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On the Algebraic Properties of Convex Bodies and Some Applications

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We extend the set of convex bodies up to differences (factorized pairs) of convex bodies; thereby (Minkowski) multiplication by real scalar is extended in a natural way. We show that differences of convex bodies form a special quasilinear space with group structure. The latter is abstractly studied by introducing analogues of linear combinations, dependence, basis, associated linear spaces etc. A theorem of H. Rådström for embedding of convex bodies in a normed vector space is generalized. Support functions and their differences are discussed in relation to quasilinear spaces.

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1. Introduction

Arithmetic computations with convex bodies and intervals have been increasingly used in convex, resp. interval analysis. The set of convex bodies is an abelian cancellative monoid with respect to addition and is a quasilinear space with respect to addition and multiplication by scalar. The latter means that multiplication by real scalar satisfies three of the axioms of a linear space — first distributive law, commutative law and identity law (with respect to multiplication by one); the fourth (last) axiom known as second distributive law is relaxed: it holds only for equally signed scalars. If the second distributive law holds for all scalars, then the quasilinear space becomes linear.

By means of the familiar extension method (used e. g. when defining negative numbers) the monoid of convex bodies can be embedded in an abelian group of factorized pairs (differences) of convex bodies. A corresponding isomorphic extension of the familiar (Minkowski) multiplication by scalar is introduced, here called "q-linear" multiplication by scalar. We thus obtain a natural generalization of the linear (vector) space, called "q-linear space". The axiomatic definition of the latter differs from the one of a linear space only by the second distributivity law, which is assumed to hold only for nonnegative scalars. Q-linear spaces have remarkable algebraic properties and can be effectively used for algebraic calculations. They can be characterized as linear spaces with an additional operation "conjugation". This is clearly seen on the case of (differences of) support functions, which form a linear space together with conjugation — a symmetry with respect to the origin. Similar extensions have been already studied to some extent in the case of one-dimensional intervals, see e. g. [5], but seem to be little investigated for convex

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bodies, for related papers see [1], [3], [12], [13], [15], [17], [18]–[20].

The difference between our approach and the one used by H. Rådström [13] is as follows. In the induced abelian group H. Rådström defines a linear multiplication by real scalar so that the group becomes a linear space. We introduce a q-linear multiplication by scalar as a natural isomorphic extension of the multiplication by real scalar in the quasilinear space. The q-linear multiplication by scalar and Rådström's linear multiplication by scalar coincide for nonnegative scalars. The q-linear multiplication can be expressed by a linear one and conjugation. The concept of q-linear space is a generalization of the concept of linear space; in particular it comprises the linear space as considered by Rådström. In a q-linear space we introduce analogues of linear combinations, linear dependence, and basis. Thus we outline a theory of q-linear spaces analogous to the theory of linear spaces. The extension of the partial order relation inclusion to q-linear spaces is briefly considered in the paper.

The paper is structured as follows: Sections 2 and 3 summarize some basic concepts of the quasilinear, resp. q-linear spaces, and Section 4 is devoted to algebraic transformations in q-linear spaces. In Section 5 the concepts of linear combination, linear dependence and basis in a q-linear space are introduced. In Section 6 the linear multiplication by scalar is considered and its relation to q-linear multiplication is discussed. In Section 7 several examples of q-linear spaces are briefly summarized. In Section 8 the q-linear space of (differences of) support functions is considered.

2. Convex Bodies and Quasilinear Spaces

Convex bodies. By $\mathbb{E}^m$, $m \geq 1$, we denote a real m-dimensional Euclidean vector space with origin (null) 0, $\mathbb{R}$ is the ordered field of reals. A convex compact subset of $\mathbb{E}^m$ is called *convex body (in $\mathbb{E}^m$)* [16]. The set of all convex bodies (in $\mathbb{E}^m$) is denoted by $\mathcal{K} = \mathcal{K}(\mathbb{E}^m)$. In the case $m = 1$ the elements of $\mathcal{K}(\mathbb{E}^m)$ are compact intervals on the real line, simply called *intervals.* The elements of $\mathbb{E}^m$ are also considered as elements of $\mathcal{K}$; the set of all such elements will be denoted by $\mathcal{K}_D \subset \mathcal{K}$, clearly $\mathcal{K}_D \cong \mathbb{E}^m$. Various names for the elements of $\mathcal{K}_D$ used in the literature are *single-point bodies, degenerate bodies, distributive (linear) elements, vectors* etc.

Addition. The *(Minkowski) addition* of two convex bodies is defined for $A, B \in \mathcal{K}$ by

$$A + B \quad = \quad \{c \mid c = a + b,\ a \in A,\ b \in B\}. \tag{2.1}$$

Addition defined by (2.1) and the null element 0 satisfy for all $A, B, C \in \mathcal{K}$, see e. g. [9], [13]:

$$\begin{aligned} (A+B)+C &= A+(B+C), && (2.2)\\ A+0 &= A, && (2.3)\\ A+B &= B+A, && (2.4)\\ A+C=B+C &\Longrightarrow A=B. && (2.5) \end{aligned}$$

Minkowski difference and summability. If $B = \{b\} \in \mathcal{K}_D$ and $A \in \mathcal{K}$, then we write $A + B$ as $A + b$. The opposite to b in $\mathcal{K}_D$ is denoted as $-b$, that is $b + (-b) = 0$; the element $A + \{-b\}$ is written as $A + (-b)$ or as $A - b$.

The *Minkowski difference* [2] is defined for $A, B \in \mathcal{K}$ by

$$A \stackrel{*}{-} B = \bigcap_{b \in B} (A - b), \text{ if } \neq \emptyset, \tag{2.6}$$

where the expression "if $\neq \emptyset$" means that $A \stackrel{*}{-} B$ is defined whenever the right-hand side is not empty. Note that $A \stackrel{*}{-} A = 0$ for all $A \in \mathcal{K}$.

We say that $B \in \mathcal{K}$ is a *summand of* $A \in \mathcal{K}$ if there exists $X \in \mathcal{K}$, such that $A = B + X$. If B is a summand of A, then $A \stackrel{*}{-} B$ is also a summand of A (see Lemma 3.1.8, [16]). Hence, if B is a summand of A, then $X = A \stackrel{*}{-} B$ is a solution of $X + B = A$. In other words B is a *summand of* A if $A \stackrel{*}{-} B \neq \emptyset$ and $A = B + (A \stackrel{*}{-} B)$.

Multiplication by real scalar. *Multiplication by real scalar* $* : \mathbb{R} \times \mathcal{K} \longrightarrow \mathcal{K}$ is defined by

$$\alpha * B \;\; = \;\; \{c \mid c = \alpha b, \; b \in B\}, \;\; \alpha \in \mathbb{R}, \;\; B \in \mathcal{K}. \tag{2.7}$$

Assume $A, B, C \in \mathcal{K}$, $\alpha, \beta, \gamma \in \mathbb{R}$. We have, see e. g. [16]:

$$\gamma * (A + B) \;\; = \;\; \gamma * A + \gamma * B, \tag{2.8}$$
$$\alpha * (\beta * C) \;\; = \;\; (\alpha\beta) * C, \tag{2.9}$$
$$1 * A \;\; = \;\; A, \tag{2.10}$$
$$(\alpha + \beta) * C \;\; = \;\; \alpha * C + \beta * C, \quad \text{if } \; \alpha\beta \geq 0. \tag{2.11}$$

Inner sum and difference. We may extend relation (2.11) for the case when the scalar multipliers are of different signs. Note that if $\alpha > 0$, $\beta > 0$, then it follows from (2.11) that $\beta * C$ is a summand of $(\alpha + \beta) * C$, and hence: $\alpha * C = (\alpha + \beta) * C \stackrel{*}{-} \beta * C$. Substituting $\alpha + \beta = \gamma > 0$ (and hence $\gamma > \beta > 0$) we have:

$$(\gamma - \beta) * C = \gamma * C \stackrel{*}{-} \beta * C, \;\; \gamma > \beta > 0. \tag{2.12}$$

Substituting in (2.12) $\beta = -\delta$, $\delta < 0$, we have $(\gamma + \delta) * C = \gamma * C \stackrel{*}{-} (-\delta) * C$, $\gamma > -\delta > 0$, cf. also [2]. Using the original notation α, β (instead of γ, resp. δ) we can write:

$$(\alpha + \beta) * C = \alpha * C \stackrel{*}{-} (-\beta) * C, \;\; \alpha > -\beta > 0,$$

which can be written more symmetrically as

$$(\alpha + \beta) * C = \begin{cases} \alpha * C \stackrel{*}{-} (-\beta) * C, & \text{if } \alpha\beta < 0, \; |\alpha| \geq |\beta|, \\ \beta * C \stackrel{*}{-} (-\alpha) * C, & \text{if } \alpha\beta < 0, \; |\alpha| < |\beta|. \end{cases}$$

Combining relation (2.11) and the above formula we can write a general expression of $(\alpha + \beta) * C$ in terms of $\alpha * C$ and $\beta * C$ valid for all $\alpha, \beta \in \mathbb{R}$:

$$(\alpha + \beta) * C = \begin{cases} \alpha * C + \beta * C, & \text{if } \alpha\beta \geq 0, \\ \alpha * C \stackrel{*}{-} (-\beta) * C, & \text{if } \alpha\beta < 0, \; |\alpha| \geq |\beta|, \\ \beta * C \stackrel{*}{-} (-\alpha) * C, & \text{if } \alpha\beta < 0, \; |\alpha| < |\beta|. \end{cases} \tag{2.13}$$

To write the above relation in a more compact form suitable for symbolic algebraic transformations let us introduce the *inner sum* [8]: for $A, B \in \mathcal{K}$

$$
\begin{aligned}
A +^- B &= \begin{cases} A \overset{*}{-} (-1) * B, & \text{if } \neq \emptyset, \\ B \overset{*}{-} (-1) * A, & \text{if } \neq \emptyset; \end{cases} \\
&= \begin{cases} \bigcap_{b \in B} (A + b), & \text{if } \neq \emptyset, \\ \bigcap_{a \in A} (B + a), & \text{if } \neq \emptyset. \end{cases}
\end{aligned}
$$

The inner sum $A +^- B$ is the solution X to the equation $X + (-1) * B = A$, if existing, or the solution Y to $Y + (-1) * A = B$, if existing; if both solutions exist, they coincide. In the one dimensional case ($m = 1$) it is a (normal) operation, called *inner* (*special, nonstandard*) *addition*. The *inner difference* $A -^- B = A +^- (\neg B)$ is an extension of Minkowski difference, $A -^- B$ is the solution X to $X + B = A$ if such solution exists, and $\neg Y$, where Y is the solution to $Y + A = B$ if existing.

Using "$+^-$" and the convention $+^+ = +$, relation (2.13) may be written in the form

$$(\alpha + \beta) * C \;=\; \alpha * C +^{\sigma(\alpha)\sigma(\beta)} \beta * C, \tag{2.14}$$

where $\sigma : \mathbb{R} \longrightarrow \{+, -\}$, is the sign of a real number:

$$\sigma(\alpha) = \begin{cases} +, & \text{if } \alpha \geq 0, \\ -, & \text{if } \alpha < 0. \end{cases} \tag{2.15}$$

and, $\sigma(\alpha)\sigma(\beta)$ is computed according to the following *sign rule*: $++ = -- = +$, $+- = -+ = -$. In the sequel we shall consider the ordered field $\mathbb{R}$ together with the function σ.

Relation (2.11) or its corollaries (2.13), (2.14) will be further referred to as *quasidistributive law*. Note that the Minkowski difference involved in (2.13) or (2.14) always exists, hence (2.14) holds true for any $\alpha, \beta \in \mathbb{R}$, $C \in \mathcal{K}$.

Negation and symmetry. The operator $\neg : \mathcal{K} \longrightarrow \mathcal{K}$ defined by $\neg A = (-1) * A = \{-a \mid a \in A\}$, $A \in \mathcal{K}$, is called *negation*. We have $\neg(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (\neg A)$ for $\gamma \in \mathbb{R}$ and $A \in \mathcal{K}$. We shall sometimes use the notations: $A \neg B = A + (\neg B)$, $A -^- B = A +^- (\neg B)$.

An element $A \in \mathcal{K}$ is called *centrally symmetric with respect to the origin*, briefly: *symmetric*, if $A = \neg A$. The set of all symmetric convex bodies is denoted $\mathcal{K}_S = \{A \in \mathcal{K} \mid A = \neg A\}$. For $A \in \mathcal{K}$, we have $A \neg A \in \mathcal{K}_S$; indeed, $\neg(A \neg A) = \neg A + A = A \neg A$. For $A \in \mathcal{K}$ the convex body $A \neg A$ is called the *difference body* of A and is denoted $\mathrm{D}A$; clearly all difference bodies are symmetric, $A \neg A \in \mathcal{K}_S$. The set $\mathcal{K}_D \cong \mathbb{E}^m$ of all single-point bodies can be written as $\mathcal{K}_D = \{A \in \mathcal{K} \mid A + (\neg A) = 0\}$.

Remark 2.1. Instead of "$\neg$" the symbol "$-$" is widely used in the literature on convex and interval analysis [5], [16]. Note that $A + (\neg A) \neq 0$ for $A \in \mathcal{K} \setminus \mathcal{K}_D$. Hence, the use of the symbol "$-$" to denote negation may lead to confusion, since "$-$" is normaly used for opposite elements, such that $A + (-A) = 0$. Using the symbol "$\neg$" we also avoid confusion with the opposite in the extended space of factorized pairs (differences) of convex bodies to be introduced in the next section.

Partial order and metric. *Inclusion* "$\subset$" is a partial order relation in $\mathcal{K}$, which is consistent with the arithmetic operations "+" and "$*$" in the sense that for all $A, B, C \in \mathcal{K}$, $\gamma \in \mathbb{R}$, $\gamma \neq 0$:

$$A \subset C \iff A + B \subset C + B, \tag{2.16}$$
$$A \subset B \iff \gamma * A \subset \gamma * B. \tag{2.17}$$

Inclusion is consistent with "$\in$" in the sense that

$$\text{For every } A \in \mathcal{K} \text{ there exists } P \in \mathcal{K}_D, \text{ such that } P \in A, \tag{2.18}$$
$$A \subset B \text{ iff } P \in A \text{ imply } P \in B. \tag{2.19}$$

Relations (2.16), (2.17) are known as *isotonicity* of addition, resp. multiplication by scalar. Note that (2.17) holds for $\gamma < 0$ as well. In particular, we have $A \subset B \iff \neg A \subset \neg B$.

A natural *metric* in $\mathcal{K}$ is the Hausdorff distance. Let $\mathcal{B} \in \mathcal{K}_S$ be a *unit ball* in $\mathbb{E}^m$. Then the Hausdorff distance is defined for $X, Y \in \mathcal{K}$ by [16]:

$$\delta_{\mathcal{B}}(X, Y) = \min\{\gamma \mid \gamma \geq 0,\ X \subset Y + \gamma * \mathcal{B},\ Y \subset X + \gamma * \mathcal{B}\}, \tag{2.20}$$

which in terms of the distance $|x - y|_{\mathcal{B}}$ in $\mathbb{E}^m$ (calibrated by $\mathcal{B}$) can be written as:

$$\delta_{\mathcal{B}}(X, Y) = \max\{\max_{x \in X} \min_{y \in Y} |x - y|_{\mathcal{B}},\ \max_{x \in Y} \min_{y \in X} |x - y|_{\mathcal{B}}\}.$$

The Hausdorff distance satisfies the axioms for distance. Furthermore, for $A, B, C \in \mathcal{K}, \gamma \in \mathbb{R}$, the Hausdorff distance satisfies [13]:

$$\delta_{\mathcal{B}}(A + C, B + C) = \delta_{\mathcal{B}}(A, B), \tag{2.21}$$
$$\delta_{\mathcal{B}}(\gamma * A, \gamma * B) = |\gamma| \delta_{\mathcal{B}}(A, B). \tag{2.22}$$

Quasilinear spaces. We conclude this section by recalling some abstract algebraic concepts to be used in the sequel. Properties (2.2)–(2.5) axiomatically define an *abelian cancellative (a. c.) monoid* $\mathcal{Q}$ with neutral element 0 [7]. That is, an a. c. monoid is a set $\mathcal{Q}$ with a neutral element 0 and a binary operation "+":$\mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{Q}$, satisfying (2.2)–(2.5) for all $A, B, C \in \mathcal{Q}$. An element $A \in \mathcal{Q}$ is *invertible*, if there exists $X \in \mathcal{Q}$ such that $A + X = 0$, in this case the (unique) element X is the *opposite* of A, we write symbolically $X = \mathrm{opp}(A)$ or $X = -A$. The set of all invertible elements of an a. c. monoid $(\mathcal{Q}, +)$ forms an abelian group, which is a nonempty subgroup of the monoid. Denote the subgroup of all invertible elements of $\mathcal{Q}$ by $\mathcal{Q}_I$; note that $\mathcal{Q}_I$ is not empty due to $0 \in Q_I$. We do not exclude the special (improper) case $\mathcal{Q} = \mathcal{Q}_I$, when the monoid $(\mathcal{Q}, +)$ becomes a group (an a. c. monoid with $\mathcal{Q} \neq \mathcal{Q}_I$ is called *proper*). A *submonoid of a monoid* $(\mathcal{Q}, +)$ is a monoid $(\mathcal{Q}', +)$, such that $\mathcal{Q}' \subset \mathcal{Q}$.

Definition 2.2. Let $(\mathcal{Q}, +)$ be an a. c. monoid. Assume that an operator multiplication by real scalar "$*$" is defined on $\mathbb{R} \times \mathcal{Q}$ satisfying (2.8)–(2.11). The algebraic system $(\mathcal{Q}, +, \mathbb{R}, *)$ is called a *(cancellative) quasilinear space (over* $\mathbb{R}$*)*. If the monoid is a group, then $(\mathcal{Q}, +, \mathbb{R}, *)$ is called a *quasilinear space over R with group structure* or, briefly, *q-linear space (over* $\mathbb{R}$*)*.

Remark 2.3. The definition of quasilinear space given in [11] does not require cancellation law and is thus more general. We do not consider noncancellative quasilinear spaces and thus change the original definition by always assuming the presence of cancellation law. Cancellative quasilinear spaces are useful for the study of the algebraic properties of convex sets and intervals. Such spaces have been considered in [15] under the name "R-semigroups with cancellation law". In [15] the authors are concerned with the general case when the monoid $(\mathcal{Q}, +)$ may be proper; in this work we restrict ourselves to the case when the monoid $(\mathcal{Q}, +)$ is a group. Note that a linear space is a special case of a q-linear space (and, hence, of a quasilinear space), such that (2.11) holds true for all α, β, not only such that $\alpha\beta \geq 0$ (the latter assumption implies that the monoid $(\mathcal{Q}, +)$ is necessarily a group).

