n,$$

for every $i = 1\ldots m$ (see [1]). Then, by the definition of $\tilde v$ and the estimate

$$|\nabla \tilde v(x)| \le \frac{1}{\alpha_n}\int_{|\omega|=1} |\nabla v(\omega\,|x|)|\,d\omega\,, \tag{3.7}$$

we deduce that $\tilde{v} \in W^{1,1}(B^n_R, \mathbb{R}^m)$. In order to prove that $\tilde{v} \in W^{1,1}_0(B^n_R, \mathbb{R}^m)$, we consider the functions

$$\tilde{\varphi}_k(x) = \frac{1}{\alpha_n} \int_{|\omega|=1} \varphi_k(\omega\,|x|)\, d\omega\,, \qquad x \in B^n_R\,,\ k \in \mathbb{N}\,,$$

where $(\varphi_k)_k$ is a sequence of functions belonging to $C^\infty_c(B^n_R, \mathbb{R}^m)$ which converges to v in the strong topology of $W^{1,1}(B^n_R, \mathbb{R}^m)$. It can be easily checked that $(\tilde{\varphi}_k)_k$ converges to $\tilde{v}$ in the same topology, which implies that $\tilde{v} \in W^{1,1}_0(B^n_R, \mathbb{R}^m)$. Finally, from the convexity of f and h, and Jensen's inequality, we conclude that $J(\tilde{v}) \leq J(v)$. □

Now our aim is to find a solution to problem (3.6). Our starting point is the fact that, even if we deal with functions not belonging to $AC([0, R], \mathbb{R}^m)$, it is possible to associate to problem (3.6) a system of differential inclusions of the Euler-Lagrange type. More precisely, if W and W^* are the sets defined in (1.3) and (1.4) respectively, then the following lemma holds.

Lemma 3.9. *Assume that* $(H1)$ *and* $(H5)$ *hold, and suppose that the pair* $(u, p) \in W \times W^*$ *satisfies the differential inclusions*

$$p'(t) \in t^{n-1}\partial h(t, u(t)), \tag{3.8}$$

$$p(t) \in t^{n-1}\partial g(t, u'(t)), \tag{3.9}$$

for a.e. $t \in [0, R]$. *Then* u *is a solution to problem* (3.6).

Proof. The case $m = 1$ is proved in [8], Remark 3.5. Although the general case does not present new difficulties, we give here a sketch of the proof for the reader's convenience.

Assume that the pair $(u, p) \in W \times W^*$ satisfies (3.8) and (3.9), let $w \in W$ be fixed, and let $z \doteq w - u$. From (3.8) and (3.9) we have that

$$F(w) - F(u) \geq \int_0^R \left[\langle p(t), z'(t)\rangle + \langle p'(t), z(t)\rangle\right] dt\,. \tag{3.10}$$

We claim that the functions $\langle p(t), z'(t)\rangle$ and $\langle p'(t), z(t)\rangle$ belong to $L^1(0, R)$, and

$$\int_0^R \langle p'(t), z(t)\rangle\; dt = -\int_0^R \langle p(t), z'(t)\rangle\; dt\,, \tag{3.11}$$

so that, by (3.10), $F(w) \geq F(u)$ for every $w \in W$.

Given $\varepsilon > 0$, we have that both z and p belong to $AC([\varepsilon, R], \mathbb{R}^m)$. Since $z(R) = 0$, integrating by parts we get

$$\int_\varepsilon^R \langle p'(t), z(t)\rangle\; dt = -\langle p(\varepsilon), z(\varepsilon)\rangle - \int_\varepsilon^R \langle p(t), z'(t)\rangle\; dt\,. \tag{3.12}$$

Moreover, z belongs to W, so that there exists $\xi \in L^1([0, R], \mathbb{R}^m)$ such that $z(t) = -\int_t^R s^{1-n}\xi(s)\, ds$ for every $t \in]0, R]$, whereas p belongs to W^*, so that there exists $q \in$

$L^1([0,R],\mathbb{R}^m)$ such that $p(t) = \int_0^t s^{n-1} q(s)\,ds$ for every $t \in [0,R]$. One can easily check that

$$|\langle p'(t), z(t)\rangle| \le \|\xi\|_{L^1} |q(t)| \,, \quad |\langle p(t), z'(t)\rangle| \le \|q\|_{L^1} |\xi(t)| \,,$$
$$|\langle p(t), z(t)\rangle| \le \|\xi\|_{L^1} \int_0^t |q(s)|\, ds \,, \quad \forall t \in]0,R] \,.$$

Then we can pass to the limit in (3.12), obtaining (3.11). □

As a first step, we prove the existence of a minimizer of F provided that h is a smooth function with respect to u.

Theorem 3.10. *Assume that* (H1)–(H7) *hold and that* $h(t,\cdot) \in C^1(\mathbb{R}^m)$ *for every* $t \in [0,R]$*. Then problem* (3.6) *admits at least one solution* $u \in W$*.*

In view of Lemma 3.9, our aim will be to find a pair $(u,p) \in W \times W^*$ satisfying (3.8) and (3.9). This goal will be achieved by introducing a multifunction defined on a subset of W, whose fixed points are solutions of (3.8) and (3.9) for a suitable $p \in W^*$.

Let $U \in L^1(0,R)$ be the function defined in (3.5) corresponding to the constant M_0 introduced in (H7), and let V be the set

$$V \doteq \{v \in AC([0,R],\mathbb{R}^m) \mid v(R) = 0\,,\ |v'(t)| \le U(t) \text{ a.e. } t \in [0,R]\} \,. \tag{3.13}$$

For every $v \in V$ we define

$$B_v(t) \doteq t^{1-n} \int_0^t s^{n-1} \nabla h \left(s, -\int_s^R \sigma^{1-n} v'(\sigma)\, d\sigma \right) ds \tag{3.14}$$

for $t \in]0,R]$, and $B_v(0) \doteq 0$. Notice that, by (H6) and (H7), we have that $\|B_v\|_{L^\infty} \le M_0$, hence, from Lemma 3.6, for every $v \in V$ the set

$$\Phi(v) \doteq \left\{u \in AC([0,R],\mathbb{R}^m);\ u(R) = 0,\ u'(t) \in t^{n-1}\partial g^*(t, B_v(t)) \text{ a.e. } t\right\} \tag{3.15}$$

is well-defined and

$$|u'(t)| \le U(t) \qquad \text{for every } u \in \Phi(v). \tag{3.16}$$

In particular, the estimate (3.16) implies that $\Phi(v) \subset V$ for every $v \in V$. We want to show that the multifunction $\Phi\colon V \to 2^V$ has a fixed point, that is there exists $v \in V$ such that $v \in \Phi(v)$.

In order to apply Proposition 2.2, we have to investigate some convergence properties of the subgradients of convex functions (see [8], Lemma 4.6).

Lemma 3.11. *Let* $\psi_k\colon [0,R] \times \mathbb{R}^m \to \overline{\mathbb{R}}$*,* $k \in \mathbb{N}$*, be a sequence of Borel measurable functions such that, for a.e.* $t \in [0,R]$*,* $\psi_k(t,\cdot)$ *is convex for every* $k \in \mathbb{N}$*, and there exists a nonempty open set* $D(t) \subset \bigcap_k \operatorname{Dom} \psi_k(t,\cdot)$*. Assume that, for a.e.* $t \in [0,R]$*, the sequence* $(\psi_k(t,\cdot))_k$ *converges pointwise on* $D(t)$ *to a finite function* $\psi(t,\cdot)$*.*

Let $q_k\colon [0,R] \to \mathbb{R}^m$*,* $k \in \mathbb{N}$*, be a sequence of measurable functions converging a.e. to a function* q*, and assume that* $q(t) \in D(t)$ *for a.e.* $t \in [0,R]$*. Furthermore, assume* *that there exists $G \in L^1(0,R)$ such that, for a.e. $t \in [0,R]$, $\partial\psi(t,q(t)) \cap B^m_{G(t)} \neq \emptyset$ and* $\sup\{|p|;\ p \in \bigcup_k \partial\psi_k(t,q_k(t))\} \leq G(t)$.