A *subspace of a quasilinear space* $(\mathcal{Q}, +, \mathbb{R}, *)$ is a quasilinear space $(\mathcal{Q}', +, \mathbb{R}, *)$, such that $\mathcal{Q}' \subset \mathcal{Q}$. If $(\mathcal{Q}', +, \mathbb{R}, *)$ is a subspace of the quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *)$ then, of course, $(\mathcal{Q}', +,)$ is an a. c. submonoid of the a. c. monoid $(\mathcal{Q}, +)$.

Multiplication by "-1" is called *negation* in $\mathcal{Q}$ and is denoted by "$\neg$"; we briefly denote $A + (\neg B)$ by $A \neg B$. An element $A \in \mathcal{Q}$ with $A = \neg A$ is called *symmetric*. The set of all symmetric elements of $\mathcal{Q}$ is denoted by $\mathcal{Q}_S = \{A \in \mathcal{Q} \mid \neg A = A\}$. The set $\mathcal{Q}_S$ is not empty due to $0 \in \mathcal{Q}_S$. An element $C \in \mathcal{Q}$, such that (2.11) holds true for all α, β, not only such that $\alpha\beta \geq 0$, is called *distributive* or *linear*. The set of all distributive elements is denoted by $\mathcal{Q}_D$, we have $0 \in \mathcal{Q}_D$, hence $\mathcal{Q}_D$ is not empty. It is easy to see that an element $C \in \mathcal{Q}$ is distributive if and only if $C \neg C = 0$ [14]; hence $\mathcal{Q}_D = \{A \in \mathcal{Q} \mid \neg A + A = 0\}$. In other words, for $A \in \mathcal{Q}_D$ we have $-A = \neg A$, i. e. the opposite elements in $\mathcal{Q}_D$ are exactly the negative ones. Compared to the symmetry relation $A = \neg A$, the relation $A \neg A = 0$ (or, equivalently, $\mathrm{opp} A = \neg A$) expresses another kind of symmetry which may be referred to as *linearity*; indeed the space $(\mathcal{Q}_D, +, *)$ is a linear subspace of $\mathcal{Q}$. For $A \in \mathcal{Q}$ the element $A \neg A$ is called the *difference element* of A and is denoted $\mathrm{D}A$; all difference elements are symmetric, $\mathrm{D}A = A \neg A \in \mathcal{Q}_S$. We have $\mathrm{D}A = 0$, iff $A \in \mathcal{Q}_D$.

Example 2.4. Due to (2.2)–(2.5) the system $(\mathcal{K}, +)$ of all convex bodies in $\mathbb{E}$ is a proper a. c. monoid with a subgroup of invertible elements $\mathcal{K}_I$ and a neutral element "0". All invertable elements are also distributive (linear), so that we have $\mathcal{K}_I = \mathcal{K}_D$ in this case. The system of convex bodies $(\mathcal{K}, +, \mathbb{R}, *)$ with "$*$" defined by (2.7) is a quasilinear space. The system of all symmetric convex bodies has a unique invertible element, namely the null element "0"; indeed "0" is the unique element contained both in $\mathcal{K}_D$ and $\mathcal{K}_S$, $0 = \mathcal{K}_D \cap \mathcal{K}_S$. Thus $(\mathcal{K}_S, +)$ is another example of an a. c. monoid, which induces a corresponding quasilinear space $(\mathcal{K}_S, +, \mathbb{R}, *)$; the latter is a subspace of the quasilinear space $(\mathcal{K}, +, \mathbb{R}, *)$.

Inclusion and metric in quasilinear spaces. Assume that a partial order relation is defined on a quasilinear space $\mathcal{Q}$, further denoted by "$\subset$", such that (2.16), (2.17) hold for all $A, B, C \in \mathcal{Q}$, $\gamma \in \mathbb{R}$, $\gamma \neq 0$.

In convex analysis convex bodies are defined as convex compact sets of single-point bodies (vectors). In a quasilinear space $\mathcal{Q}$ the role of single-point bodies is played by the (distributive) elements of $\mathcal{Q}_D$. Note that a distributive element A is not an element of some other (not distributive) element B of $\mathcal{Q}$ (in set-theoretical sense) but a distributive element A may be included in the (nondistributive) element B. In fact, we can make the analogy with the situation in convex analysis complete by means of a suitable notation as follows.

If $P \subset A$ and $P \in \mathcal{Q}_D$ (that is $P \neg P = 0$), then we shall write $P \in A$. This notation brings no confusion, since A is not a set, and hence P is not an element of A; in this case the symbol "$\in$" means only a special case of the inclusion "$\subset$", leading to useful analogies. We shall further assume that (2.18) and (2.19) hold.

Note that the identity relation $A = B$ satisfies (2.16)–(2.17) but does not generally satisfy (2.18), (2.19), and hence is not a special case of the inclusion relations "$\subset$".

It can be easily checked that any relation satisfying conditions (2.16)–(2.19) is a partial order in $\mathcal{Q}$. A relation "$\subset$" satisfying (2.16)–(2.19) will be called *inclusion* (or *enclosure*) in $\mathcal{Q}$. The rules (2.16), (2.17) are known as *isotonicity (monotone homomorphism)* of addition, resp. multiplication by scalar. Note that (2.17) holds for $\gamma < 0$ as well. In particular, we have $A \subset B \Longleftrightarrow \neg A \subset \neg B$; $0 \subset B \Longleftrightarrow 0 \subset \neg B$.

If a partial order relation inclusion "$\subset$" is defined on a quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *)$, so that (2.16)–(2.19) are satisfied, then $(\mathcal{Q}, +, \mathbb{R}, *, \subset)$ is a *partially ordered quasilinear space (over* $\mathbb{R}$*)*; the latter has a natural metric (2.20).

Using (2.16), (2.17) and the transitive property of the partial order relation "$\subset$" we have for all $\lambda \in [0, 1]$

$$A \subset C, B \subset C \Longrightarrow \lambda * A + (1 - \lambda) * B \subset \lambda * C + (1 - \lambda) * C = C. \tag{2.23}$$

In the special case $A, B \in \mathcal{Q}_D$ relation (2.23) obtains the form $A \in C, B \in C \Longrightarrow \lambda * A + (1 - \lambda) * B \subset C$, which corresponds to the well-known convexity condition on C.

It is easy to check that every symmetric element of $\mathcal{Q}$ contains 0. Indeed, to show that $A \in \mathcal{Q}_S$ implies $0 \in A$, assume $A \in \mathcal{Q}_S$, that is $A = \neg A$, and let $P \in A$, $P \neg P = 0$; according to (2.18) there exists at least one such P. The inclusion $P \in A$ implies $\neg P \in \neg A = A$. Using (2.23) we obtain $0 = P + (\neg P) \in A$.

Note that, if A is a summand of B, $A + T = B$, then $0 \in T$ is equivalent to $A \subset B$.

A natural *metric* δ can be introduced in a quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *, \subset)$. For a given element $\mathcal{B} \in \mathcal{Q}_S$, called *unit ball*, the *Hausdorff distance* is defined for $X, Y \in \mathcal{Q}$ by (2.20). It is easy to check that the Hausdorff distance satisfies the axioms for distance and relations (2.21), (2.22). We shall further assume that the unit ball $\mathcal{B}$ is fixed and shall write δ instead of $\delta_{\mathcal{B}}$.

The results in this Section show that calculation in a quasilinear space is problematic: not all elements are invertible and (Minkowski) difference does not always exist, that is, the equation $A + X = B$ is not always solvable with respect to X. Calculations become more familiar if we extend the concept of convex body in the way negative numbers are defined. Then, as shown in the sequel, all (extended) elements are invertible, an equation of the form $a + x = b$ is always solvable, etc. In what follows we shall usually denote elements of a group by $a, b, c, ..., x, y, z$, and elements of a monoid (which may not be a group) by $A, B, C, ..., X, Y, Z$; that is $-a, -b, ..., -z$ will exist, whereas the opposite of $A, B, ..., Z$ may not exist.

Abelian semigroups can be embedded in groups. If the abelian semigroup is an a. c. monoid, then the algebraic structure of the induced group becomes richer; the operation "$*$" can be also naturally extended in the group. We summarize the necessary abstract algebraic foundation in the next section.

3. Q-linear Spaces

It is well-known, see e. g. [6], [13], that every abelian semigroup $(\mathcal{Q}, +)$ can be embedded in an abelian group $(\mathcal{G}, +)$; we briefly recall the familiar construction below for the special case of an a. c. monoid $(\mathcal{Q}, +)$ and introduce some notation.

The extension method. Let $(\mathcal{Q}, +)$ be an a. c. monoid. Denote by $\mathcal{G} = (\mathcal{Q} \times \mathcal{Q})/\rho = \mathcal{Q}^2/\rho$ the set of all pairs (A, B), $A,\ B \in \mathcal{Q}$, factorized by the equivalence (congruence) relation $\rho : (A, B)\rho(C, D) \iff A + D = B + C$; we shall further write "$=$" instead of ρ. Denote the equivalence class in $\mathcal{G}$, represented by the pair (A, B), again by (A, B), we have $(A, B) = (A + X, B + X)$. Define addition in $\mathcal{G}$ by means of:

$$(A, B) + (C, D) = (A + C, B + D). \tag{3.1}$$

The neutral (null) element of $\mathcal{G}$ is the class (Z, Z), $Z \in \mathcal{Q}$; due to the existence of null element in $\mathcal{Q}$, we have $(Z, Z) = (0, 0)$. The opposite element to $(A, B) \in \mathcal{G}$ is denoted $-(A, B)$. We have $-(A, B) = (B, A)$; indeed $(A, B) + (-(A, B)) = (A, B) + (B, A) = (A + B, B + A) = (0, 0)$. Instead of $(A, B) + (-(C, D))$, we may write $(A, B) - (C, D)$; we have $(A, B) - (C, D) = (A, B) + (D, C) = (A + D, B + C)$. The system $(\mathcal{G}, +)$ thus obtained is an abelian group. Due to $(A, B) = (A, 0) - (B, 0)$ the pair (A, B) is called the *difference* of A and B and the set $\mathcal{G}$ is called the *difference set* of $\mathcal{Q}$ and denoted $\mathcal{G} = \operatorname{dis} \mathcal{Q} = \mathcal{Q}^2/\rho$; we shall also say that $\mathcal{G}$ is the group generated (induced) by the monoid $\mathcal{Q}$ (using the extension method).

It is easy to check that

$$-((A, B) + (C, D)) = -(A, B) - (C, D). \tag{3.2}$$

The mapping $\varphi : \mathcal{Q} \longrightarrow \mathcal{G}$ defined for $A \in \mathcal{Q}$ by $\varphi(A) = (A, 0) \in \mathcal{G}$, is isomorphic and hence is an *embedding* of semigroups. We *embed* $\mathcal{Q}$ in $\mathcal{G}$ by identifying $A \in \mathcal{Q}$ with the equivalence class $(A, 0) = (A + X, X)$, $X \in \mathcal{Q}$; all elements of $\mathcal{G}$ admitting the form $(A, 0)$ are called *proper*. Hence, the proper elements of $\mathcal{G}$ are all pairs (U, V), $U,\ V \in \mathcal{Q}$, such that $V + Y = U$ for some $Y \in \mathcal{Q}$, i. e. $(U, V) = (V + Y, V) = (Y, 0)$. The set of all proper elements of $\mathcal{G}$ is $\varphi(\mathcal{Q}) = \{(A, 0) \mid A \in \mathcal{Q}\} \cong \mathcal{Q}$. The image of $\mathcal{Q}_D$ under the embedding φ is: $\varphi(\mathcal{Q}_D) = \{(A, 0) \mid A \in \mathcal{Q}_D\} \cong \mathcal{Q}_D$. Note that if $\mathcal{Q}$ is an abelian group itself, then $\mathcal{G} \cong \mathcal{Q}$; hence the extension method makes sense only if $(\mathcal{Q}, +)$ is a proper monoid (not a group).

Multiplication by scalar. We next extend multiplication by scalar "$*$" from $\mathbb{R} \times \mathcal{Q}$ to $\mathbb{R} \times \mathcal{G}$. A natural definition of $* : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ is

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A,\ B \in \mathcal{Q}, \quad \gamma \in \mathbb{R}. \tag{3.3}$$

In what follows we shall consider the difference system $(\mathcal{G}, +, \mathbb{R}, *)$, $\mathcal{G} = \operatorname{dis} \mathcal{Q}$, induced by a quasilinear system $(\mathcal{Q}, +, \mathbb{R}, *)$ by means of (3.1), (3.3). For the sake of simplicity we use lower case roman letters to denote the elements of $\mathcal{G} = \operatorname{dis} \mathcal{Q}$, writing e. g. $a = (A', A'') \in \mathcal{G}$.

Theorem 3.1. *Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasilinear system, and let $(\mathcal{G}, +)$, $\mathcal{G} = \operatorname{dis} \mathcal{Q}$, be the induced abelian group. Let $* : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ be multiplication by scalar defined by* (3.3). *Then for $a, b, c \in \mathcal{G}$, $\alpha, \beta, \gamma \in \mathbb{R}$ we have:*

$$\begin{aligned} \alpha * (\beta * c) &= (\alpha\beta) * c, && (3.4)\\ \gamma * (a + b) &= \gamma * a + \gamma * b, && (3.5)\\ 1 * a &= a, && (3.6)\\ (\alpha + \beta) * c &= \alpha * c + \beta * c, \ \alpha\beta \geq 0. && (3.7) \end{aligned}$$

Proof. Relations (3.4)–(3.6) are easily verified. To check relation (3.7) substitute $c = (U, V) \in \mathcal{G}$ with $U, V \in \mathcal{Q}$; the right-hand side becomes $\alpha * (U, V) + \beta * (U, V) = (\alpha * U + \beta * U,\ \alpha * V + \beta * V)$. If $\alpha\beta \geq 0$, using (2.11) we see that the latter expression is identical to the left-hand side $(\alpha + \beta) * (U, V) = ((\alpha + \beta) * U, (\alpha + \beta) * V)$ of (3.7). □

Theorem 3.1 shows that the algebraic system $(\mathcal{G}, +, \mathbb{R}, *)$ is a q-linear space (quasilinear space with group structure). The q-linear space $\mathcal{G}$ is an abelian group with respect to addition and satisfies relations (3.4)–(3.7) — or, equivalently, (2.8)–(2.11). Note that $(\mathcal{G}, +, \mathbb{R}, *)$ is required to satisfy the distributive relation $(\alpha + \beta) * c = \alpha * c + \beta * c$ only for equally signed real scalars ($\alpha\beta \geq 0$), whereas a linear space must satisfy this relation for all real scalars. Since a q-linear space is a special case of a quasilinear space all definitions valid for quasilinear spaces apply also for q-linear spaces, in particular for the induced difference space $\mathcal{G} = \operatorname{dis} \mathcal{Q}$; below we shall discuss in some detail the concepts of symmetric and linear elements of a q-linear space.

Negative, symmetric and conjugate elements. From (3.3) for $\gamma = -1$ we obtain *negation* in $\mathcal{G}$, $\neg a = (-1) * a$, pairwise:

$$\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B), \quad A,\ B \in \mathcal{Q}, \quad \gamma \in \mathbb{R}.$$

Relation (3.4) implies $\neg(\neg a) = a$. As before, we write $a \neg b = a + (\neg b) = a + (-1) * a$. From (3.5) we have

$$\neg(a + b) = \neg a \neg b. \tag{3.8}$$

A *symmetric element* $a \in \mathcal{G}$ is defined by the property $a = \neg a$; the set of all symmetric elements of $\mathcal{G}$ is $\mathcal{G}_S = \{a \in \mathcal{G} \mid a = \neg a\}$. Let $a = (A', A'') \in \mathcal{G}$ be symmetric, then $a = \neg a \iff (A', A'') = (\neg A', \neg A'') \iff A' \neg A'' = A'' \neg A' \iff A' \neg A'' = \neg(A' \neg A'') \iff A' \neg A'' \in \mathcal{Q}_S$. Hence, $a = (A', A'')$ is symmetric (as an element of $\mathcal{G}$), if $A' \neg A''$ is symmetric, $A' \neg A'' \in \mathcal{Q}_S$.

The operators opposite and negation are defined for $a = (A', A'') \in \mathcal{G}$ by:

$$\begin{aligned} -a &= -(A', A'') = (A'', A'),\\ \neg a &= \neg(A', A'') = (\neg A', \neg A''). \end{aligned}$$

A distributive (linear) element $a \in \mathcal{G}$ possesses the property $-a = \neg a$, or, equivalently, $a \neg a = 0$. The set of all distributive elements of $\mathcal{G}$ is $\mathcal{G}_D = \{a \in \mathcal{G} \mid -a = \neg a\} = \{a \in \mathcal{G} \mid a \neg a = 0\}$. We have $-a = \neg a \iff (A'', A') = (\neg A', \neg A'') \iff A'' \neg A'' = A' \neg A' \iff \mathrm{D}A' = \mathrm{D}A''$. The latter is true, in particular, if $A', A'' \in Q_D$, since then $\mathrm{D}A' = \mathrm{D}A'' = 0$. More generally, if $A' = A'' + P$, $P \neg P = 0$ (i. e. $P \in \mathcal{Q}_D$), then $\mathrm{D}A' = \mathrm{D}A''$ as well.

Elements from $\mathcal{G}_D \cup \mathcal{G}_S$, which are either symmetric or linear, will be further called *axial*. The null element 0 is the only element, which is both symmetric and linear. Elements from $\mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ are called *nonaxial*.

The composition of "$-$" and "$\neg$" in $\mathcal{G}$ is called *conjugation* or *dualization* and is denoted by a_- (read: "a dual" or "a conjugate"), symbolically: $a_- = -(\neg a) = \neg(-a)$. Pair-wise we have: $a_- = (A', A'')_- = \neg(-(A', A'')) = \neg(A'', A') = (\neg A'', \neg A')$. Using conjugation we may express the opposite by $-a = \neg a_- = (-1) * a_-$. In the sequel we shall usually write $\neg a_-$ instead of $-a$ in order to avoid confusion due to the double usage of the symbol "$-$" in the literature (once for opposite and once for negation). Thus, instead of $a - b = a + (-b)$ we shall write $a \neg b_- = a + (-1) * b_-$. Respectively, the equality $a = a$ or $a - a = 0$ may be written as $a \neg a_- = 0$. We have $(a_-)_- = a$; also from (3.2) and (3.8) we obtain

$$(a + b)_- = a_- + b_-. \tag{3.9}$$

Note that the equalities $a = \neg a$ and $-a = a_-$ are algebraically equivalent, and such are the equalities $-a = \neg a$ and $a = a_-$.