Let us define the sets $L_k \doteq \{\eta \in L^1(0,R);\ \eta(t) \in \partial\psi_k(t,q_k(t)),\ a.e.\ t\}$, $k \in \mathbb{N}$, and $L \doteq \{\eta \in L^1(0,R);\ \eta(t) \in \partial\psi(t,q(t))\ a.e.\ t\}$. Then, for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $L_k \subset L + \mathcal{B}^1_\varepsilon$ for every $k \geq k_\varepsilon$. Furthermore, if $\xi_k \in L_k$ for every $k \in \mathbb{N}$, and $\xi_k \rightharpoonup \xi$ w-L^1, then $\xi \in L$.

Lemma 3.12. *For every $v \in V$, $\Phi(v)$ has nonempty convex compact values in $L^\infty([0,R],\mathbb{R}^m)$.*

Proof. The convexity is trivial. Moreover, for every $v \in V$, $\Phi(v)$ is a nonempty relatively compact subset of $L^\infty([0,R],\mathbb{R}^m)$. Namely, if $\xi(t)$ is a measurable selection of the multifunction $t \mapsto t^{n-1}\partial g^*(t,B_v(t))$, then by Lemma 3.6, $\xi \in L^1([0,R],R^m)$ and $u(t) \doteq -\int_t^R \xi(s)\,ds$ belongs to $\Phi(v)$. Furthermore, from (3.16) we deduce that the set $\{u';\ u \in \Phi(v)\}$ is equiabsolutely integrable so that, by the Ascoli-Arzelà theorem, $\Phi(v)$ is relatively compact in $L^\infty([0,R],\mathbb{R}^m)$ (see [3], 10.2.i).

It remains to prove that the set $\Phi(v)$ is closed in the same space. Assume that $(w_k)_k$ is a sequence in $\Phi(v)$ converging to a function w strongly in $L^\infty([0,R],\mathbb{R}^m)$. From (3.16) we easily infer that $w \in AC([0,R],\mathbb{R}^m)$ and $(w'_k)_k$ converges to w' in the weak topology of $L^1([0,R],\mathbb{R}^m)$. Hence, applying Lemma 3.11 to $\psi_k(t,\cdot) = \psi(t,\cdot) = t^{n-1}g^*(t,\cdot)$, $q_k = B_v$, $D(t) =]-M,M[$ for every $t \in [0,R]$, $L_k = L = \{u';\ u \in \Phi(v)\}$, $\xi_k = w'_k$, we obtain that $w \in \Phi(v)$. □

Lemma 3.13. *The multifunction Φ is upper semicontinuous in the strong topology of $L^\infty([0,R],\mathbb{R}^m)$.*

Proof. Since, by Lemma 3.12, the multifunction Φ has convex compact values, it is enough to show that for every sequence $(v_k)_k \subset V$ converging strongly to v in $L^\infty([0,R],\mathbb{R}^m)$, and for every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\Phi(v_k) \subseteq \Phi(v) + \mathcal{B}^\infty_\varepsilon$ for every $k \geq k_\varepsilon$. Since $v_k \in V$, we have that $|v'_k(t)| \leq U(t)$ a.e. in $[0,R]$, which implies that $(v'_k)_k$ converges to v' in the weak topology of $L^1([0,R],\mathbb{R}^m)$ (notice that the whole sequence converges to v' due to the fact that the sequence $(v_k)_k$ converges to v in $L^\infty([0,R],\mathbb{R}^m)$). Hence $(B_{v_k}(t))_k$ converges to $B_v(t)$ for a.e. $t \in [0,R]$, and we can apply Lemma 3.11 to $\psi_k(t,\cdot) = \psi(t,\cdot) = t^{n-1}g^*(t,\cdot)$, $q_k = B_{v_k}$, $D(t) =]-M,M[$ for every $t \in [0,R]$, $L_k = \{u';\ u \in \Phi(v_k)\}$, $L = \{u';\ u \in \Phi(v)\}$, obtaining the required property for Φ. □

Proof of Theorem 3.10. By Lemmas 3.12 and 3.13 we infer that Φ is a upper semicontinuous multifunction with nonempty convex compact values in the space $L^\infty([0,R],\mathbb{R}^m)$. On the other hand one can easily check that V is convex and compact in the same space. Hence we can apply Proposition 2.2, obtaining that there exists $v \in V$ such that $v \in \Phi(v)$. If we define

$$u(t) \doteq -\int_t^R s^{1-n}v'(s)\,ds\,, \qquad p(t) \doteq t^{n-1}B_v(t)\,,$$

then $u \in W$, $p \in W^*$, $u'(t) \in \partial g^*(t,t^{1-n}p(t))$ and $p'(t) = t^{n-1}\nabla h(t,u(t))$ for a.e. $t \in [0,R]$. Hence (3.8) and (3.9) are fulfilled, so that, by Lemma 3.9, u is a solution of problem (3.6). □

In order to deal with nonsmooth h, we need the following notion of solution to problem (3.6) obtained by approximation. In Proposition 3.17 we shall show that every solution of this kind is actually a solution to (3.6).

Definition 3.14. We say that $u \in W$ is a solution to (3.6) obtained by approximation if there exist three sequences $(h_k)_k$, $(u_k)_k$ and $(p_k)_k$ such that:

(i) $\lim_k u_k(t) = u(t)$ for a.e. $t \in [0, R]$;
(ii) for every $k \in \mathbb{N}$, $h_k\colon [0, R] \times \mathbb{R}^m \to \mathbb{R}$ is a measurable function, $h_k(t, \cdot)$ is convex and of class C^1 for every $t \in [0, R]$, and $(h_k(t, \cdot))_k$ converges pointwise to $h(t, \cdot)$ for a.e. $t \in [0, R]$;
(iii) for every $k \in \mathbb{N}$, $(u_k, p_k) \in W \times W^*$ is a solution of the Euler–Lagrange inclusions

$$p_k'(t) = t^{n-1} \nabla h_k(t, u_k(t)), \tag{3.17}$$

$$p_k(t) \in t^{n-1} \partial g(t, u_k'(t)), \tag{3.18}$$

for a.e. $t \in [0, R]$.

Remark 3.15. One can easily check that, if u is a solution obtained by approximation, and $(h_k)_k$, $(u_k)_k$, $(p_k)_k$ are the sequences appearing in Definition 3.14, then, for every $k \in \mathbb{N}$, u_k is a solution to the problem

$$\min_{v \in W} \int_0^R t^{n-1} \left[f(t, |v'(t)|) + h_k(t, v(t)) \right] dt \,. \tag{3.19}$$

In the following lemma we list all the a-priori estimates on the approximating sequences that will be needed in the last part of this section.

Lemma 3.16. *Assume that* (H1)–(H7) *hold, and let* $(u_k, p_k) \in W \times W^*$ *be a solution to* (3.17)–(3.18). *Then*

$$|p_k'(t)| \le \gamma(t), \qquad a.e.\ t \in [0, R], \tag{3.20}$$

$$\left\| t^{1-n} p_k(t) \right\|_{L^\infty} \le M_0, \tag{3.21}$$

$$|u_k'(t)| \le t^{1-n} U(t), \qquad a.e.\ t \in [0, R]. \tag{3.22}$$

Proof. The estimate (3.20) follows from (3.17) and (H6), while (3.21) is a direct consequence of (3.20) and (H7). Finally, by (3.21), (3.18), Proposition 2.1(iv) and Lemma 3.6 we obtain (3.22). □

Proposition 3.17. *Assume that* (H1)–(H7) *hold. Then every solution to* (3.6) *obtained by approximation is a solution to* (3.6).