New notation. We now introduce a symbolic notation to be systematically used in the sequel. For $a \in \mathcal{G}$ denote $a_+ = a$; then for $\lambda \in \{+, -\}$ the element $a_\lambda \in \mathcal{G}$ (read: "a dualized by λ" or "a conjugated by λ") is either a or a_- according to the binary value of λ. Using this notation we can write (3.9) in the form $(a + b)_\lambda = a_\lambda + b_\lambda$. Note also that the equalities $a = b$ and $a_\lambda = b_\lambda$ are algebraically equivalent; each one is obtained from the other via conjugation (dualization) by λ. Using the sign rules ("$++ = -- = +$", "$+- = -+ = -$"), we can write formulas like $(a_\lambda)_\mu = a_{\lambda\mu}$, $(a_\mu + b_\nu)_\lambda = a_{\lambda\mu} + b_{\lambda\nu}$. Another similar notation is $\lambda a = \{a, \lambda = +; -a, \lambda = -\}$, where $a \in \mathcal{G}$, $\lambda \in \{+, -\}$ (to avoid confusion with multiplication by scalar, the latter will be always denoted by a special sign, namely "$*$" or "$\cdot$"). Similarly, we may write: $\lambda\alpha = \{\alpha, \lambda = +; -\alpha, \lambda = -\}$, where $\alpha \in \mathbb{R}$, $\lambda \in \{+, -\}$.

Proposition 3.2. *For $a \in \mathcal{G}$ we have $a + a_- \in \mathcal{G}_D$, $a \neg a \in \mathcal{G}_S$. For $a \in \mathcal{G}_S$, $\gamma \in \mathbb{R}$, we have $\gamma * a = (-\gamma) * a = |\gamma| * a$. Also: $a + a_- = 0$, iff $a = \neg a$, that is, $a \in \mathcal{G}_S$; and $a \neg a = 0$, iff $a = a_-$, i. e. $a \in \mathcal{G}_D$. For each $t \in \mathcal{G}_D$ there exists $x \in \mathcal{G}$, such that $t = x + x_-$; x can be written in the form $x = (1/2) * t + s$, where $s \in \mathcal{G}_S$ is arbitrary. We have $\mathcal{G}_D = \{x + x_- \mid x \in \mathcal{G}\}$. For each $s \in \mathcal{G}_S$ there exists $y \in \mathcal{G}$, such that $s = y \neg y$; namely $y = (1/2) * s + t$, where $t \in \mathcal{G}_D$ is arbitrary. We have $\mathcal{G}_S = \{y \neg y \mid y \in \mathcal{G}\}$.*

Proof. Let $a \in \mathcal{G}$. We have $(a + a_-)_- = a_- + a = a + a_-$, hence $a + a_- \in \mathcal{G}_D$. Also: $\neg(a \neg a) = \neg a \neg (\neg a) = \neg a + a = a \neg a$, hence $a \neg a \in \mathcal{G}_S$. The rest follows also easily from the definitions of $\mathcal{G}_D$ and $\mathcal{G}_S$. $\square$

Let φ be the embedding $\varphi(\mathcal{Q}) = \{(A, 0) \mid A \in \mathcal{Q}\}$ mentioned at the beginning of the section: $\varphi(\mathcal{Q}_S) = \{(A, 0) \mid A \in \mathcal{Q}_S\} = \{(A, 0) \mid A = \neg A\} = \{(A, 0) \mid A \neg 0 = 0 \neg A\}$.

Compared to $\mathcal{G}_S = \{(A,B) \mid (A,B) = \neg(A,B)\} = \{(A,B) \mid A \neg B = B \neg A\}$, we obtain $\varphi(\mathcal{Q}_S) \subset \mathcal{G}_S$. Note that $(A,B) \in \mathcal{G}_S$, iff $A \neg B \in \mathcal{Q}_S$.

We have $\varphi(\mathcal{Q}_D) = \{(A,0) \mid A \neg A = 0\}$, $\mathcal{G}_D = \{(A,B) \mid -(A,B) = \neg(A,B)\} = \{(A,B) \mid A \neg A = B \neg B\} \cong \varphi(\mathcal{Q}_D)$.

It is easy to see that in a q-linear space $\mathcal{G}$ the symmetry property $a = -a$, or, equivalently, $\neg a = a_-$, is satisfied only by the null element of $\mathcal{G}$, $0 = (X,X)$, $X \in \mathcal{Q}$.

Rules for calculation. Recall that every q-linear space is an abelian group. From the group properties (like: $0+a = a$, $-a+a = 0$, $-(a+b) = -a+(-b)$, $a+b = a+c \Longrightarrow b = c$, etc.) and relations (3.4)–(3.9) one may derive rules for calculation in a q-linear system. A set of such rules is summarized in the following

Proposition 3.3. *Let $(\mathcal{G}, +, \mathbb{R}, *)$ be a q-linear system over $\mathbb{R}$. For all $\alpha, \beta, \gamma \in \mathbb{R}$ and for all $a, b, c \in \mathcal{G}$ the following properties hold true:*

(1) $\gamma * 0 = 0$, *in particular,* $(-1) * 0 = \neg 0 = 0$;
(2) $0 * a = 0$;
(3) $\neg(\gamma * a) = (-\gamma) * a$;
(4) $(\alpha - \beta) * c = \alpha * c + (-\beta) * c = \alpha * c \neg \beta * c,\ \alpha\beta \geq 0$;
(5) $\gamma * (a \neg b) = \gamma * a \neg \gamma * b$, *i. e.* $\gamma * (a + (-1) * b) = \gamma * a + (-\gamma) * b$;
(6) $\gamma * (a + b_-) = \gamma * a + \gamma * b_-$;
(7) $\gamma * (a - b) = \gamma * a - \gamma * b$;
(8) $\gamma * a = 0 \Longrightarrow \gamma = 0$ *or* $a = 0$;
(9) $\alpha * c = \beta * c \Longrightarrow \alpha = \beta$ *or* $c = 0$;
(10) $\gamma * a = \gamma * b \Longrightarrow \gamma = 0$ *or* $a = b$;
(11) $a - a = a \neg a_- = \neg a + a_- = 0$;
(12) $(\sum_{i=1}^n \alpha_i) * c = \sum \alpha_i * c,\ \alpha_i \geq 0,\ i = 1, ..., n$;
(13) $\alpha * \sum_{i=1}^n c_i = \sum_{i=1}^n \alpha * c_i$;
(14) $(a_\mu + b_\nu)_\lambda = a_{\lambda\mu} + b_{\lambda\nu}$, *in particular,* $(a+b)_\lambda = a_\lambda + b_\lambda$;
(15) $(\alpha * c_\mu)_\nu = \alpha * c_{(\mu\nu)}$, *in particular,* $(\alpha * c_\mu)_\mu = \alpha * c$, *and* $(\alpha * c)_- = \alpha * c_-$;
(16) $a + \gamma * b = 0 \Longleftrightarrow a = (-\gamma) * b_- = \neg(\gamma * b_-)$, *in particular,* $a + b = 0 \Longleftrightarrow a = \neg b_-$;
(17) $a \in \mathcal{G}_D \Longleftrightarrow \neg a = -a \Longleftrightarrow a = a_- \Longleftrightarrow a \neg a = 0 \Longleftrightarrow \exists c \in \mathcal{G} : a = c + c_-$;
(18) $b \in \mathcal{G}_S \Longleftrightarrow \neg b = b \Longleftrightarrow -b = b_- \Longleftrightarrow b + b_- = 0 \Longleftrightarrow \exists d \in \mathcal{G} : b = d \neg d$.

Proof. Relations (1)–(18) are trivially verified. For example we prove relations (1), (8) and (11). Proof of (1). We have $\gamma * 0 + 0 = \gamma * 0 = \gamma * (0+0) = \gamma * 0 + \gamma * 0$, hence by the cancellation law $\gamma * 0 = 0$. Proof of (8). Let $\gamma * a = 0$. If $\gamma \neq 0$, $a = 1 * a = (\gamma^{-1}\gamma) * a = \gamma^{-1} * (\gamma * a) = \gamma^{-1} * 0 = 0$. Proof of (11). Setting $\alpha = 1$, $\beta = -1$ in (3.10), implies $a + (\neg a_-) = 0$. $\square$

The quasidistributive law (3.7) and its consequences like rules 4 and 12 of Proposition 3.3 are inconvenient for symbolic algebraic transformations, since we need to know in advance the signs of the scalars involved. We shall next derive a consequence of (3.7) which will enable symbolic manipulations and will be systematically used throughout the paper.

Theorem 3.4. *Let $(\mathcal{G}, +, \mathbb{R}, *)$ be a q-linear space over $\mathbb{R}$. Then for all $c \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$, we have*

$$(\alpha+\beta) * c_{\sigma(\alpha+\beta)} \quad = \quad \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \tag{3.10}$$

Proof. The case $\sigma(\alpha) = \sigma(\beta)$, and hence $\sigma(\alpha) = \sigma(\alpha+\beta)$, follows directly from (3.7). Consider now the case $\sigma(\alpha) = -\sigma(\beta)$. Substitute $c = (U, V) \in \mathcal{G}$ with $U, V \in \mathcal{Q}$; the right-hand side of (3.10) becomes:

$$\begin{aligned} r &= \alpha * (U,V)_{\sigma(\alpha)} + \beta * (U,V)_{\sigma(\beta)} \\ &= (\alpha * (U,V) + \beta * (U,V)_-)_{\sigma(\alpha)} \\ &= (\alpha * (U,V) + \beta * (\neg V, \neg U))_{\sigma(\alpha)} \\ &= ((\alpha * U, \alpha * V) + ((-\beta) * V, (-\beta) * U))_{\sigma(\alpha)} \\ &= (\alpha * U + (-\beta) * V,\ \alpha * V + (-\beta) * U)_{\sigma(\alpha)}. \end{aligned}$$

Now we have to consider a number of subcases. Consider, e. g. the subcase $\sigma(\alpha) = +$, $\sigma(\beta) = -$, $\sigma(\alpha+\beta) = +$. In this subcase we have $\alpha \geq -\beta > 0$, hence $\alpha = (\alpha+\beta) + (-\beta)$ with $\alpha+\beta \geq 0$, $-\beta > 0$. Using the quasidistributive law (2.11) we obtain:

$$\begin{aligned} r &= (\alpha * U + (-\beta) * V,\ \alpha * V + (-\beta) * U) \\ &= ((\alpha+\beta) * U + (-\beta) * U + (-\beta) * V,\ (\alpha+\beta) * V + (-\beta) * V + (-\beta) * U) \\ &= ((\alpha+\beta) * U, (\alpha+\beta) * V) + ((-\beta) * U + (-\beta) * V, (-\beta) * U + (-\beta) * V) \\ &= (\alpha+\beta) * (U,V) + (0,0) = l, \end{aligned}$$

where l is the left hand-side of (3.10); in the last line we use that $((-\beta) * U + (-\beta) * V, (-\beta) * U + (-\beta) * V) = (W, W) = (0, 0)$.

The rest of the cases are treated analogously. This concludes the proof. □

We conclude from the proof of Theorem 3.4 that both relations (3.10) and (3.7) are algebraically equivalent. Relation (3.10) can be also written in one of the following forms:

$$(\alpha+\beta) * c \quad = \quad (\alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)})_{\sigma(\alpha+\beta)}; \tag{3.11}$$

$$(\alpha+\beta) * c \quad = \quad \alpha * c_\lambda + \beta * c_\mu, \quad \lambda = \sigma(\alpha)\sigma(\alpha+\beta),\ \mu = \sigma(\beta)\sigma(\alpha+\beta); \tag{3.12}$$

$$(\alpha-\beta) * c_{\sigma(\alpha-\beta)} \quad = \quad \alpha * c_{\sigma(\alpha)} + (-\beta) * c_{-\sigma(\beta)} = \alpha * c_{\sigma(\alpha)} \neg\, \beta * c_{-\sigma(\beta)}. \tag{3.13}$$

Relation (3.10) or its corollaries (3.11)–(3.13) will be referred to as *quasidistributive law in q-linear space* or, briefly, *q-distributive law*. Note that (3.10)–(3.13) permit symbolic algebraic transformations for any parameters α, β, whereas (3.7) does not. Relation (3.10) can be readily generalized for more than two summands as follows:

$$(\sum_{i=1}^{n} \alpha_i) * c_{\sigma(\Sigma_{i=1}^n \alpha_i)} = \sum \alpha_i * c_{\sigma(\alpha_i)}. \tag{3.14}$$

4. Algebraic Transformations

To demonstrate equivalent algebraic transformations in a q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ consider expressions of one variable $x \in \mathcal{G}$ using finitely many operations "+", "$*$", "$_-$", such as $3 * (x + x_-) + 2 * x \neg x + 2 * x_- \neg x_-$ (recall that $\neg x = (-1) * x$). Every expression involving a variable $x \in \mathcal{G}$ can be simplified to an expression of the form: $\alpha * x_\mu + \beta * x_\nu$, $\alpha, \beta \in \mathbb{R}$, $\mu, \nu \in \{+, -\}$, hence every equation of an unknown variable $x \in \mathcal{G}$ can be written in the form

$$\alpha * x + \beta * x_\lambda = d. \tag{4.1}$$

The left-hand side of this general q-linear equation of one variable is an expression of x and possibly of its conjugate x_- (if $\lambda = -$). Due to the q-distributive law we cannot further simplify the left-hand side of (4.1) without additional assumptions on the variables involved; for instance we may simplify using (3.10) under the assumption $\lambda\sigma(\beta) = +$. We shall next show that using a Gaussian-like elimination technique we can algebraically transform equation (4.1), and find an explicit formula for the solution x.

Indeed, multiplying (4.1) consecutively by α and $-\beta$ we obtain, resp.:

$$\begin{aligned} \alpha^2 * x + (\alpha\beta) * x_\lambda &= \alpha * d, \\ (-\alpha\beta) * x + (-\beta^2) * x_\lambda &= (-\beta) * d. \end{aligned}$$

Conjugating the second equation by $-\lambda$ we obtain $(-\alpha\beta) * x_{-\lambda} + (-\beta^2) * x_- = (-\beta) * d_{-\lambda}$. We now consider different cases. For $\alpha = \beta$ or $\alpha = -\beta$ both equations coincide. Clearly in these special cases equation (4.1) has no solutions or has infinitely many solutions depending on the right-hand side. Assume now that $\alpha^2 - \beta^2 \neq 0$. Summing up the two (different) equations and using rule 11 of Proposition 3.3 we obtain the relation:

$$\alpha^2 * x + (-\beta^2) * x_- = \alpha * d + (-\beta) * d_{-\lambda}.$$

Using the q-distributive law (3.10) we simplify the left-hand side of the above relation to obtain:

$$(\alpha^2 - \beta^2) * x = \alpha * d + (-\beta) * d_{-\lambda} = \alpha * d \neg \beta * d_{-\lambda}.$$

The above arguments lead to the following:

Proposition 4.1. *Let $\alpha, \beta \in \mathbb{R}$, $\lambda \in \{+, -\}$, $d \in \mathcal{G}$. If $\gamma = \alpha^2 - \beta^2 \neq 0$, then equation* (4.1) *has a unique solution*

$$x = \gamma^{-1} * (\alpha * d \neg \beta * d_{-\lambda})_{\sigma(\gamma)}. \tag{4.2}$$

If $\gamma = \alpha^2 - \beta^2 = 0$, then (4.1) *has no solutions or has infinitely many solutions depending on the right-hand side.*

Proof. The above derivation of (4.2) presents actually a proof. However, we give below a direct proof in order to demostrate the technique of algebraic transformations. Assuming $\gamma = \alpha^2 - \beta^2 \neq 0$ denote $\sigma = \sigma(\gamma)$ and substitute (4.2) into (4.1):

$$\begin{aligned} \alpha * x \ &+ \ \beta * x_\lambda \\ &= (\alpha\gamma^{-1}) * (\alpha * d \neg \beta * d_{-\lambda})_\sigma + (\beta\gamma^{-1}) * (\alpha * d \neg \beta * d_{-\lambda})_{\lambda\sigma} \\ &= (\alpha^2\gamma^{-1}) * d_\sigma + (-\alpha\beta\gamma^{-1}) * d_{-\lambda\sigma} + (\beta\alpha\gamma^{-1}) * d_{\lambda\sigma} + (-\beta^2\gamma^{-1}) * d_{-\sigma} \\ &= (\alpha^2\gamma^{-1}) * d_\sigma + (-\beta^2\gamma^{-1}) * d_{-\sigma} = \gamma^{-1} * (\alpha^2 * d + (-\beta^2) * d_-)_\sigma \\ &= \gamma^{-1} * ((\alpha^2 - \beta^2) * d_\sigma)_\sigma = (\gamma^{-1}(\alpha^2 - \beta^2)) * d = d, \end{aligned}$$

where for the equivalent transformation between the last two lines we use (3.10). Noticing that all above algebraic tranformations are equivalent, we see that (4.2) is the unique solution to (4.1).

If $\gamma = \alpha^2 - \beta^2 = 0$, then equation (4.1) reduces to the equation $x + x_- = d$ (if $\alpha = \beta$ and $\lambda = -$) or $x \neg x = d$ (if $\alpha = -\beta$ and $\lambda = +$). Due to $x + x_- \in \mathcal{G}_D$, $x \neg x \in \mathcal{G}_S$, see Proposition 3.2, these equations have (infinitely many) solutions for $d \in \mathcal{G}_D$, resp. $d \in \mathcal{G}_S$, and no solution in the remaining cases. The solutions in the case $\alpha = \beta$ are of the form $x = (1/2) * d + s$, where $s \in \mathcal{G}_S$ is arbitrary, and in the case $\alpha = -\beta$ are: $x = (1/2) * d + t$, where $t \in \mathcal{G}_D$ is arbitrary. □

Proposition 4.1 shows that the q-distributive law can be used essentially in the same manner as the usual distributive law is used for equivalent algebraic transformations. Proposition 4.1 also shows that a q-linear equation can be solved by algebraic transformations no matter that the left-hand side cannot be generally simplified.

We know from (3.10) that an expression of the form

$$\alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)} \tag{4.3}$$

can be always (for all α, β, x) simplified to $(\alpha + \beta) * x_{\sigma(\alpha+\beta)}$; we shall say that the form (4.3) is *reducible*. One should distinguish the reducible form (4.3) from other forms, which are not reducible. For instance, the expression $\alpha * x_{\sigma(\alpha)} \neg \gamma * x_{-\sigma(\gamma)}$ is in reducible form, which is obvious after substituting $\gamma = -\beta$.

Consider the following four expressions obtained from the expression $\alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)}$ by putting a sign "$-$" at an appropriate place:

$$\begin{gathered} (-\alpha) * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)}, \quad \alpha * x_{\sigma(-\alpha)} + \beta * x_{\sigma(\beta)}, \\ \alpha * x_{\sigma(\alpha)} + (-\beta) * x_{\sigma(\beta)}, \quad \alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(-\beta)}. \end{gathered}$$

We may notice that neither of the above four expressions is in reducible form. Since each one of these expressions can be transformed into any other (by a substitution of a parameter, like $\alpha' = -\alpha$), we chose one of the above forms, namely

$$q(x) = \alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} \tag{4.4}$$

as a "standard" irreducible form. Let us investigate (4.4) in certain special cases.