Proof. Let $u \in W$ be a solution to (3.6) obtained by approximation, and let $(h_k)_k$, $(u_k, p_k)_k$ be the sequences appearing in Definition 3.14.

From (3.20) we infer that there exists a subsequence, still denoted by $(p_k')_k$, which converges to a function z in the weak topology of $L^1([0, R], \mathbb{R}^m)$. It is easy to see that the function $p(t) \doteq \int_0^t z(s)\, ds$ belongs to W^*, and $\lim_k p_k(t) = p(t)$ for every $t \in [0, R]$.

Since $\lim_k u_k(t) = u(t)$ for a.e. $t \in [0, R]$, from (3.22) we deduce that

$$t^{n-1} u_k'(t) \rightharpoonup t^{n-1} u'(t) \quad \text{weak-}L^1([0, R], \mathbb{R}^m). \tag{3.23}$$

From Proposition 2.1(iv), the inclusion (3.18) can be rewritten as

$$t^{n-1}u_k'(t) \in t^{n-1}\partial g^*(t, t^{1-n}p_k(t)), \qquad a.e.\ t \in [0,R]. \tag{3.24}$$

Since $\lim_k p_k(t) = p(t)$ for every $t \in [0,R]$, from (3.23), (3.24) and Lemma 3.11 applied to $\psi_k(t,\cdot) = \psi(t,\cdot) = t^{n-1}g^*(t,\cdot)$, $q_k(t) = t^{1-n}p_k(t)$, $q(t) = t^{1-n}p(t)$, $D(t) =]-M, M[$ for every $t \in [0,R]$, $\xi_k(t) = t^{n-1}u_k'(t)$, $\xi(t) = t^{n-1}u'(t)$, we obtain

$$t^{n-1}u'(t) \in t^{n-1}\partial g^*(t, t^{1-n}p(t)), \qquad a.e.\ t \in [0,R]\,, \tag{3.25}$$

which implies, thanks to Lemma 2.1(iv), that (u,p) satisfies (3.9). Finally, from (3.17), and Lemma 3.11 applied to $\psi_k(t,\cdot) \doteq t^{n-1}h_k(t,\cdot)$, $\psi(t,\cdot) \doteq t^{n-1}h(t,\cdot)$, $D(t) = \mathbb{R}^m$ for every $t \in [0,R]$, $q_k = u_k$, $q = u$, $G = H$, $\xi_k = p_k'$, and $\xi = p'$, we conclude that the pair (u,p) also satisfies the inclusion (3.8). Hence, by Lemma 3.9, u is a solution to problem (3.6). □

Now we are in a position to prove that, even if $h(t,\cdot)$ is not a smooth function, there exists at least one solution obtained by approximation of problem (3.6).

Theorem 3.18. *Assume that* (H1)–(H7) *hold. Then problem* (3.6) *admits at least one solution obtained by approximation.*

Proof. Let $(\phi_k)_k$ be the standard mollifiers and let $h_k(t,\cdot) \doteq \phi_k * h(t,\cdot)$. It can be easily checked that, for every $k \in \mathbb{N}$, $h_k(t,\cdot)$ is a smooth convex function which satisfies (H6) and (H7). Hence, by Theorem 3.10, there exists $u_k \in W$ solution of (3.19), and there exists $p_k \in W^*$ such that (u_k, p_k) is a solution to (3.17) and (3.18).