Proposition 4.2. *If $x \in \mathcal{G}_D \cup \mathcal{G}_S$, then* (4.4) *can be reduced to the form $q(x) = \gamma * x_{\sigma(\gamma)} \in \mathcal{G}_D \cup \mathcal{G}_S$. More specifically,* (i) *for $x \in \mathcal{G}_D$ we have: $q(x) = \gamma * x_{\sigma(\gamma)} = \gamma * x_{\sigma(|\gamma|)} = \alpha * x$;* (ii) *for $x \in \mathcal{G}_S$ we can write: $q(x) = \gamma * x_{\sigma(\gamma)} = |\gamma| * x_{\sigma(\gamma)}$. If $\alpha^2 - \beta^2 = 0$, then $q(x) \in \mathcal{G}_D \cup \mathcal{G}_S$ for any $x \in \mathcal{G}$.*

Proof. Assume $x \in \mathcal{G}_D$, i. e. $x = x_-$. Then we have: $\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = \alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)} = (\alpha + \beta) * x_{\sigma(\alpha+\beta)} = (\alpha + \beta) * x$. Let now $x \in \mathcal{G}_S$, i. e. $\neg x = x$; then $\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = \alpha * x_{\sigma(\alpha)} + \beta * (\neg x)_{-\sigma(\beta)} = \alpha * x_{\sigma(\alpha)} + (-\beta) * x_{\sigma(-\beta)} = (\alpha - \beta) * x_{\sigma(\alpha-\beta)} = |(\alpha - \beta)| * x_{\sigma(\alpha-\beta)}$ (using that $x = \neg x$). To prove the last part of the proposition, assume $\alpha^2 - \beta^2 = 0$. The case when x is axial is clear, hence consider the case $x \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$, that is $x \neq \neg x$, $x \neq x_-$. If $\alpha = \beta$, then we have: $q(x) = \alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = \alpha * (x + x_-)_{\sigma(\alpha)} = \alpha * (x + x_-) \in \mathcal{G}_D$. If $\alpha = -\beta$, then $q(x) = \alpha * (x \neg x)_{\sigma(\alpha)} = |\alpha| * (x \neg x)_{\sigma(\alpha)} \in \mathcal{G}_S$. □

Note that, from the point of view of formal algebraic transformations, calculations in a q-linear space differ from those in a linear space, where an expression of the form $\alpha \cdot x + \beta \cdot x$ can be always simplified to $\gamma \cdot x$, with $\gamma = \alpha + \beta$. In a q-linear space we cannot generally factor out x, that is, in general there is no γ such that $\alpha * x + \beta * x = \gamma * x$. The latter is true if and only if: (i) $x \in \mathcal{G}_D \cup \mathcal{G}_S$; (ii) $x \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\alpha\beta \geq 0$. We next demonstrate that a general expression of one variable, can be written in the form (4.4).

Proposition 4.3. *Every expression of the form $\alpha * x_\lambda + \beta * x_\mu$ can be presented in the form* (4.4).

Proof. We have to prove that for every fixed $\alpha, \beta \in \mathbb{R}$, $\lambda, \mu \in \{+, -\}$ there exist $\alpha', \beta' \in \mathbb{R}$ such that

$$\alpha * x_\lambda + \beta * x_\mu = \alpha' * x_{\sigma(\alpha')} + \beta' * x_{-\sigma(\beta')}. \tag{4.5}$$

Consider separately the following four cases according to the values of the parameters α, β, λ and μ involved in the left-hand side of (4.5):

(i) $\lambda = \sigma(\alpha)$, $\mu = -\sigma(\beta)$. In this case the assertion is obvious with $\alpha' = \alpha$, $\beta' = \beta$.

(ii) $\lambda = \sigma(\alpha)$, $\mu = \sigma(\beta)$; we have

$$\begin{aligned}\alpha * x_\lambda + \beta * x_\mu &= \alpha * x_{\sigma(\alpha)} + \beta * x_{\sigma(\beta)} \\ &= (\alpha + \beta) * x_{\sigma(\alpha+\beta)} = \alpha' * x_{\sigma(\alpha')} + \beta' * x_{-\sigma(\beta')}\end{aligned}$$

with $\alpha' = \alpha + \beta$, $\beta' = 0$.

(iii) $\lambda = -\sigma(\alpha)$, $\mu = \sigma(\beta)$; we have

$$\alpha * x_\lambda + \beta * x_\mu = \alpha * x_{-\sigma(\alpha)} + \beta * x_{\sigma(\beta)} = \alpha' * x_{\sigma(\alpha')} + \beta' * x_{-\sigma(\beta')}$$

with $\alpha' = \beta$, $\beta' = \alpha$.

(iv) $\lambda = -\sigma(\alpha)$, $\mu = -\sigma(\beta)$; in this case

$$\begin{aligned}\alpha * x_\lambda + \beta * x_\mu &= \alpha * x_{-\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} \\ &= (\alpha + \beta) * x_{-\sigma(\alpha+\beta)} = \alpha' * x_{\sigma(\alpha')} + \beta' * x_{-\sigma(\beta')}\end{aligned}$$

with $\alpha' = 0$, $\beta' = \alpha + \beta$. We thus proved the existence of α', β' for the presentation (4.5). □

Proposition 4.3 shows that if $x \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\alpha^2 - \beta^2 \neq 0$, then the choice of α', β' in (4.5) is unique. There are no rules for a further simplification of (4.4) for nonaxial elements, that is the form (4.4) is not reducible. We shall call expression (4.4) *q-linear form*; this form will often appear in the sequel.

The q-linear form (4.4) uses the operations "+", "$*$" and "$_-$". Since "$\neg$" is a special case of "$*$" we use negation anyway. However, we have the option to use either conjugate "$_-$" or opposite "$-$" (using both of them is redundant due to $a_- = \neg(-a)$, resp., $-a = \neg a_-$). If we chose to use opposite, and avoid conjugate, then by means of the notation $\lambda b = \{b, \lambda = +; -b, \lambda = -\}$ and the identity $\alpha * b_- = \alpha * \neg(-b) = (-\alpha) * (-b)$ we may write: $\alpha * b_\lambda = (\lambda\alpha) * (\lambda b)$, in particular, $\alpha * b_{\sigma(\alpha)} = (\sigma(\alpha)\alpha) * (\sigma(\alpha)b) = |\alpha| * (\sigma(\alpha)b)$. The q-linear form (4.4) becomes

$$\begin{aligned} \alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} &= |\alpha| * (\sigma(\alpha)x) + |\beta| * (-\sigma(\beta)x) \\ &= |\alpha| * (\sigma(\alpha)x) - |\beta| * (\sigma(\beta)x). \end{aligned} \tag{4.6}$$

Note that the left-hand side of (4.6) looks simpler than its right-hand side. It seems to be a common situation that the use of opposite instead of dual brings no simplification to the form of the expressions.

The solution of an equation of one variable whenever the left-hand side is presented in the form (4.4) is given by the following modification of Proposition 4.1.

Proposition 4.4. *Let $\alpha, \beta \in \mathbb{R}$, $d \in \mathcal{G}$. If $\gamma = \alpha^2 - \beta^2 \neq 0$, then the equation for $x \in \mathcal{G}$:*

$$\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = d \tag{4.7}$$

is equivalent to $\gamma * x_{\sigma(\gamma)} = \alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)}$, resp. to

$$x = \gamma^{-1} * (\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)})_{\sigma(\gamma)}, \tag{4.8}$$

which is the unique solution of (4.7). If $\beta = \alpha \neq 0$, then (4.7) is equivalent to $x + x_- = \alpha^{-1} * d_{\sigma(\alpha)}$ and, hence, is solvable iff $d \in \mathcal{G}_D$; we have $x = (1/2) * d + s$, $s \in \mathcal{G}_S$. If $\beta = -\alpha \neq 0$, then (4.7) is equivalent to $x \neg x = \alpha^{-1} * d_{\sigma(\alpha)}$ and is solvable iff $d \in \mathcal{G}_S$; we have $x = (1/2) * d + t$, $t \in \mathcal{G}_D$.

Remark 4.5. Note that the expression $\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)} = \alpha * d_{\sigma(\alpha)} + (-\beta) * d_{-\sigma(-\beta)}$ in the right-hand side of (4.8) is of the q-linear form $\alpha * d_{\sigma(\alpha)} + \beta' * d_{-\sigma(\beta')}$, $\beta' = -\beta$, which is irreducible, that is a q-linear form of d. Using opposite we may write the solution (4.8) of (4.7) as $x = \gamma^{-1} * (\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)})_{\sigma(\gamma)} = \gamma^{-1} * (\alpha * d_{\sigma(\alpha)} - \beta * d_{\sigma(-\beta)})_{\sigma(\gamma)} = \gamma^{-1} * (|\alpha| * (\sigma(\alpha)d) - |\beta| * (\sigma(\beta)d))_{\sigma(\gamma)} = |\gamma|^{-1} * (|\alpha| * (\sigma(\alpha)\sigma(\gamma)d) - |\beta| * (\sigma(\beta)\sigma(\gamma)d))$. This shows again that the use of opposite leads to clumsy expressions; this is one more motivation to use conjugate (dual) and avoid the use of opposite in the sequel. Recall also the double usage of the sign "$-$" in the available literature, which may lead to possible confusion.

Proposition 4.6. *Assume that $\alpha^2 - \beta^2 \neq 0$, then the equation for $c \in \mathcal{G}$*

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0 \tag{4.9}$$

has the unique solution $c = 0$.

Proof. If $\alpha^2 - \beta^2 \neq 0$, according to Proposition 4.4 equation (4.9) is equivalent to $(\alpha^2 - \beta^2) * c_{\sigma(\alpha^2-\beta^2)} = 0$, and hence to $(\alpha^2 - \beta^2) * c = 0$, implying $c = 0$. The proposition follows also from (4.8) with $d = 0$. $\square$

Proposition 4.7. *If $(\alpha, \beta) \neq (0,0)$, then (4.9) implies $c \in \mathcal{G}_D \cup \mathcal{G}_S$.*

Proof. Assume $(\alpha, \beta) \neq (0,0)$ and consider separately the two cases: $\alpha^2 - \beta^2 \neq 0$ and $\alpha^2 - \beta^2 = 0$. If $\alpha^2 - \beta^2 \neq 0$, then from Proposition 4.6 we have $c = 0 \in \mathcal{G}_D \cup \mathcal{G}_S$. Assume now that $\alpha^2 - \beta^2 = 0$. Let $\alpha = -\beta \neq 0$; we then have $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \alpha * c_{\sigma(\alpha)} + (-\alpha) * c_{\sigma(\alpha)} = \alpha * (c \neg c)_{\sigma(\alpha)} = 0$, which using Proposition 3.2 implies $c \in \mathcal{G}_D$, hence any linear $c \in \mathcal{G}_D$ satisfies (4.9). Let $\alpha = \beta \neq 0$; we then have $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \alpha * c_{\sigma(\alpha)} + \alpha * c_{-\sigma(\alpha)} = \alpha * (c + c_{-})_{\sigma(\alpha)} = 0$, and using again Proposition 3.2 we obtain $c \in \mathcal{G}_S$, that is (4.9) holds true for any symmetric element c. $\square$

Proposition 4.8. *Assume that $c \in \mathcal{G}$, $c \neq 0$ and equation (4.9) holds. Then $\alpha^2 - \beta^2 = 0$. Furthermore, if $\alpha \neq 0$ (and hence $\beta \neq 0$), then $c \in (\mathcal{G}_D \cup \mathcal{G}_S) \setminus \{0\}$.*

Proof. Assume that $\alpha^2 - \beta^2 \neq 0$. Then, by Proposition 4.6 we obtain the contradiction $c = 0$. Hence, $\alpha^2 - \beta^2 = 0$. Therefore, either $\alpha = \beta$, or $\alpha = -\beta$. In the first case equation (4.9) reduces to $c + c_{-} = 0$, which according to Proposition 3.2 means $c \in \mathcal{G}_S \setminus \{0\}$. In the second case equation (4.9) reduces to $c \neg c = 0$, which according to Proposition 3.2 means $c \in \mathcal{G}_D \setminus \{0\}$. $\square$

Proposition 4.9. *Let $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. Then equation (4.9) implies $\alpha = \beta = 0$.*

Proof. If $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$, then $c \neq 0$. Then according to the Proposition 4.8 equation (4.9) implies $\alpha^2 = \beta^2$. Consider now the two cases $\alpha = \beta$ and $\alpha = -\beta$. If $\alpha = \beta$, then from (4.9) we have $\alpha * (c + c_{-}) = 0$, and $c \in \mathcal{G} \setminus \mathcal{G}_S$ implies $c + c_{-} \neq 0$ and $\alpha = 0 = \beta$. If $\alpha = -\beta$, then from (4.9) we obtain $\alpha * (c \neg c) = 0$, and $c \in \mathcal{G} \setminus \mathcal{G}_D$ implies $c \neg c \neq 0$ and again $\alpha = 0 = \beta$. $\square$

Proposition 4.9 shows that for nonaxial c equality (4.9) can hold only with trivial coefficients ($\alpha = \beta = 0$). For comparison, Proposition 4.7 says that if the coefficients of (4.9) are nontrivial, then c should be axial.

Proposition 4.10. *For $c \in \mathcal{G}$ the set $P(c) = \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\}$ is closed under the operations $+, *, {}_{-}$. Every $d \in P(c)$ has a unique representation in the form $d = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}$ if and only if $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$.*

Proof. The closeness follows easily from the rules for calculation in $\mathcal{G}$. We prove now the uniqueness of the representation. Assume that $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. We have to show that

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \alpha' * c_{\sigma(\alpha')} + \beta' * c_{-\sigma(\beta')}$$

implies $\alpha = \alpha'$, $\beta = \beta'$. Rearranging the terms in the above equation and using the quasidistributive law, we obtain:

$$\begin{aligned}&\alpha * c_{\sigma(\alpha)} + (-\alpha') * c_{-\sigma(\alpha')} + (\beta * c_{\sigma(\beta)} + (-\beta') * c_{-\sigma(\beta')})_{-} \\ &\qquad = (\alpha - \alpha') * c_{\sigma(\alpha-\alpha')} + (\beta - \beta') * c_{-\sigma(\beta-\beta')} = \lambda * c_{\sigma(\lambda)} + \mu * c_{-\sigma(\mu)} = 0.\end{aligned}$$

Using $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and Proposition 4.9 we obtain that $\lambda = \mu = 0$, that is $\alpha = \alpha'$, $\beta = \beta'$.

Assume now that $c \in \mathcal{G}$ is such that every element d of the form

$$d = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}, \tag{4.10}$$

that is of $P(c)$, has unique presentation, that is the scalars α, β in (4.10) are unique. We shall prove that c is nonaxial. Indeed, consider the presentation (4.10) for $d = 0$, that is, assume that for some scalars α, β, the equality

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0 \tag{4.11}$$

holds. From (4.11) we have the obvious equality

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0 * c + 0 * c_{-}.$$

From the assumed uniqueness of the representation we obtain $\alpha = \beta = 0$. Hence (4.11) may only hold with $(\alpha, \beta) = (0, 0)$. Using Proposition 4.7 we conclude that c is nonaxial. □

Similarly, we can prove:

Proposition 4.11. *For $c \in \mathcal{G}_D \cup \mathcal{G}_S$ the set $S(c) = \{\alpha * c_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\}$ is closed under the operations $+, *, \,_-$. Every element of $S(c)$ has a unique representation in the form $\alpha * c_{\sigma(\alpha)}$ if and only if $c \neq 0$.*

In particular, if $c \in \mathcal{G}_D$, $c \neq 0$, then, due to $c = c_-$, every element of the set $S(c) = \{\alpha * c_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\}$ has a unique representation of the "linear" form $\alpha * c$. In fact $\mathcal{G}_D$ is a linear space and $S(c)$ is a subspace of $\mathcal{G}_D$. We shall discuss this in more detail in Section 6.

Metric, norm. Inclusion is naturally extended from a quasilinear space $\mathcal{Q}$ to the induced q-linear space $\mathcal{G} = \text{dis}\,\mathcal{Q}$ by means of:

$$(A, B) \subset (C, D) \Longleftrightarrow A + D \subset B + C. \tag{4.12}$$

A metric can be also naturally extended from a quasilinear space into the induced q-linear space; moreover, due to the presence of inverse elements, it is possible to introduce a norm by means of the metric. Indeed, let $(\mathcal{Q}, +, \mathbb{R}, *, \subset)$ be a quasilinear space with partial order inclusion "$\subset$", satisfying (2.16)–(2.19), and let δ be the natural metric, defined by (2.20), such that relations (2.21), (2.22) hold true. Then δ can be naturally extended from $\mathcal{Q}$ into $\mathcal{G} = \text{dis}\,\mathcal{Q}$ by means of

$$\delta((A, B), (C, D)) = \delta(A + D, B + C). \tag{4.13}$$

It is easy to see that, if $\mathcal{B}$ is a unit ball in $\mathcal{Q}$ and $\mathbf{b} = (\mathcal{B}, 0) \in \mathcal{G}$, then for $x, y \in \mathcal{G}$ we have

$$\delta(x, y) = \min\{\gamma \mid \gamma \geq 0,\ x \subset y + \gamma * \mathbf{b},\ y \subset x + \gamma * \mathbf{b}\}.$$

The latter formula can be used to define a distance in $\mathcal{G}$ using more general unit ball elements — not necessarily proper ones of the form $(\mathcal{B}, 0)$.

We have $(A, 0) \subset (B, 0) \iff A \subset B$ and $\delta((A, 0), (B, 0)) = \delta(A, B)$, showing that the inclusion (4.12) and the metric (4.13) on $\mathcal{G}$ are extensions of the inclusion, resp. the metric, on $\mathcal{Q}$. Note the relations: $(A, B) \subset (0, 0) \iff A \subset B$ and $(0, 0) \subset (A, B) \iff B \subset A$.

Theorem 4.12. *Let* $(\mathcal{Q}, +, \mathbb{R}, *, \subset, \delta)$ *be a quasilinear space endowed with inclusion and metric, such that relations* (2.16)–(2.19), (2.21), (2.22), *hold true. Let* $(\mathcal{G}, +, \mathbb{R}, *)$, $\mathcal{G} = \operatorname{dis} \mathcal{Q}$, *be the induced q-linear space in the sense of* Theorem 3.1. *Then*

(i) *inclusion in* $\mathcal{G}$ *defined by* (4.12) *is isotone with respect to addition and scalar multiplication, i. e. for* $a, b, c \in \mathcal{G}$, $\gamma \in \mathbb{R}$: $a \subset b \iff a+c \subset b+c$; $a \subset b \iff \gamma*a \subset \gamma*b$.