From (3.22) there exist a subsequence of $(u_k)_k$, still denoted by $(u_k)_k$, and a function $u \in W$ such that $\lim_k u_k(t) = u(t)$ for a.e. $t \in [0,R]$. Since $(h_k(t,\cdot))_k$ converges pointwise to $h(t,\cdot)$ for every $t \in [0,R]$, then u is a solution to (3.6) obtained by approximation. □

4. Euler-Lagrange inclusions

In the previous section we exhibited a solution to problem (3.6) which satisfies also the Euler–Lagrange inclusions (3.8) and (3.9). Now we want to prove that the solvability of these inclusions provides a necessary and sufficient condition for minimality.

Theorem 4.1. *Assume that* (H1)–(H6) *hold. Then* $u \in W$ *is a solution of problem* (3.6) *if and only if there exists* $p \in W^*$ *such that* (u,p) *is a solution of the Euler-Lagrange inclusions* (3.8) *and* (3.9).

We shall use the following result concerning necessary minimality conditions for functionals defined on absolutely continuous functions (see [15], Corollary 1).

Proposition 4.2. *Let us consider the minimization problem*

$$\min_{v \in AC([a,b],\mathbb{R}^m)} \left\{ l(v(a), v(b)) + \int_a^b L(t, v(t), v'(t))\, dt \right\} \tag{4.1}$$

where

(i) *l and $L(t,\cdot,\cdot)$ are convex functions, lower semicontinuous and not identically $+\infty$;*

(ii) *L is a normal convex integrand;*

(iii) *there exist $p \in L^\infty([a,b],\mathbb{R}^m)$, $s \in L^1([a,b],\mathbb{R}^m)$, and $\alpha \in L^1(a,b)$ such that*

$$L(t,u,\xi) \geq \langle u, s(t)\rangle + \langle \xi, p(t)\rangle - \alpha(t)$$

for a.e. $t \in [a,b]$;

(iv) *for every $u \in \mathbb{R}^m$ there exist $v \in L^1([a,b],\mathbb{R}^m)$, and $\beta \in L^1(a,b)$ such that $L(t,u,v(t)) \leq \beta(t)$ for a.e. $t \in [a,b]$;*

(v) *let $C_l \doteq \{(c_a,c_b) \in \mathbb{R}^m \times \mathbb{R}^m;\ l(c_a,c_b) < +\infty\}$ and let C_L be the set of all pairs $(c_a,c_b) \in \mathbb{R}^m \times \mathbb{R}^m$ such that there exists $v \in AC([a,b],\mathbb{R}^m)$ with $v(a) = c_a$, $v(b) = c_b$ and $(v(t),v'(t)) \in \overline{\mathrm{Dom}} L(t,\cdot,\cdot)$ for a.e. $t \in [a,b]$. Assume that $\mathrm{ri} C_l \cap \mathrm{ri} C_L \neq \emptyset$.*

Then $u \in AC([a,b],\mathbb{R}^m)$ is a solution to (4.1) if and only if there exists $p \in AC([a,b],\mathbb{R}^m)$ such that

$$p'(t) \in \partial_2 L(t,v(t),v'(t)), \tag{4.2}$$

$$p(t) \in \partial_3 L(t,v(t),v'(t)), \tag{4.3}$$

for a.e. $t \in [a,b]$ and

$$p(a) \in \partial_1 l(v(a),v(b)), \tag{4.4}$$

$$-p(b) \in \partial_2 l(v(a),v(b)), \tag{4.5}$$

where ∂_i denotes the subdifferential with respect to the $i-th$ variable.

Proof of Theorem 4.1. If $(u,p) \in W \times W^*$ is a solution of (3.8) and (3.9), then the minimality of u follows from Lemma 3.9.