(ii) *the function* δ *defined by* (4.13) *is a metric on* $\mathcal{G}$, *which satisfies for* $x, y \in \mathcal{G}$ *the identities:* $\delta(\gamma * x, \gamma * y) = |\gamma|\delta(x, y)$ *and* $\delta(x + z, y + z) = \delta(x, y)$. *Also:* $\delta(x, y) = \delta(x - y, 0) = \delta(x \neg y_-, 0)$, *so that* $\| z \| = \delta(z, 0)$ *is a norm in* $\mathcal{G}$.

Proof. The proof of Part (i) can be obtained as in [4]; Part (ii) is proved in [13]. □

The system $(\mathcal{G}, +, \mathbb{R}, *, \subset, \| \cdot \|)$ considered in Theorem 4.12 is a *normed* q-linear space, which is a generalization of the familiar normed linear (vector) spaces.

As a consequence of the isotonicity with respect to "$*$" we obtain that negation is *isotone* in the sense that $a \subset b \iff \neg a \subset \neg b$. However, opposite and conjugation are *antitone*, that is $a \subset b \iff -a \supset -b$, and $a \subset b \iff a_- \supset b_-$.

5. Linear Dependence and Basis

Subspaces of q-linear spaces. Let $\mathcal{G}$ be a q-linear space over $\mathbb{R}$. A nonempty subset $\mathcal{H}$ of $\mathcal{G}$ is a *subspace* of $\mathcal{G}$ if $\mathcal{H}$ is a subgroup of $\mathcal{G}$ under addition (inherited from $\mathcal{G}$) and is closed under multiplication by scalar (inherited from $\mathcal{G}$). That is, $\mathcal{H}$ is a subspace of the q-linear space $\mathcal{G}$ if and only if $\mathcal{H} \subset \mathcal{G}$ and:

- $-a \in \mathcal{H}$ and $a + b \in \mathcal{H}$ for all $a, b \in \mathcal{H}$ (i. e. $\mathcal{H}$ is a subgroup of $\mathcal{G}$ under addition);
- $\alpha * c \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$, $c \in \mathcal{H}$ (i. e. $\mathcal{H}$ is closed under multiplication by scalar).

Clearly every subspace $\mathcal{H}$ of a q-linear space $\mathcal{G}$ is itself a q-linear space. Observe that $\alpha * c \in \mathcal{H}$ implies $\neg a \in \mathcal{H}$, which together with $-a \in \mathcal{H}$ implies $a_- \in \mathcal{H}$. Thus the assumptions $-a \in \mathcal{H}$ and $\alpha * c \in \mathcal{H}$ in the above definition imply $a_- \in \mathcal{H}$. Vice versa, $a_- \in \mathcal{H}$ and $\neg a \in \mathcal{H}$ imply $-a \in \mathcal{H}$. We formulate this in the form of the following Lemma:

Lemma 5.1. $\mathcal{H}$ *is a subspace of the q-linear space* $\mathcal{G}$ *if and only if* $\mathcal{H} \subset \mathcal{G}$ *and* $\mathcal{H}$ *is closed under "+", "$*$", "$_-$", i. e.:*

(i) $a + b \in \mathcal{H}$ *for all* $a, b \in \mathcal{H}$;
(ii) $\alpha * c \in \mathcal{H}$ *for all* $\alpha \in \mathbb{R}$ *and* $c \in \mathcal{H}$;
(iii) $a_- \in \mathcal{H}$ *for all* $a \in \mathcal{H}$.

Lemma 5.1 allows to check for q-linear spaces without the use of opposite. As an example we show that $\mathcal{G}_S$ and $\mathcal{G}_D$ are subspaces of $\mathcal{G}$.

Proposition 5.2. *Given a q-linear space $\mathcal{G}$, the sets $\mathcal{G}_D, \mathcal{G}_S$ are subspaces of $\mathcal{G}$.*

Proof. We apply the subspace criterion of Lemma 5.1. (i) Assume that $a, b \in \mathcal{G}_D = \{c \in \mathcal{G} \mid c = c_-\}$, that is $a = a_-$ and $b = b_-$. Then $a + b = a_- + b_- = (a+b)_-$, hence $a + b \in \mathcal{G}_D$. Further we have $\gamma * a = \gamma * a_- = (\gamma * a)_- \in \mathcal{G}_D$ and $a_- = a = (a_-)_- \in \mathcal{G}_D$. (ii) Assume now that $a, b \in \mathcal{G}_S = \{c \in \mathcal{G} \mid c = \neg c\}$, that is $a = \neg a$ and $b = \neg b$. Then $a + b = \neg a + \neg b = \neg(a+b)$, hence $a + b \in \mathcal{G}_S$; also $\gamma * a = \gamma * \neg a = \neg(\gamma * a) \in \mathcal{G}_S$ and $a_- = (\neg a)_- = \neg(a_-) \in \mathcal{G}_S$. □

In the previous section the reader may have noticed that in a q-linear space the expression $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}$ plays the role of a linear combination $\alpha \cdot c$ in a linear space. The following definition of "linear combination" generalizes the concept of linear combination in a linear space.

Definition 5.3. Let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be finitely many (not necessarily distinct) elements of a q-linear space $\mathcal{G}$ over the field $\mathbb{R}$. An element d of $\mathcal{G}$ of the form

$$d = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \beta_1 * c^{(1)}_{-\sigma(\beta_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \beta_2 * c^{(2)}_{-\sigma(\beta_2)} + \\ + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} + \beta_k * c^{(k)}_{-\sigma(\beta_k)}, \quad (5.1)$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2, ..., \alpha_k, \beta_k \in \mathbb{R}$, is called a *(linear) combination* of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$.

Note that there will be no confusion if we use the name linear combination for (5.1), because if $\mathcal{G}$ is linear ($\mathcal{G} = \mathcal{G}_S$), then (5.1) obtains the form of the usual linear combination. In other words, (5.1) is an extension of the concept of linear combination (as defined in a linear space) to a q-linear space. When using the name "linear" we should be careful with the particular space in question (linear or q-linear).

As usually, the pair $(0,0)$ is called *trivial*, a pair (α, β), $\alpha, \beta \in \mathbb{R}$, such that $(\alpha, \beta) \neq (0,0)$ (that is, at least one of the numbers α, β is not zero), is called *nontrivial*. Likewise, any system of k pairs $\{(0,0), ..., (0,0)\}$ is called *trivial* and a system of k pairs $\{(\alpha_i, \beta_i)\}_{i=1}^k$ is *nontrivial* if at least one pair of the system is nontrivial, that is $(\alpha_i, \beta_i) \neq (0,0)$ for some $i = 1, ..., k$. A linear combination (5.1) having a trivial system of pairs, is called *trivial*. Clearly, every trivial combination has the value 0.

A pair (α, β), $\alpha, \beta \in \mathbb{R}$, such that $\alpha^2 - \beta^2 = 0$ will be further called *axial*, otherwise, when $\alpha^2 - \beta^2 \neq 0$, — *nonaxial*. Similarly, a system of k pairs $\{(\alpha_i, \beta_i)\}_{i=1}^k$ is *axial* if all pairs (α_i, β_i), $i = 1, 2, ..., k$, are axial, and a system is *nonaxial* if at least one pair is nonaxial. Recall that if a pair (α_i, β_i) in (5.1) is axial, then $\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)} \in \mathcal{G}_D \cup \mathcal{G}_S$ for any $c \in \mathcal{G}$.

In a q-linear space every trivial linear combination is, of course, axial, and every nonaxial linear combination is nontrivial.

We may write symbolically (5.1) as $d = \sum_{i=1}^k (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)})$, $\alpha_i, \beta_i \in \mathbb{R}$.

Let d be a linear combination of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$. Then $\mu * d_{\sigma(\mu)} + \nu * d_{-\sigma(\nu)}$, $\mu, \nu \in \mathbb{R}$, is also a linear combination of $c^{(1)}, c^{(2)}, ..., c^{(k)}$. More generally, we have:

Proposition 5.4. *Let $d^{(1)}, d^{(2)}, ..., d^{(l)}$ be linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$ and let g be a linear combination of $d^{(1)}, d^{(2)}, ..., d^{(l)}$. Then g is a linear combination of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ as well.*

Proof. Denote

$$\begin{aligned} d^{(j)} &= \alpha_{j1} * c^{(1)}_{\sigma(\alpha_{j1})} + \beta_{j1} * c^{(1)}_{-\sigma(\beta_{j1})} + \alpha_{j2} * c^{(2)}_{\sigma(\alpha_{j2})} + \beta_{j2} * c^{(2)}_{-\sigma(\beta_{j2})} + ... \\ &+ \alpha_{jk} * c^{(k)}_{\sigma(\alpha_{jk})} + \beta_{jk} * c^{(k)}_{-\sigma(\beta_{jk})}, \end{aligned} \tag{5.2}$$

where $\alpha_{j1}, \beta_{j2}, \alpha_{j2}, \beta_{j2}, ..., \alpha_{jk}, \beta_{jk} \in \mathbb{R}$, $j = 1, 2, ..., l$. Denote also

$$\begin{aligned} g &= \mu_1 * d^{(1)}_{\sigma(\mu_1)} + \nu_1 * d^{(1)}_{-\sigma(\nu_1)} + \mu_2 * d^{(2)}_{\sigma(\mu_2)} + \nu_2 * d^{(2)}_{-\sigma(\nu_2)} + ... \\ &+ \mu_l * d^{(l)}_{\sigma(\mu_l)} + \nu_l * d^{(l)}_{-\sigma(\nu_l)}, \end{aligned} \tag{5.3}$$

where $\mu_1, \nu_2, \mu_2, \nu_2, ..., \mu_l, \nu_l \in \mathbb{R}$. Substituting (5.2) in (5.3) we obtain

$$\begin{aligned} g &= \mu'_1 * c^{(1)}_{\sigma(\mu'_1)} + \nu'_1 * c^{(1)}_{-\sigma(\nu'_1)} + \mu'_2 * c^{(2)}_{\sigma(\mu'_2)} + \nu'_2 * c^{(2)}_{-\sigma(\nu'_2)} + ... \\ &+ \mu'_k * c^{(k)}_{\sigma(\mu'_k)} + \nu'_k * c^{(k)}_{-\sigma(\nu'_k)}, \end{aligned}$$

where $\mu'_i = \sum_{j=1}^{l} \mu_j \alpha_{ij}$, $\nu'_i = \sum_{j=1}^{l} \nu_j \beta_{ij}$, $i = 1, ..., k$, which proves the Proposition. □

Proposition 5.5. *Let $\mathcal{G}$ be a q-linear space over $\mathbb{R}$ and let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be a finite nonempty subset of elements of $\mathcal{G}$. Then the set*

$$\begin{aligned} \mathcal{H} &= \operatorname{span}\{c^{(1)}, c^{(2)}, ..., c^{(k)}\} \\ &\stackrel{Def}{=} \{\sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) \mid \alpha_i, \beta_i \in \mathbb{R}\} \\ &= \{\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} \\ &+ \beta_1 * c^{(1)}_{-\sigma(\beta_1)} + ... + \beta_k * c^{(k)}_{-\sigma(\beta_k)} \mid \alpha_i, \beta_i \in \mathbb{R}\} \end{aligned}$$

of all linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ is a subspace of $\mathcal{G}$.

Indeed, it is easy to check that the subspace criterion of the Lemma 5.1 holds. We say that the space $\mathcal{H}$ defined in Proposition 5.5 is *spanned* by $c^{(1)}, c^{(2)}, ..., c^{(k)}$.

Definition 5.6. Let $\mathcal{G}$ be a q-linear space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$, $k \geq 1$, are called *(linearly) dependent (over $\mathbb{R}$)* if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i. e. if there exist a nontrivial system $\{(\alpha_i, \beta_i)\}_{i=1}^{k}$, such that

$$\begin{aligned} \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \beta_1 * c^{(1)}_{-\sigma(\beta_1)} &+ \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \beta_2 * c^{(2)}_{-\sigma(\beta_2)} + ... \\ &+ \alpha_k * c^{(k)}_{\sigma(\alpha_k)} + \beta_k * c^{(k)}_{-\sigma(\beta_k)} = 0. \end{aligned} \tag{5.4}$$

Elements which are not linearly dependent, are called *(linearly) independent*. In other words the elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$ are linearly independent, if (5.4) is possible only for the trivial linear combination, such that $\alpha_i = \beta_i = 0$ for all $i = 1, ..., k$.

Using Propositions 4.7 and 4.9 we can conclude that a single element is linearly dependent iff it is axial (that is linear or symmetric), and is linearly independent iff nonaxial.

Proposition 5.7. *Let $\mathcal{G}$ be a q-linear space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$, $k \geq 2$, are linearly dependent, iff at least one of the elements is a linear combination of the other elements.*

Proof. We first assume that $c^{(1)}, c^{(2)}, ..., c^{(k)}$ are linearly dependent. Then there exist scalars α_i, β_i, $i = 1, ..., n$, not all of them zero, such that (5.4) holds. Let us suppose that $\alpha_1 \neq 0$. Denoting

$$f = \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \beta_2 * c^{(2)}_{-\sigma(\beta_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} + \beta_k * c^{(k)}_{-\sigma(\beta_k)} \tag{5.5}$$

we can write (5.4) as

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \beta_1 * c^{(1)}_{-\sigma(\beta_1)} = \neg f_-. \tag{5.6}$$

We consider now separately the cases: (i) $\gamma_1 = \alpha_1^2 - \beta_1^2 \neq 0$; (ii) $\alpha_1 = \beta_1$; (iii) $\alpha_1 = -\beta_1$. In case (i) we can write (5.6) as $\gamma_1 * c^{(1)} = \neg f_-$, resp. as $c^{(1)} = (-\gamma_1^{(-1)}) * f_-$, which shows that $c^{(1)}$ is a linear combination of the other elements. In the case (ii) we obtain $\alpha_1 * (c^{(1)} + c^{(1)}_-)_\lambda = \neg f_-$, and in case (iii) we obtain $\alpha_1 * (c^{(1)} \neg c^{(1)})_\lambda = \neg f_-$. Both in case (ii) and (iii) we see that $c^{(1)}$ is a linear combination of the other elements.

Conversely, if one of the elements is a linear combination of the rest, then it follows immediately that all elements are linearly dependent. Indeed, let (5.5) holds with $f = c^{(1)}$. Then we can rewrite (5.5) in the form (5.4) with $(\alpha_1, \beta_1) \neq (0, 0)$. □

Similarly, it can be proved that, given k elements $c^{(1)}, c^{(2)}, ..., c^{(k)}$, if $l < k$ elements of them are linearly dependent, then all k elements are also linearly dependent. In particular, if $c^{(i)}$ for some i, $1 \leq i \leq k$, is axial, then the elements $c^{(1)}, c^{(2)}, ..., c^{(k)}$ are linearly dependent.

If a q-linear space is linear or symmetric, then the form of the linear combination is simplified. The linear case is, of course, well-known, and we shall come back to the symmetric case again later on. We first consider the general case, assuming that the q-linear space $\mathcal{G}$ is not axial (not linear and not symmetric) and thus contains nonaxial elements. For such spaces we shall extend the concept of basis as known from the theory of linear spaces in the following way.

Definition 5.8. Let $\mathcal{G}$ be a q-linear space. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a *basis* of $\mathcal{G}$, if $c^{(i)}$ are linearly independent and $\mathcal{G}$ is spanned by $\{c^{(i)}\}_{i=1}^k$, $\mathcal{G} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$.

Theorem 5.9. *Let $\mathcal{G}$ be a q-linear space over $\mathbb{R}$. A set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a basis of $\mathcal{G}$ iff every element d of $\mathcal{G}$ can be written in the form* (5.1) *in a unique way (i. e. with unique scalars α_i, β_j).*

Proof. Assume first that $\{c^{(i)}\}_{i=1}^k$ is a basis of $\mathcal{G}$, that is $\mathcal{G} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$, and every element of $\mathcal{G}$ can be written in the form (5.1) with suitable scalars α_i, β_j. We have to show the uniqueness of this representation, that is we are to show that, if

$$\sum_{i=1}^k (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) = \sum_{i=1}^k (\alpha'_i * c^{(i)}_{\sigma(\alpha'_i)} + \beta'_i * c^{(i)}_{-\sigma(\beta'_i)}), \tag{5.7}$$

then $\alpha_i = \alpha_i'$, $\beta_i = \beta_i'$ for $i = 1, ..., k$. Indeed, from (5.7), by means of the quasidistributive law, we obtain

$$\sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + (-\alpha_i') * c^{(i)}_{\sigma(-\alpha_i')}) \quad + \quad (\beta_i * c^{(i)}_{\sigma(\beta_i)} + (-\beta_i') * c^{(i)}_{\sigma(-\beta_i')})_-$$

$$= \sum_{i=1}^{k} (\lambda_i * c^{(i)}_{\sigma(\lambda_i)} + \mu_i * c^{(i)}_{-\sigma(\mu_i)}) = 0, \tag{5.8}$$

where $\lambda_i = \alpha_i - \alpha_i'$, $\mu_i = \beta_i - \beta_i'$. From (5.8), using that $\{c^{(i)}\}_{i=1}^{k}$ are linearly independent, we have $\lambda_i = \alpha_i - \alpha_i' = 0$, $\mu_i = \beta_i - \beta_i' = 0$, which proves uniqueness.

Conversely, let us suppose that the system $\{c^{(i)}\}_{i=1}^{k}$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is chosen in such a way that every element d of $\mathcal{G}$ can be written in the form

$$d = \sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) \tag{5.9}$$

with unique scalars $\alpha_i, \beta_i \in \mathbb{R}$. Then $\mathcal{G} = \mathrm{span}\{c^{(i)}\}_{i=1}^{k}$. It remains to prove that $\{c^{(i)}\}_{i=1}^{k}$ are linearly independent. Indeed, assume that for some scalars α_i, β_i, $i = 1, ..., k$, the equality

$$\sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) = 0 \tag{5.10}$$

holds. Equality (5.10) also holds for the trivial linear combination, hence

$$\sum_{i=1}^{k} (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \beta_i * c^{(i)}_{-\sigma(\beta_i)}) = \sum_{i=1}^{k} (0 * c^{(i)} + 0 * c^{(i)}_{-}).$$

Using the assumed uniqueness of the representation we obtain $\alpha_i = \beta_i = 0$, $i = 1, ..., k$, that is, (5.10) can only hold for a trivial linear combination. Therefore the elements $\{c^{(i)}\}_{i=1}^{k}$ are linearly independent, and hence are a basis of $\mathcal{G}$. □

Let us now interpret some results from the previous section in the light of the newly introduced concepts.

A (system consisting of a) single element $c \in \mathcal{G}$ is linearly dependent iff $c \in \mathcal{G}_D \cup \mathcal{G}_S$ (in particular the element $c = 0$ is linearly dependent). Indeed, let $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = 0$ and $(\alpha, \beta) \neq (0, 0)$, so that the resp. linear combination of c is nontrivial. Then from Proposition 4.7 we obtain that there exist axial solutions $c \in \mathcal{G}_D \cup \mathcal{G}_S$ to equation (4.9), corresponding to nontrivial linear combinations.

Proposition 3.2 and Proposition 4.2 imply that if $a \in \mathcal{G}_D$, $a \neq 0$ (that is $a = a_- \neq 0$), then $\mathrm{span}\{a\} = \{\gamma * a \mid \gamma \in \mathbb{R}\}$. Also, if $b \in \mathcal{G}_S$, $b \neq 0$, then $\mathrm{span}\{b\} = \{\gamma * b_{\sigma(\gamma)} \mid \gamma \in \mathbb{R}\} = \{|\gamma| * b_{\sigma(\gamma)} \mid \gamma \in \mathbb{R}\}$.

Proposition 5.10. *Let $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. Then*

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_D \cup \mathcal{G}_S \tag{5.11}$$

iff $\alpha^2 - \beta^2 = 0$. More specifically: (i) $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_D$, *iff $\alpha = \beta$;* (ii) $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_S$, *iff $\alpha = -\beta$.*

Proof. If $\alpha = \beta$, then $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \alpha * (c + c_-) \in \mathcal{G}_D$. Let $z \in \mathcal{G}_D$. Using Proposition 5.9 we can write $z = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \in \mathcal{G}_D$ with unique α, β. However, every element of $\mathcal{G}_D$ can be written as $\gamma * (c + c_-)$ with some $\gamma \in \mathbb{R}$. From $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \gamma * (c + c_-) = \gamma * c_{\sigma(\gamma)} + \gamma * c_- - \sigma(\gamma)$ we obtain $\alpha = \beta$. Part (ii) is proved similarly. $\square$

Assume $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\mathcal{H} = \operatorname{span}\{c\} = \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\}$. Using Proposition 4.2 we can write

$$\begin{aligned}
\mathcal{H}_S &= \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha = -\beta\} \\
&= \{\alpha * (c \neg c)_{\sigma(\alpha)} \mid \alpha \in \mathbb{R}\} = \{a \in \mathcal{H} \mid a = \neg a\} \\
\mathcal{H}_D &= \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha = \beta\} \\
&= \{\alpha * (c + c_-) \mid \alpha \in \mathbb{R}\} = \{a \in \mathcal{H} \mid a = a_-\}.
\end{aligned}$$

$\mathcal{H}_S$ is a symmetric subspace of $\mathcal{H}$ and $\mathcal{H}_D$ is a linear subspace of $\mathcal{H}$. In what follows the direct sum of the q-linear spaces $\mathcal{G}_1, \mathcal{G}_2$ is denoted by $\mathcal{G}_1 + \mathcal{G}_2$.

Proposition 5.11. *Assume $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\mathcal{H} = \operatorname{span}\{c\} = \{\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} \mid \alpha, \beta \in \mathbb{R}\}$. We have $\mathcal{H}_S \cap \mathcal{H}_D = 0$, and*

$$\mathcal{H} \;=\; \mathcal{H}_S + \mathcal{H}_D = \{u + v \mid u \in \mathcal{H}_S, v \in \mathcal{H}_D\}.$$

Every element of $\mathcal{H}$ can be presented uniquely in the form

$$\alpha * s_{\sigma(\alpha)} + \beta * t_{-\sigma(\beta)} = \alpha * s_{\sigma(\alpha)} + \beta * t = |\alpha| * s_{\sigma(\alpha)} + \beta * t, \tag{5.12}$$

where $s = (c \neg c)/2 \in \mathcal{H}_S$, $t = (c + c_-)/2 \in \mathcal{H}_D$.

The elements $s = (c \neg c)/2 \in \mathcal{H}_S$, $t = (c + c_-)/2 \in \mathcal{H}_D$ are projections of $c \in \mathcal{H} = \mathcal{H}_S + \mathcal{H}_D$ on the subspaces $\mathcal{H}_S$, resp. $\mathcal{H}_D$, $c = s + t$.

Let $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\mathcal{H} = \operatorname{span}\{c\}$, that is c is a basis of $\mathcal{H}$. Let $d = \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}$ be such that $\alpha^2 - \beta^2 \neq 0$. The latter is equivalent (in view of Proposition 5.10) to the assumption $d \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. It is easy to see that $\mathcal{H} = \operatorname{span}\{d\}$, that is, any element of $\operatorname{span}\{c\}$ belongs to $\operatorname{span}\{d\}$ and vice versa. Indeed, recall that, according to Proposition 4.4, $c = (\alpha^2 - \beta^2)^{-1} * (\alpha * d_{\sigma(\alpha)} + (-\beta) * d_{\sigma(\beta)})_{\sigma(\alpha^2 - \beta^2)}$. We thus see that the basis c can be replaced by the basis d. Transformation formulae can be derived as follows.

Assume that the q-linear space $\mathcal{G}$ is spanned over a basis consisting of a single element $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. Let $d \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$ and $\gamma = \alpha^2 - \beta^2 \neq 0$, so that the solution of (4.7): $\alpha * x_{\sigma(\alpha)} + \beta * x_{-\sigma(\beta)} = d$ is, according to Proposition 4.4, $x = \gamma^{-1} * (\alpha * d_{\sigma(\alpha)} \neg \beta * d_{\sigma(\beta)})_{\sigma(\gamma)}$. Assume that the right-hand side d of (4.7) is presented in the basis c by

$$d = \delta * c_{\sigma(\delta)} + \varepsilon * c_{-\sigma(\varepsilon)}.$$

Note that $\delta^2 - \varepsilon^2 \neq 0$ due to $d \in \mathcal{G} \setminus (\mathcal{G}_S \cup \mathcal{G}_D)$. We want to present x as a linear combination of c. Thus we need to express the unknown "coordinates" ξ, η of

$$x = \xi * c_{\sigma(\xi)} + \eta * c_{-\sigma(\eta)}$$

in terms of the given "coordinates" δ, ε of d with respect to the basis c. Substituting the expressions for d and x in (4.7) we arrive to the following system for ξ, η:

$$\alpha\xi + \beta\eta = \delta, \quad \beta\xi + \alpha\eta = \varepsilon,$$

which, due to $\alpha^2 - \beta^2 \neq 0$, has a unique solution $\xi = (\alpha\delta - \beta\varepsilon)/\gamma$, $\eta = (\alpha\varepsilon - \beta\delta)/\gamma$. This shows that the solution of an equation of one variable is reduced to a system of two equations for the two "coordinates" (ξ, η) of the unknown variable x.

The above considerations can be easily generalized for the case of a basis of k elements $c^{(1)}, \ldots, c^{(k)}$, such that $c^{(i)} \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$.

Propositions 5.9, 5.11 imply that, if the elements $\{c^{(i)}\}_{i=1}^k$ are a basis of a q-linear space $\mathcal{G}$, i. e. $\mathcal{G} = \mathrm{span}\{c^{(1)}, c^{(2)}, ..., c^{(k)}\}$, then every $\mathcal{G}_i = \mathrm{span}\{c^{(i)}\}$ is a q-linear subspace of $\mathcal{G}$. We also have $\mathcal{G}_i \cap \mathcal{G}_j = 0$, $i, j = 1, 2, ..., k$. Hence $\mathcal{G}$ is a *direct sum* of $\mathcal{G}_i$: $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + ... + \mathcal{G}_n = \{g_1 + g_2 + ... + g_n \mid g_i \in \mathcal{G}_i\}$.

A q-linear space, which is spanned over a finite basis, is a direct sum of q-linear spaces spanned over single elements. Hence, it can be decomposed as a direct sum of linear and a symmetric q-linear spaces. Using Proposition 5.11 we have $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + ... + \mathcal{G}_n = (\mathcal{G}_1)_S + (\mathcal{G}_1)_D + ... + (\mathcal{G}_n)_S + (\mathcal{G}_n)_D = (\mathcal{G}_1)_S + (\mathcal{G}_2)_S + ... + (\mathcal{G}_n)_S + (\mathcal{G}_1)_D + (\mathcal{G}_2)_D + ... + (\mathcal{G}_n)_D = \mathcal{G}_S + \mathcal{G}_D$.

Proposition 5.12. *Let $\mathcal{G}$ be a q-linear space spanned over nonaxial $c^{(1)}, ..., c^{(k)}$. Then $\mathcal{G} = \mathcal{G}_S + \mathcal{G}_D$, where $\mathcal{G}_S$ is the symmetric q-linear space spanned over $s^{(i)} = (c^{(i)} \neg c^{(i)})/2$, $i = 1, ..., k$, and $\mathcal{G}_D$ is the linear space spanned over $t^{(i)} = (c^{(i)} + c_-^{(i)})/2$, $i = 1, ..., k$.*

Proposition 5.12 shows the need of a systematic study of symmetric q-linear spaces.

Symmetric q-linear spaces. Assume that the q-linear space $\mathcal{G}$ is symmetric, $\mathcal{G} = \mathcal{G}_S$. In a symmetric q-linear space the definitions of linear combination, linear dependence etc. follows as special cases from the definitions for general q-linear spaces. The concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathcal{G}$ ($\mathcal{G} = \mathcal{G}_S$). An element f of $\mathcal{G}$ of the form

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}, \tag{5.13}$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, is a *linear combination* of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$.

Proposition 5.13. *Let $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$, $k \geq 1$. Then the set*

$$\mathcal{H} = \{\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R}\}$$

of all linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ is a subspace of $\mathcal{G}$.

The elements $c^{(1)}, c^{(2)}, ..., c^{(k)}$ form a *generating set* of $\mathcal{H}$. We also say that the subspace $\mathcal{H}$ defined in Proposition 5.13 is *spanned* by $c^{(1)}, c^{(2)}, ..., c^{(k)}$ and write $\mathcal{H} = \text{span}\{c^{(1)}, c^{(2)}, ..., c^{(k)}\}$.

Let $\mathcal{G}$ be a symmetric q-linear space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$, $k \geq 1$, are *linearly dependent (over* $\mathbb{R}$*)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i. e. if there exist a nontrivial system $\{\alpha_i\}_{i=1}^k$, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{5.14}$$

Elements of $\mathcal{G}$, which are not linearly dependent, are *linearly independent*. That is, the elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{G}$ are *linearly independent*, if (5.14) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, ..., k$.

Clearly a single element $c \in \mathcal{G} = \mathcal{G}_S$ is linearly dependent, iff $c = 0$, and is linearly independent, iff $c \neq 0$.

Basis in a symmetric q-linear space. Let $\mathcal{G} = \mathcal{G}_S$ be a symmetric q-linear space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a *basis* of $\mathcal{G}$, if $c^{(i)}$ are linearly independent and $\mathcal{G} = \text{span}\{c^{(i)}\}_{i=1}^k$.

The following result is a special case of Theorem 5.9:

Theorem 5.14. *Let $\mathcal{G}$ be a symmetric quasilinear space with group structure over $\mathbb{R}$. A set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{G}$, $k \geq 1$, is a basis of $\mathcal{G}$, iff every $f \in \mathcal{G}$ can be presented in the form* (5.13) *in a unique way (i. e. with unique scalars α_i).*

The above uniqueness theorem can be used to compute with k-tuples representing the elements of the symmetric q-linear space $\mathcal{G} = \mathcal{G}_S = \text{span}\{c^{(i)}\}_{i=1}^k$. Let

$$a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)}, \quad b = \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)} \tag{5.15}$$

be two elements of $\mathcal{G}$. Their sum is

$$a + b = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)} = \sum_{i=1}^k (\alpha_i + \beta_i) * c^{(i)}_{\sigma(\alpha_i+\beta_i)}. \tag{5.16}$$

Multiplication with a real number is given by

$$\gamma * a = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(|\gamma|\alpha_i)}. \tag{5.17}$$

To every $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \in \mathcal{G} = \mathcal{G}_S$ we allocate a k-tuple $(\alpha_1, \alpha_2, ..., \alpha_k)$. Then, minding formulae (5.16), (5.17), we define

$$(\alpha_1, \alpha_2, ..., \alpha_k) + (\beta_1, \beta_2, ..., \beta_k) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_k + \beta_k) \tag{5.18}$$

$$\gamma * (\alpha_1, \alpha_2, ..., \alpha_k) = (|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_k). \tag{5.19}$$

It is easy to see that the system $(\mathbb{R}^k, +, \mathbb{R}, *)$ with addition defined by (5.18) and multiplication by real scalar defined by (5.19) is a q-linear space. As we know, negation in $\mathcal{G} = \mathcal{G}_S$ is same as identity. Conjugation in $\mathcal{G} = \mathcal{G}_S$ coincides with opposite: $a_- = -a = \sum_{i=1}^k \alpha_i * c^{(i)}_{-\sigma(\alpha_i)} = \sum_{i=1}^k (-\alpha_i) * c^{(i)}_{\sigma(-\alpha_i)}$. This implies in terms of $(\mathbb{R}^k, +, \mathbb{R}, *)$:

$$(\alpha_1, \alpha_2, ..., \alpha_k)_- = -(\alpha_1, \alpha_2, ..., \alpha_k) = (-\alpha_1, -\alpha_2, ..., -\alpha_k). \tag{5.20}$$

We can define dimension of a symmetric q-linear space $\mathcal{G}_S$ spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. It can be shown that the number k of terms in the expression for the span does not change with the particular basis, hence can be called *dimension* of $\mathcal{G}_S$.

Let the basis of $\mathcal{G}$ consists of the k elements $c^{(i)} \in \mathcal{G} \setminus (\mathcal{G}_S \cup \mathcal{G}_D)$, $i = 1, ..., k$. Take the corresponding projections of $c^{(i)}$ on $\mathcal{G}_S$ and $\mathcal{G}_D$, that is the k symmetric elements $s^{(i)} = (c^{(i)} \neg c^{(i)})/2 \in \mathcal{G}_S$, and k linear elements $t^{(i)} = (c^{(i)} + c^{(i)}_-)/2 \in \mathcal{G}_D$, $i = 1, ..., k$. Using (5.12) we have

$$c^{(i)} = |\alpha_i| * s^{(i)}_{\sigma(\alpha_i)} + \beta_i * t^{(i)}.$$

Let $\mathcal{G}_S$ be a symmetric q-linear space of finite dimension k, so that $\mathcal{G}_S \cong (\mathbb{R}^k, +, \mathbb{R}, *)$. Consider the symmetric quasilinear space $((\mathbb{R}^+)^k, +, \mathbb{R}, *)$. We obviously have $\mathcal{G}_S = \mathrm{dis}((\mathbb{R}^+)^k, +, \mathbb{R}, *)$, meaning that given a finite symmetric q-linear space we can reconstruct the underlying (proper) quasilinear space. This can be generalized for any finite q-linear space, such that $\mathcal{G} = \mathcal{G}_S + \mathcal{G}_D$, where the subpaces $\mathcal{G}_S$, $\mathcal{G}_D$ are finite. Indeed, let $\mathcal{G} = \mathcal{G}_S + \mathcal{G}_D$ and $\mathcal{G}_S \cong (\mathbb{R}^k, +, \mathbb{R}, *)$. We ask is there a (proper) quasilinear space $\mathcal{Q}$, such that $\mathcal{G} = \mathrm{dis}\,\mathcal{Q}$? Take $\mathcal{Q} = ((\mathbb{R}^+)^k, +, \mathbb{R}, *) \times \mathcal{G}_D$, then $\mathcal{Q}_S = ((\mathbb{R}^+)^k, +, \mathbb{R}, *)$. The space $\mathcal{Q}$ is the underlying quasilinear space, since $\mathcal{Q}_D \cong \mathcal{G}_D$.

6. Linear Multiplication and Associated Linear Spaces

Linear multiplication. Recall that a linear space is a system $(\mathcal{G}, +, \mathbb{R}, \cdot)$, such that $(\mathcal{G}, +)$ is an abelian group, and $\cdot : \mathcal{G} \times \mathbb{R} \longrightarrow \mathcal{G}$ satisfies for all $a, b, c \in \mathcal{G}$, $\alpha, \beta, \gamma \in \mathbb{R}$:

$$\begin{aligned}
\alpha \cdot (\beta \cdot c) &= (\alpha\beta) \cdot c, & (6.1)\\
\gamma \cdot (a + b) &= \gamma \cdot a + \gamma \cdot b, & (6.2)\\
1 \cdot a &= a, & (6.3)\\
(\alpha + \beta) \cdot c &= \alpha \cdot c + \beta \cdot c. & (6.4)
\end{aligned}$$

In a linear space $(-1) \cdot a + a = 0$, that is, $-a = (-1) \cdot a$ (opposite and negation coincide). The operation "$\cdot$" is further called *linear multiplication by scalar.*

Let $(\mathcal{G}, +, \mathbb{R}, *)$ be the q-linear space. We next show that the operation "$\cdot$": $\mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$, defined by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)},\ \alpha \in \mathbb{R},\ c \in \mathcal{G}, \tag{6.5}$$

is linear multiplication by scalar.

Theorem 6.1. *Let $(\mathcal{G}, +, \mathbb{R}, *)$ be a q-linear space. Then $(\mathcal{G}, +, \mathbb{R}, \cdot)$, where "$\cdot$" is defined by (6.5), is a linear space.*

Proof. We have to prove that "$\cdot$" defined by (6.5) satisfies the axioms for multiplication by scalar in a linear space.

1. Let us prove relation (6.1): $\alpha \cdot (\beta \cdot d) = (\alpha\beta) \cdot d$. Substitute $c = d_{\sigma(\beta)}$ in the relation $\alpha * (\beta * c) = (\alpha\beta) * c$ to obtain $\alpha * (\beta * d_{\sigma(\beta)}) = (\alpha\beta) * d_{\sigma(\beta)}$. Using (6.5) we have $\alpha * (\beta \cdot d) = (\alpha\beta) * d_{\sigma(\beta)}$. "Conjugating" by $\sigma(\alpha)$ we obtain $\alpha * (\beta \cdot d)_{\sigma(\alpha)} = (\alpha\beta) * d_{\sigma(\beta)\sigma(\alpha)}$ $= (\alpha\beta) * d_{\sigma(\beta\alpha)}$, or $\alpha \cdot (\beta \cdot d) = (\alpha\beta) \cdot d$, for all $d \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$.

2. To prove the relation (6.2): $\gamma \cdot (a + b) = \gamma \cdot a + \gamma \cdot b$, substitute $a = c_{\sigma(\gamma)}$, $b = d_{\sigma(\gamma)}$ in $\gamma * (a + b) = \gamma * a + \gamma * b$. We obtain $\gamma * (c_{\sigma(\gamma)} + d_{\sigma(\gamma)}) = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$, or $\gamma * (c + d)_{\sigma(\gamma)} = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$. This implies that $\gamma \cdot (c + d) = \gamma \cdot c + \gamma \cdot d$, for all $c, d \in \mathcal{G}$, $\gamma \in \mathbb{R}$.

3. Relation (6.3): $1 \cdot a = a$ is obviously true.

4. Relation (6.4): $(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c$, resp. $(-1) \cdot a + a = 0$, follows directly from the quasidistributive law (3.10) using the definition (6.5).