Conversely, assume that u is a solution of problem (3.6). For every $k \in \mathbb{N}$, $k > 1/R$, let us consider the set

$$W_k \doteq \{v \in AC([1/k,R]) \mid v(R) = 0\}$$

and the functional

$$F_k(v) \doteq \int_{1/k}^{R} t^{n-1}[f(t,|v'(t)|) + h(t,v(t))]\,dt$$

defined for $v \in W_k$. Let us define $\varepsilon_k \doteq \int_0^{1/k} \gamma(t)\,dt$, where γ is the function defined in (H6), and $l_k(x) \doteq \varepsilon_k\,|x - u(1/k)|$ for every $x \in \mathbb{R}^m$. We remark that, from (H6) and the absolute continuity of the Lebesgue integral,

$$\lim_{k\to\infty} \varepsilon_k = 0. \tag{4.6}$$

We want to prove that the restriction u_k of u to the interval $[1/k,R]$ is a solution to the problem

$$\min_{v\in W_k} \{F_k(v) + l_k(v(1/k))\} \tag{4.7}$$

for every $k > 1/R$. To do so, fix $v \in W_k$, and define $\delta \doteq v(1/k) - u(1/k)$ and

$$w(t) \doteq \begin{cases} u(t) + \delta, & \text{if } t \in]0, 1/k], \\ v(t), & \text{if } t \in [1/k, R]. \end{cases}$$

Since w belongs to W and u is a solution of (3.6), we have that $F(u) \le F(w)$, which implies that

$$F_k(u_k) \le F_k(v) + \int_0^{1/k} t^{n-1}[h(t, u(t) + \delta) - h(t, u(t))]\, dt \le F_k(v) + \varepsilon_k |\delta| \,, \tag{4.8}$$

where the last inequality follows from (H6) and the definition of ε_k. From (4.8) we obtain

$$F_k(u_k) + l_k(u_k(1/k)) = F_k(u_k) \le F_k(v) + l_k(v(1/k))$$

for every $v \in W_k$, so that u_k is a solution of (4.7).

Now we claim that it is possible to apply Proposition 4.2 to problem (4.7). Namely (i) and (ii) follow from (H1) and (H5), (iii) is satisfied with $p \equiv 0$, $\alpha(t) = -t^{n-1}\,[f(t,0) + h(t,0)]$, which belongs to $L^1(1/k, R)$ due to (H2) and (H5), and $s(t)$ a measurable selection of $t \mapsto t^{n-1}\partial h(t, 0)$, which is integrable since $|s(t)| \le \gamma(t)$ for a.e. $t \in [0, R]$. Condition (iv) is satisfied with $v \equiv 0$ and $\beta(t) = t^{n-1}\,[f(t,0) + h(t,u)]$, which belongs to $L^1(1/k, R)$ due to (H2) and Remark 3.2. It remains to prove that (v) is satisfied. Clearly $\mathrm{ri} C_{l_k} = C_{l_k} = \mathbb{R}^m \times \{0\}$, so that we have to prove that the set of starting points of arcs $v \in AC([1/k, R], \mathbb{R}^m)$ with $v(R) = 0$ and such that $v'(t) \in \overline{\mathrm{Dom}} f(t,\cdot)$ for a.e. $t \in [1/k, R]$ contains an open subset. From (H2) we infer that $0 \in \overline{\mathrm{Dom}} f(t,\cdot)$ for a.e. $t \in [0, R]$.

We can assume, without loss of generality, that there exists $v_0 \in W$ with $v_0'(t) \in \overline{\mathrm{Dom}} f(t,\cdot)$ for a.e. $t \in [0, R]$ and $v_0 \not\equiv 0$. Indeed, if this is not true, then $\mathrm{Dom}\, f(t,\cdot) = \{0\}$ for a.e. $t \in [0, R]$, so that $\partial g(t, 0) = \mathbb{R}^m$ for a.e. $t \in [0, R]$. Hence, choosing a measurable selection $z(t)$ of $t \mapsto t^{n-1}\partial h(t, 0)$, the Euler–Lagrange inclusions (3.8) and (3.9) are fulfilled with $p(t) \doteq \int_0^t z(s)\, ds$.