We thus proved that $(\mathcal{G}, +, \mathbb{R}, \cdot)$ is a linear space. □

Note that $(-1) \cdot a = (-1) * a_- = \neg a_- = -a$ is the opposite in $\mathcal{G}$, i. e. for $a \in \mathcal{G}$ we have: $a + (-1) \cdot a = a + (-a) = 0$. Note also that (6.5) is defined for all elements of $\mathcal{G}$ and all scalars from $\mathbb{R}$. Theorem 6.1 shows that every q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ involves an *associated linear space* $(\mathcal{G}, +, \mathbb{R}, \cdot)$. This fact justifies the name "linear multiplication by scalar". To better distinguish between the two multiplications by scalar "$*$" and "$\cdot$", the operation "$*$" will be referred to as *q-linear multiplication by scalar*. Let us also note that $(\mathcal{G}, +, \mathbb{R}, \cdot)$ is a *normed* linear space under the norm inherited from the q-linear space, i. e. $\| z \| = \delta(z, 0)$ with $\delta(x, y) = \delta(x - y, 0) = \delta(x + (-1) \cdot y, 0)$.

Let $(\mathcal{G}, +, \mathbb{R}, *)$ be a q-linear space induced by the quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *)$, so that $\mathcal{G} = \operatorname{dis} \mathcal{Q}$. The (normed) linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ can be obtained directly from the underlying quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *)$, without passing through the q-linear system $(\mathcal{G}, +, \mathbb{R}, *)$. This has been actually done in [13], where the linear multiplication by scalar has been defined by

$$\gamma \cdot (A, B) = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geq 0, \\ (|\gamma| * B, |\gamma| * A), & \text{if } \gamma < 0. \end{cases} \tag{6.6}$$

Formula (6.6) can be written without introducing q-linear multiplication by scalar, resp. the q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$. If the space $(\mathcal{G}, +, \mathbb{R}, *)$ is introduced, then the right-hand sides of (6.6) and (6.5) are equivalent, indeed we have:

$$\gamma * (A, B)_{\sigma(\gamma)} = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geq 0, \\ ((-\gamma) * B, (-\gamma) * A), & \text{if } \gamma < 0. \end{cases} \tag{6.7}$$

Note that the q-linear multiplication by scalar can be expressed by the linear one as

$$\alpha * c = \alpha \cdot c_{\sigma(\alpha)}, \quad \alpha \in \mathbb{R}, \ c \in \mathcal{G}. \tag{6.8}$$

Relation (6.8) suggests that one can try to replace the q-linear multiplication by the two operations — linear multiplication and conjugation. This idea has a far reaching consequences, in particular, one may reformulate the theory of q-linear spaces starting from such a point of view. We shall call this approach a “linear form”.

Linear form. Note that $\alpha * c \in \mathcal{G}$ implies $\neg c \in \mathcal{G}$ and the assumptions $-a \in \mathcal{G}$ and $\neg a \in \mathcal{G}$ imply $a_{-} \in \mathcal{G}$; hence every expression in a q-linear space using “+”, “$*$”, “$\cdot$”, “$-$”, “$\neg$” and “$_-$” can be rewritten by means of the (linear) operations “+”, “$\cdot$”, “$-$”and additionally conjugation “$_-$”, without using “$*$” (and, in particular, “$\neg$”). The idea is to consider a q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ as a linear space involving an additional operation “conjugation”, that is a system of the form $(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$; in this system “$*$” can be defined as a composition of “$\cdot$” and “$_-$” using (6.8). The next result in certain sense inverses Theorem 6.1:

Theorem 6.2. *Let $(\mathcal{G}, +, \mathbb{R}, \cdot)$ be a linear space. Assume that i is an involution (dual automorphism) in $\mathcal{G}$, that is a function $i : \mathcal{G} \longrightarrow \mathcal{G}$, such that for $a, b \in \mathcal{G}$: $i(a + b) = i(a) + i(b)$, $i(a) = 0$ iff $a = 0$, $i(\alpha \cdot c) = \alpha \cdot i(c)$, $i(i(a)) = a$. Then $(\mathcal{G}, +, \mathbb{R}, *)$, where $* : \mathcal{G} \times \mathbb{R} \longrightarrow \mathcal{G}$ is given by*

$$\alpha * c = \begin{cases} \alpha \cdot c, & \text{if } \alpha \geq 0, \\ \alpha \cdot i(c), & \text{if } \alpha < 0, \end{cases} \tag{6.9}$$

is a q-linear space.

Proof. We are given that $(\mathcal{G}, +)$ is an abelian group and “$\cdot$” satisfies the relations (6.1)–(6.4). We shall prove that relations (3.4)–(3.7) hold true with “$*$” defined by (6.9).

To prove (3.4) we consider various cases. Let $\alpha \geq 0$, $\beta \geq 0$. In this case (3.4) holds because $\alpha * c = \alpha \cdot c$ for all $c \in \mathcal{G}$. Let $\alpha \geq 0$, $\beta < 0$. In this case we have $\beta * c = \beta \cdot i(c)$ for all $c \in \mathcal{G}$, resp., using the properties of i, we obtain $\beta \cdot c = \beta * i(c)$ and $(\alpha\beta) \cdot c = (\alpha\beta) * i(c)$. Replacing in (6.1) we obtain $\alpha * (\beta * i(c)) = (\alpha\beta) * i(c)$, which implies (3.4). The case $\alpha < 0$, $\beta \geq 0$ is proved analogously, the case $\alpha < 0$, $\beta < 0$ requires the property $i(i(a) = a$. Property (3.4) is proved, properties (3.5)–(3.7) are proved similarly. $\square$

Setting $\alpha = -1$ in (6.9) we obtain $(-1) * c = (-1) \cdot i(c) = -i(c)$, or, using the familiar concept negation $\neg c = (-1) * c$, we see that i is the familiar operation conjugation (dual): $i(c) = -(\neg c) = c_{-}$.

Theorem 6.2 can be expressed by saying that a q-linear space is a linear space endowed with an involution (conjugation). In the special case, when this involution coincides with the involutions in the associated linear space (identity and opposite), then the q-linear space is axial (this may also happen on some subspace of $\mathcal{G}$). The q-linear space $(\mathcal{G}, +, \mathbb{R}, \cdot, i)$ is nonaxial, if i is neither identity nor opposite. A nonaxial q-linear space has four involutions: identity, opposite, conjugation (i) and negation ($-i$).

In the case when the q-linear system $\mathcal{G}$ is induced by a quasilinear system $\mathcal{Q}$, $\mathcal{G} = \operatorname{dis} \mathcal{Q}$, the role of the involution i is played by the operator $i(A, B) = -(-1) * (A, B) = ((-1) * B, (-1) * A) = (\neg B, \neg A)$.

As before we shall write “$_-$” instead of “i” to denote conjugation. Theorems 6.1 and 6.2 show that we can consider a q-linear space as a linear system endowed with conjugation:

$(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$; then "$*$" is defined by (6.8). Using this linear form we can reformulate the theory of q-linear spaces. For instance the subspace lemma (Lemma 5.1) obtains the form:

Lemma 6.3. *A system $(\mathcal{H}, +, \mathbb{R}, \cdot, _-)$ is a subspace of the q-linear space $(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$, if and only if $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}$ is closed under the operations "$+$", "$\cdot$", "$_-$", i. e.:*

(i) $a + b \in \mathcal{H}$ *for all* $a, b \in \mathcal{H}$;
(ii) $\alpha \cdot c \in \mathcal{H}$ *for all* $\alpha \in \mathbb{R}$, $c \in \mathcal{H}$;
(iii) $a_- \in \mathcal{H}$ *for all* $a \in \mathcal{H}$.

Using Lemma 6.3, the linear combination (5.1) of k given elements $c^{(1)}, ..., c^{(k)} \in \mathcal{G}$ can be written in linear form as

$$\begin{aligned} c &= \sum_{i=1}^{k} (\alpha_i \cdot c^{(i)} + \beta_i \cdot c_-^{(i)}) \\ &= \alpha_1 \cdot c^{(1)} + \beta_1 \cdot c_-^{(1)} + \alpha_2 \cdot c^{(2)} + \beta_2 \cdot c_-^{(2)} + ... + \alpha_k \cdot c^{(k)} + \beta_k \cdot c_-^{(k)}, \end{aligned} \tag{6.10}$$

wherein $\alpha_i, \beta_i \in \mathbb{R}$.

The system $(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$ is the associated linear system extended by conjugation, to be briefly called *extended linear system.* We may tacitly assume that the extended linear system also has the operation "$*$" defined by (6.8). Then every linear combination (5.1) of given elements of $\mathcal{G}$ can be presented in the form (6.10), and vice versa. Hence we may say that the q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ is *equivalent* to the extended linear space $(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$. This equivalency follows from the transition formulae (6.5), (6.8) and the equality: $a_- = \neg(-a)$. Note that the associated linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ (without conjugation!), is generally not equivalent to the q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$. Indeed, assume that the elements $c^{(1)}, ..., c^{(k)} \in \mathcal{G}$ are linearly independent (as elements of $(\mathcal{G}, +, \mathbb{R}, *)$) and consider the q-linear space $(\mathcal{H}, +, \mathbb{R}, *)$ spanned over these elements: $\mathcal{H} = \mathrm{span}\{c^{(1)}, ..., c^{(k)}\}$. The elements $c^{(1)}, ..., c^{(k)}$ are also linearly independent as elements of $(\mathcal{G}, +, \mathbb{R}, \cdot)$. Consider now the *linear* space $(\mathcal{H}', +, \mathbb{R}, \cdot)$ spanned (in the sense of linear spaces) over these elements, $\mathcal{H}' = \mathrm{span}\{c^{(1)}, ..., c^{(k)}\}$, which is a subspace of the associated linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$. The q-linear subspace $\mathcal{H}$ is larger than the linear subspace $\mathcal{H}'$ — the first one contains linear combinations of the elements $c_-^{(1)}, ..., c_-^{(k)}$ and the second one does not. Indeed, consider the linear subspace $(\mathcal{H}'', +, \mathbb{R}, \cdot)$ spanned over the conjugate basic elements: $\mathcal{H}'' = \mathrm{span}\{c_-^{(1)}, ..., c_-^{(k)}\}$. We have $\mathcal{H} = \mathcal{H}' \cup \mathcal{H}''$. Note that $\mathcal{H}'$ and $\mathcal{H}''$ are both linear subspaces of the linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ but none of them is a q-linear subspace of the q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$.

We may reformulate any expression (problem) from a q-linear system into linear form, that is in terms of linear multiplication and conjugation. As an example, Proposition 4.8 in linear form states: "*Let $c \in \mathcal{G} \setminus (\mathcal{G}_D \cup \mathcal{G}_S)$. Then $\alpha \cdot c + \beta \cdot c_- = 0$ implies* $\alpha = 0$, $\beta = 0$."

Rules for calculation in linear form. The q-linear system $(\mathcal{G}, +, \mathbb{R}, \cdot, _-)$ comprises the linear system $(\mathcal{G}, +, \mathbb{R}, \cdot)$, where familiar rules for calculations hold. The rules involving conjugation, like $(a_-)_- = a$, $(a + b)_- = a_- + b_-$, and more generally $(a_\lambda)_\mu = a_{\lambda\mu}$, $(a_\mu + b_\nu)_\lambda = a_{\mu\lambda} + b_{\nu\lambda}$, are known from Section 3. We also have:

(1) $(\alpha \cdot c)_- = \alpha \cdot c_-$;
(2) $\delta(a_-, b_-) = \delta(a, b)$, $\| a_- \| = \| a \|$;
(3) $a \subset b \Longleftrightarrow \gamma \cdot a \subset \gamma \cdot b$, if $\gamma \geq 0$;
(4) $a \subset b \Longleftrightarrow \gamma \cdot a \supset \gamma \cdot b$, if $\gamma < 0$;
(5) If $c \in \mathcal{G}_S$, then $(\alpha \cdot c)_- = (-\alpha) \cdot c$.

The linear multiplication is useful for the interpretation of properties of the q-linear spaces in terms of known linear concepts. However, we should keep in mind that the linear multiplication does not correspond to the natural Minkowski multiplication and its use is practically limited to the case of nonnegative scalars from the above properties related to inclusion. Indeed, linear multiplication is not inclusion isotone; neither is inclusion antitone.

As we already mentioned we may reformulate any q-linear expression (problem) into a linear form, that is as linear expression (problem) using additionally conjugation. For example equation (4.7) in linear form becomes $\alpha \cdot y + \beta \cdot y_- = d$. The following is a reformulation of Proposition 4.4: *Let $\alpha, \beta \in \mathbb{R}$, $d \in \mathcal{G}$. If $\gamma = \alpha^2 - \beta^2 \neq 0$, then the equation $\alpha \cdot x + \beta \cdot x_- = d$ has a unique solution $x = \gamma^{-1} \cdot (\alpha \cdot d - \beta \cdot d_-)$. If $\beta = \alpha \neq 0$, then $x + x_- = \alpha^{-1} \cdot d$. If $\beta = -\alpha \neq 0$, then $x - x_- = \alpha^{-1} \cdot d$.*

Considering the property related to symmetric elements, it is easy to see that the linear space $(\mathcal{G}_S, +, \mathbb{R}, \cdot)$ is equivalent to the q-linear space $(\mathcal{G}_S, +, \mathbb{R}, *)$ under the relations $\gamma \cdot a = |\gamma| * a_{\sigma(\gamma)} = \sigma(\gamma)(|\gamma| * a)$ and $\gamma * a_\lambda = \lambda|\gamma| \cdot a = |\gamma| \cdot a_\lambda$, wherein $a \in \mathcal{G}$, $\gamma \in \mathbb{R}$, $\lambda \in \{+, -\}$. Indeed, we know that the q-linear spaces $(\mathcal{G}_S, +, \mathbb{R}, \cdot, _-)$ and $(\mathcal{G}_S, +, \mathbb{R}, *)$ are equivalent. Note that $(\mathcal{G}_S, +, \mathbb{R}, \cdot, _-)$ and $(\mathcal{G}_S, +, \mathbb{R}, \cdot)$ are also equivalent due to $(-1) \cdot a = -a = a_-$ for $a \in \mathcal{G}_S$. Let $a \in \mathcal{G}_S$. Using $\neg a = a$, that is $(-1) * a = a = 1 * a$, we have $\gamma * a = |\gamma| * a$. Using (6.5) we have $\gamma \cdot a = \gamma * a_{\sigma(\gamma)} = |\gamma| * a_{\sigma(\gamma)}$. From $a_- = -a$ we have: $a_\sigma = \sigma a = \{a, \text{ if } \sigma = +; -a, \text{ if } \sigma = -\}$, hence $\gamma \cdot a = |\gamma| * a_{\sigma(\gamma)} = \sigma(\gamma)(|\gamma| * a)$. Also: $\gamma * c = (\gamma \cdot c)_{\sigma(\gamma)} = \sigma(\gamma)\gamma \cdot c = |\gamma| \cdot c$, resp. $\gamma * c_- = -|\gamma| \cdot c$, or, summarizing both cases: $\gamma * c_\lambda = \lambda|\gamma| \cdot c$. Therefore $(\mathcal{G}_S, +, \mathbb{R}, \cdot)$ is equivalent to $(\mathcal{G}_S, +, \mathbb{R}, *)$ under the transition formulae: $\gamma \cdot c = \sigma(\gamma)(|\gamma| * c)$ and $\gamma * c_\lambda = \lambda|\gamma| \cdot c$.

7. Examples of Q-linear Spaces

Let us briefly summarize some special cases of quasilinear, resp. q-linear spaces.

1. A basic example is the space $\mathcal{K}$ of all convex bodies in $\mathbb{E}^m$ discussed in Section 2. Due to (2.2)–(2.5) the system of convex bodies $(\mathcal{K}, +)$, is an a. c. monoid with a subgroup $\mathcal{K}_D \cong \mathbb{E}^m$ of invertible elements and a neutral element "0". The system $(\mathcal{K}, +, \mathbb{R}, *)$ with "$*$" defined by (2.7) is a (proper) quasilinear space; the natural Hausdorff metric is induced by the inclusion relation "$\subset$". Denote by $\mathcal{L} = \operatorname{dis} \mathcal{K} = (\mathcal{K} \times \mathcal{K})/\rho$ the set of extended convex bodies, i. e. factorized pairs (differences) of convex bodies as defined in Section 3. Using the extension method we obtain the q-linear space $(\mathcal{L}, +, \mathbb{R}, *)$ involving conjugation "$_-$". The partial order inclusion induces a norm, we thus arrive to the normed space $(\mathcal{L}, +, \mathbb{R}, *, \subset, \| \cdot \|)$. The latter incorporates the associated normed linear space $(\mathcal{L}, +, \mathbb{R}, \cdot, \| \cdot \|)$, considered in [13].

2. Another example of a quasilinear, resp. q-linear space, can be constructed from the set $\mathcal{K}_S$ of all symmetric bodies in $\mathbb{E}^m$. The system $(\mathcal{K}_S, +)$ is an a. c. monoid; the subgroup of invertible (and distributive) elements of $\mathcal{K}_S$ is the trivial group $\{0\}$. The quasilinear system of symmetric elements is $(\mathcal{K}_S, +, \mathbb{R}, *)$. Due to $\neg B = (-1)*B = 1*B = B$ we have $\alpha*B = |\alpha|*B$ for $B \in \mathcal{K}_S$. Using (2.7), this implies $\alpha*B = \{\alpha x | x \in B\} = \{|\alpha|x \mid x \in B\}$. By Theorem 3.1 the symmetric quasilinear space $(\mathcal{K}_S, +, \mathbb{R}, *)$ can be embedded in a q-linear space, $(\mathcal{L}_S, +, \mathbb{R}, *)$, with $\mathcal{L}_S = \operatorname{dis} \mathcal{K}_S = (\mathcal{K}_S \times \mathcal{K}_S)/\rho$. For $a \in \mathcal{L}_S$ we have $\neg a = a$, and hence $a_- = -a$. Due to $\neg a = a$, that is $(-1) * a = a$, we have that $\gamma * a = |\gamma| * a$, $a \in \mathcal{L}_S$. Let $(\mathcal{L}_S, +, \mathbb{R}, \cdot)$ be the associated linear system with linear multiplication by scalar. We have $\alpha \cdot c = \alpha * c_{\sigma(\alpha)}$, cf. (6.5). Using $a_\sigma = \sigma a = \{a, \text{ if } \sigma = +; -a, \text{ if } \sigma = -\}$, we have $\alpha \cdot c = \alpha * c_{\sigma(\alpha)} = \sigma(\alpha)(|\alpha| * c)$. Also, $\alpha * c = (\alpha \cdot c)_{\sigma(\alpha)} = \sigma(\alpha)\alpha \cdot c = |\alpha|c$, resp. $\alpha * c_- = -|\alpha|c$. The above formulae can be used to rewrite any q-linear expression in $\mathcal{L}_S$ in linear form. For example the expression $\alpha * y + \beta * y_-$ in $\mathcal{L}_S$ obtains the linear form $|\alpha| \cdot y + (-|\beta|) \cdot y = (|\alpha| - |\beta|) \cdot y$.

3. Consider the quasilinear system $(\mathbb{R}^+, +, \mathbb{R}, *)$, where $(\mathbb{R}^+, +)$ is the monoid of non-negative reals. Embedding the monoid into a group we arrive to a q-linear system $(\mathbb{R}, +, \mathbb{R}, *) = \operatorname{dis}(\mathbb{R}^+, +, \mathbb{R}, *)$, where $\alpha * c = |\alpha| \cdot c$. Since the set of scalars and the set of elements of the system coincide (both are $\mathbb{R}$) the q-linear multiplication is similar to the usual multiplication "$\cdot$" of real numbers with a difference in computing the signs. Indeed, "$*$" is noncommutative: negative by negative results in negative, negative by positive is positive, and positive by negative is negative. This system is isomorphic to the one-dimensional space of (differences of) symmetric bodies. In the k-dimensional case we obtain the q-linear space of k-tuples $(\mathbb{R}^k, +, \mathbb{R}, *)$, cosidered in Section 5.

4. Let $\mathcal{Q}$ be the monoid of all one-dimensional intervals $\mathcal{Q} = \mathcal{K}(\mathbb{E}^1)$, usually denoted $\mathcal{Q} = I(\mathbb{R})$. The system $(I(\mathbb{R}), +, \mathbb{R}, *)$ is quasilinear. The space $(\mathcal{D}, +, \mathbb{R}, *)$, $\mathcal{D} = \operatorname{dis} I(\mathbb{R})$ is the induced q-linear space of extended (other terms used in the literature are: generalized, directed, modal etc.) intervals [5], [8], [10]. Using end-points, we write $[a, b] \in \mathcal{D}$ with $a, b \in \mathbb{R}$; using center/radius, we write $(a', a'') \in \mathcal{D}$ with $a', a'' \in \mathbb{R}$. The presentations of the operators opposite, negation and dualization, both in end-point and center-radius form are as follows:

$$\begin{aligned} -[a^-, a^+] &= [-a^-, -a^+], & -(a', a'') &= (-a', -a''); \\ \neg[a^-, a^+] &= [-a^+, -a^-], & \neg(a', a'') &= (-a', a''); \\ [a^-, a^+]_- &= [a^+, a^-], & (a', a'')_- &= (a', -a''). \end{aligned}$$

Generalizations for the many-dimensional case are obtained in a straightforward way.

In the next section we consider one more example of a q-linear space often used in convex analysis.

8. Spaces of Support Functions

The quasilinear space of sublinear (support) functions. Denote by $\mathcal{K}^*$ the set of all Lipschitz continuous sublinear functions; recall that a function $h : \mathbb{E}^m \longrightarrow \mathbb{R}$ is *sublinear*, if it is positively homogeneous, i. e. $h(\alpha x) = \alpha h(x)$, for $\alpha \geq 0, x \in \mathbb{E}^m$, and subadditive, i. e. $h(x + y) \leq h(x) + h(y)$ for $x, y \in \mathbb{E}^m$. To each $A \in \mathcal{K}$ corresponds a unique $h \in \mathcal{K}^*$ which is the *support function* of A, defined by $h(A, x) = \max_{\alpha \in A}\langle \alpha, x\rangle$, where $\langle \cdot, \cdot \rangle$ is the dot product in $\mathbb{E}^m$; conversely, every function of $\mathcal{K}^*$ defines a unique convex body. To establish an isomorphism between $\mathcal{K}$ and $\mathcal{K}^*$ note that

$$\begin{aligned} h(A+B,x) &= h(A,x)+h(B,x), && (8.1)\\ h(\gamma * A,x) &= |\gamma|\cdot h(A,\sigma(\gamma)x), && (8.2)\end{aligned}$$

where "$\cdot$" is the usual linear multiplication of a function by real scalar.

Relation (8.2) justifies the introduction of a special *q-linear* multiplication "$*$" of a function by real scalar, defined by:

$$\gamma * h(A,x) = |\gamma|\cdot h(A,\sigma(\gamma)x) = \begin{cases} \gamma\cdot h(A,x), & \text{if } \gamma \ge 0,\\ |\gamma|\cdot h(A,-x), & \text{if } \gamma < 0. \end{cases} \tag{8.3}$$

Using (8.3) we have: $(\mathcal{K},+,\mathbb{R},*) \cong (\mathcal{K}^*,+,\mathbb{R},*)$. Note that the support function of a single point set in $\mathbb{E}^m$ is linear; the linear functions $h:\mathbb{E}^m \longrightarrow \mathbb{R}$ are the invertable (and distributive w. r. t. "$*$") elements of $\mathcal{K}^*$. The set of all linear functions is $\mathcal{K}^*_D$. The set $\mathcal{K}^*$ is closed with respect to addition and q-linear multiplication by scalar (note that $\mathcal{K}^*$ is closed under the usual linear multiplication by scalar).

In particular, the q-linear multiplication "$*$" of a function by $\gamma=-1$, called *negation*, gives:

$$(-1)*h(A,x) = h(\neg A,x) = h(A,-x),$$

that is negation changes the sign of the argument of the function.

As in $\mathcal{K}$ we shall briefly denote: $\neg h(A,x) = (-1)*h(A,x) = h(A,-x)$. Then $h(A\neg A,x)$ $= h(A,x)+h(A,-x)$ is symmetric with respect to the y axis.

Due to $(\mathcal{K},+,\mathbb{R},*) \cong (\mathcal{K}^*,+,\mathbb{R},*)$ the system of sublinear functions is quasilinear and we may use the rules for calculation in quasilinear spaces [8], [9]. We next extend the space $(\mathcal{K}^*,+,\mathbb{R},*)$ up to a q-linear space following the scheme outlined in Sections 3–5.

The q-linear space of differences of support functions. Recall that the a. c. monoid $(\mathcal{K},+)$ of convex bodies is extended up to an abelian group $(\mathcal{L},+)$ consisting of factorized pairs of convex bodies (extended convex bodies, differences of convex bodies) $(A,B)\in\mathcal{L}$, with $A,B\in\mathcal{K}$. Due to $(\mathcal{K}^*,+)\cong(\mathcal{K},+)$, the set $\mathcal{K}^*$ is an a. c. monoid, and we may apply the extension method to obtain an abelian group $(\mathcal{L}^*,+) = \mathrm{dis}(\mathcal{K}^*,+)$ of new elements — *extended support functions*, or, *differences of support functions*. To each pair (A,B) there corresponds a unique function from $\mathcal{L}^*$, further denoted $h((A,B),x)$. Every function from $\mathcal{L}^*$ is indeed a difference of two usual support functions. To see this, note that the equivalence relation "ρ" between pairs of convex bodies

$$(A,B)\rho(C,D) \Longleftrightarrow A+D=B+C$$

induces a corresponding equivalence relation "$\sim$" between functions from $\mathcal{L}^*$:

$$h((A,B),x)\sim h((C,D),x) \Longleftrightarrow h(A+D,x)=h(B+C,x),$$

or, using (8.1),

$$h((A,B),x)\sim h((C,D),x) \Longleftrightarrow h(A,x)-h(B,x)=h(C,x)-h(D,x),$$

showing that any function $h((A,B),x) \in \mathcal{L}^*$ can be presented as a difference of two (usual) support functions:

$$h((A,B),x) = h(A,x) - h(B,x). \tag{8.4}$$

We may consider (8.4) as a definition of $h((A,B),x)$. In particular we obtain: $h((A,0),x) = h(A,x)$ and $h((0,B),x) = -h(B,x)$.

The functions from $\mathcal{L}^*$ are positively homogeneous, but not necessarily subadditive, and hence not necessarily sublinear. Some characterization of the class $\mathcal{L}^*$ is given in [18], see also [16], [19], [20].

Addition of functions from $\mathcal{L}^*$ is the usual addition; indeed for $(A,B),(C,D) \in \mathcal{L} = \operatorname{dis}\mathcal{K}$ we have

$$\begin{aligned} h((A,B)+(C,D),x) &= h((A+C,B+D),x) = h(A+C,x) - h(B+D,x) \\ &= h(A,x) - h(B,x) + h(C,x) - h(D,x) = h((A,B),x) + h((C,D),x). \end{aligned}$$

Since the opposite of $(A,B) \in \mathcal{L}$ is (B,A), the opposite of the function $h((A,B),x) = h(A,x) - h(B,x) \in \mathcal{L}^*$ is the function $h(-(A,B),x) = h((B,A),x) = h(B,x) - h(A,x) = -(h(A,x) - h(B,x)) \in \mathcal{L}^*$, that is the function $-h((A,B),x)$. We have, as expected,

$$\begin{aligned} h((A,B),x) = h((A,0)+(0,B),x) = h((A,0),x) + h((0,B),x) = \\ = h((A,0),x) - h((B,0),x). \end{aligned}$$

Multiplication by scalar in $\mathcal{K}$: $\gamma * (A,B) = (\gamma * A, \gamma * B)$ induces a corresponding *q-linear* multiplication by scalar in $\mathcal{L}^*$:

$$\begin{aligned} \gamma * h((A,B),x) &= \gamma * (h(A,x) - h(B,x)) \\ &= |\gamma| \cdot (h(A,\sigma(\gamma)x) - h(B,\sigma(\gamma)x)) = |\gamma| \cdot h((A,B),\sigma(\gamma)x). \end{aligned}$$

In particular, we have

$$\begin{aligned} (-1) * h((A,B),x) = (-1) * (h(A,x) - h(B,x)) = h(A,-x) - h(B,-x) = \\ = h((A,B),-x). \end{aligned}$$

Denoting (as before) the extended convex bodies by lower case letters, e. g. $a = (A', A'')$, we may write for $a, b \in \mathcal{L}, h(a,x), h(b,x) \in \mathcal{L}^*$, $\gamma \in \mathbb{R}$:

$$\begin{aligned} h(a+b,x) &= h(a,x) + h(b,x), \\ h(-a,x) &= -h(a,x), \\ h(\gamma * a,x) &= \gamma * h(a,x) = |\gamma| \cdot h(a,\sigma(\gamma)x), \\ h(\neg a,x) &= \neg h(a,x) = h(a,-x), \\ h(a_-,x) &= -h(a,-x), \\ h(\gamma * a_{\sigma(\gamma)},x) &= h(\gamma \cdot a,x) = \gamma \cdot h(a,x). \end{aligned}$$

This exhausts the list of operations in the q-linear space $(\mathcal{L}^*, +, \mathbb{R}, *)$, $\mathcal{L}^* = \operatorname{dis}\mathcal{K}^* = (\mathcal{K}^* \times \mathcal{K}^*)/\sim$. Using the general results from Section 6 we see that the q-linear space

$(\mathcal{L}^*, +, \mathbb{R}, *)$ is equivalent to the extended linear space $(\mathcal{L}^*, +, \mathbb{R}, \cdot, _-)$, where conjugation is given by $h_-(a,x) = h(a_-, x) = -h(a, -x)$. Note that the space $\mathcal{L}^*$ is extended linear, but not linear; indeed in a linear functional space no transformation of the form $Tf(x) = f(-x)$ is assumed. Solving problems involving conjugation needs methods characteristic for q-linear spaces. As a simple example consider the reduction of an expression of the form $\alpha \cdot h(c,x) + \beta \cdot h(c, -x)$. A reduction is not possible in general. Let us solve the equation

$$\alpha \cdot h(x) + \beta \cdot h(-x) = g(x), \tag{8.5}$$

with respect to the unknown function h (g is given function), under the assumption $\alpha^2 - \beta^2 \neq 0$. The solution is: $h(x) = (\alpha^2 - \beta^2)^{-1} \cdot (\alpha \cdot g(x) - \beta \cdot g(-x))$.

Let us characterize functions from $\mathcal{L}^*$ corresponding to axial (symmetric or distributive) extended convex bodies.

A *symmetric* function from $\mathcal{L}^*$ satisfies $h(a,x) = h(\neg a, x)$, that is $h(a,x) = h(a,-x)$. This means symmetry of the graphics of the function with respect to the y-axis, that is the symmetric functions from $\mathcal{L}^*$ are even. We denote $\mathcal{L}^*_S = \{h \in \mathcal{L}^* \mid h(x) = h(-x)\}$.

A "distributive" (w. r. t. "$*$") function from $\mathcal{L}^*$ satisfies $h(a,x) = h(a_-, x)$, that is $h(a,x) = -h(a,-x)$. This means symmetry of the graphics of the function with respect to the origin, that is odd functions. The set of all odd functions from $\mathcal{L}^*$ is $\mathcal{L}^*_D = \{h \in \mathcal{L}^* \mid h(x) = h(-x)\}$.

Linear combinations, linear dependence, basis and dimension in $\mathcal{L}^*$ follow from the general definitions. Using the isomorphism $(\mathcal{L}^*, +, \mathbb{R}, *) \cong (\mathcal{L}, +, \mathbb{R}, *)$, resp. the direct sum $\mathcal{L}^* = \mathcal{L}^*_S + \mathcal{L}^*_D$, and Proposition 5.12 we can formulate the following

Proposition 8.1. *Every function from $\mathcal{L}^*$ can be decomposed in a unique way as a sum of an even and an odd functions from $\mathcal{L}^*$.*

Order, norm. Inclusion in $\mathcal{K}$ induces the usual partial order in $\mathcal{K}^*$.

Lemma 8.2. *Let $B \in \mathcal{K}$. Then $b \in B \Longleftrightarrow \langle b, t\rangle \leq h(B,t)$, for all $t \in \mathbb{E}^m$.*

Proof. Lemma 8.2 is obvious under the observation that $b \in B$ is equivalent to $\langle b, t\rangle \leq h(B,t) = \max_{b \in B} \langle b, t\rangle$, $t \in \mathbb{E}^m$. □

Using Lemma 8.2 we obtain the following well-known result:

Proposition 8.3. *Let $A, B \in \mathcal{K}$. Then $B \subset A \Longleftrightarrow h(B,t) \leq h(A,t)$, $t \in \mathbb{E}^m$.*

Proof. (1) "$\Longrightarrow$": Let $B \subset A$, that is $b \in B \Longrightarrow b \in A$. From $b \in A$ it follows $\langle b, t\rangle \leq h(A,t)$, $t \in \mathbb{E}^m$. Taking "max" with respect to $b \in B$ we obtain: $h(B,t) \leq h(A,t)$, $t \in \mathbb{E}^m$.

(2) "$\Longleftarrow$": Let $h(B,t) \leq h(A,t)$. Assume that there exists $b \in B$, such that $b \notin A$, that is $\langle b, t\rangle > h(A,t)$ for some t, leading to contradiction. □

The quasilinear space $\mathcal{K}^*$ is normed by

$$\| h \| = \max_{x \in S^{n-1}} |h(x)|,$$

where S^{n-1} is the unit sphere in $\mathbb{E}^m$. For $A, B \in \mathcal{K}$ we have

$$\delta(A,B) = \| \ h(A,x) - h(B,x) \ \|, \ A, B \in \mathcal{K},$$

the above relation extends in $\mathcal{L} \cong \mathcal{L}^*$ as

$$\delta(a,b) = \| \ h(a,x) - h(b,x) \ \|, \ a, b \in \mathcal{L}.$$

Concluding remarks. In this work the algebraic properties of the convex bodies are studied by considering the abelian group of factorized (extended, differences of) convex bodies with a new multiplication by real scalar. Special quasilinear systems, called q-linear spaces, are introduced and some algebraic foundation is developed, which generalizes the theory of linear spaces. By introducing suitable symbolic notation we obtain a simple distributive-like relation with general validity (valid for all scalars, not just equally signed scalars as in the familiar relation). This opens the development of an algebra for convex bodies in $\mathbb{E}^m$ in the lines of the familiar linear algebra for vectors.

The q-linear multiplication by scalar (3.3) seems not have been used by now in the case of extended convex bodies; the only known to us source where this operation is briefly mentioned is [14]. Similar is the situation with inclusion (4.12) in a q-linear space — it has been studied by now only in the case of one-dimensional intervals [4].

Theorems 3.1 and 4.12 extend Rådström's embedding theorem (cf. Theorem 1 of [13]) in the following sense: a) no restriction for the signs of the scalar multipliers in the quasidistributive law is imposed (leading to embedding of cones in Rådström's formulation), b) the naturally induced q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ (not considered in [13]) incorporates the associated linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$, and c) a natural extension of the inclusion relation is given.

We show that a finite dimensional q-linear space is a direct sum of a linear subspace and a q-linear symmetric subspace isomorphic to $(\mathbb{R}^k, +, \mathbb{R}, *)$. This allows to decompose any algebraic problem in a q-linear space into linear problems. A theory for the solution of such problems can be developed in the lines of [10], where the interval case is studied. Also problems in q-linear spaces can be "approximated" by proper/improper problems; we can solve such problems as we do this in the space of (extended) one-dimensional intervals.

The relation between a q-linear space $(\mathcal{G}, +, \mathbb{R}, *)$ and the associated linear space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ is clarified in both directions by studying the role of conjugation in the relation between the linear and q-linear multiplication by scalar.

The application of the theory of q-linear spaces outlined in this work seems to be similar to the application of the well-known theory of linear spaces. Given a problem in a quasilinear space (of, say, convex bodies) it is necessary to reformulate this problem as a corresponding q-linear problem, that is, a problem in the q-linear space. After (possibly) solving the q-linear problem by means of the presented theory we have to come back to the original quasilinear system in order to interpret the results. Such interpretation may not be simple and should be a subject of further investigations. A step in this direction is done in [1]. As in interval analysis, one may expect that proper and improper elements will play an important role for the interpretation of the results. Therefore outward/inward (optimal) approximations by such elements, as mentioned above, seem to be practically useful.

The theory of support functions has been developed to enable calculations with convex bodies whenever such calculations are problematic (e. g. when Minkowski difference is not defined we can pass to differences between support functions). The equivalency between the q-linear space of convex bodies and the space of extended support functions shows that we can calculate with extended convex bodies in the same way as with extended support functions. If our problem does not involve inclusions, then it seems easier to work using the linear form (as done in the case of support functions). However, it does not make much sense to pass completely to linear multiplication, resp. linear spaces (as done in [13]). Indeed, ignoring conjugation corresponds to ignoring Minkowski multiplication in favor to an exclusive use of linear multiplication; however we know that Minkowski q-linear multiplication is the natural operation arising in problems related to convex bodies. Also, in the associated linear space inclusion is not isotone and we lose the possibility to handle easily this important relation. This is probably the reason why Rådström's normed linear spaces does not find much application. The theory of q-linear spaces can be applied to Brunn-Minkowski theory [16], such application will be considered elsewhere.

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