Let v_0 be as above and let $r_k \doteq \int_{1/k}^R |v_0'(t)|\, dt$. Since $v_0' \not\equiv 0$, there exists $k_0 \in \mathbb{N}$ such that $r_k > 0$ for every $k \ge k_0$. For every $\xi \in B_1^m$, let us define

$$v_\xi(t) \doteq \xi \int_t^R |v_0'(s)|\, ds \,.$$

We have $v_\xi(1/k) = r_k \xi$ and, thanks to the monotonicity of the map $s \mapsto f(t, s)$, $s \ge 0$, we obtain that $f(t, |v_\xi'(t)|) \le f(t, |v_0'(t)|)$ for a.e. $t \in [1/k, R]$. Hence, if $k \ge k_0$, $B_{r_k}^m \times \{0\} \subset \mathrm{ri} C_{l_k} \cap \mathrm{ri} C_{L_k}$, and (v) in Proposition 4.2 is satisfied.

Thus we can conclude that for every $k \ge k_0$ there exists $p_k \in AC([1/k, R], \mathbb{R}^m)$ such that

$$p_k'(t) \in t^{n-1}\partial h(t, u(t)), \qquad a.e.\ t \in [1/k, R], \tag{4.9}$$

$$p_k(t) \in t^{n-1}\partial g(t, u'(t)), \qquad a.e.\ t \in [1/k, R], \tag{4.10}$$

$$p_k(1/k) \in \partial l_k(u(1/k)) \,. \tag{4.11}$$

Notice that (4.11) can be rewritten as

$$|p_k(1/k)| \le \varepsilon_k \,. \tag{4.12}$$

Let $\xi(t)$ be a fixed summable selection of the multifunction $t \mapsto t^{n-1}\partial h(t,u(t))$, $t \in [0,R]$. We extend p_k to $[0,R]$ by setting

$$p_k(t) \doteq p_k(1/k) - \int_t^{1/k} \xi(s)\,ds, \qquad t \in [0,1/k]. \tag{4.13}$$

Hence the inclusion (4.9) is satisfied for a.e. $t \in [0,R]$, so that, by (H6),

$$|p_k'(t)| \le \gamma(t), \qquad \text{for a.e. } t \in [0,R]. \tag{4.14}$$

Then $p_k \in AC([0,R],\mathbb{R}^m)$ for every $k \ge k_0$. Moreover, from (4.13), (4.14), and (4.12) we get

$$|p_k(0)| \le |p_k(1/k)| + \int_0^{1/k} \gamma(s)\,ds \le 2\varepsilon_k, \tag{4.15}$$

which implies, together with (4.6), that $\lim_k p_k(0) = 0$. Then, by the Ascoli–Arzelà compactness theorem, there exists a subsequence of $(p_k)_k$ which converges uniformly to a function $p \in AC([0,R],\mathbb{R}^m)$ with $p(0) = 0$, $|p'(t)| \le \gamma(t)$ a.e. $t \in [0,R]$, and $(p_k')_k$ converges to p' weakly in $L^1([0,R],\mathbb{R}^m)$. From (4.9) and Lemma 3.11 we deduce that $p'(t) \in t^{n-1}\partial h(t,u(t))$, for a.e. $t \in [0,R]$. Finally, for every $k \ge k_0$, let us consider the set $N_k \subseteq [1/k,R]$ with Lebesgue measure zero such that (4.10) holds for every $t \in [1/k,R]\setminus N_k$, and define $N \doteq \bigcup_k N_k$. For each $t \in]0,R] \setminus N$, we have $p_k(t) \in t^{n-1}\partial g(t,u'(t))$ for every $k > 1/t$. Since $\lim_k p_k(t) = p(t)$ and, by Proposition 2.1(ii), the set $\partial g(t,u'(t))$ is closed, we infer that $p(t) \in t^{n-1}\partial g(t,u'(t))$ for every $t \in]0,R] \setminus N$, which concludes the proof. □

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