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# Summability of the Solutions to Nonlinear Parabolic Equations with Measure Data*

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In this paper, we give an improvement of the summability of solutions to nonlinear parabolic equations with measure data.

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## 1. Introduction

Let $\Omega$ be a bounded domain in $R^N$, $N \geq 2$. For $T > 0$, let us denote by $Q$ the cylinder $\Omega \times (0,T)$, and by $\Gamma$ the lateral surface $\partial\Omega \times (0,T)$. Let $p$ be a real number such that $p > 2 - 1/(N+1)$.

We consider the following nonlinear parabolic Cauchy-Dirichlet problem:

$$\begin{cases} \frac{\partial u}{\partial t} - div(a(x,t,u,Du)) = f & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Gamma, \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{P}$$

Here $f$ and $u_0$ belong to $M(Q)$ and $M(\Omega)$, the space of bounded Radon measures on $Q$ and $\Omega$ respectively. The function $a(x,t,s,\xi) : Q \times R \times R^N \to R^N$ is a *Carathéodory* function (i.e., it is continuous with respect to s and $\xi$ for almost every $(x,t) \in Q$, and measurable with respect to $(x,t)$ for every $s \in R$ and $\xi \in R^N$). We assume that there exist two real positive constants $\lambda_0, \alpha$ and a positive function $h \in L^{p'}(Q)$, such that for any $s \in R$, $\xi \in R^N$, $\eta \in R^N$, $\xi \neq \eta$ and for almost every $(x,t) \in Q$,

$$a(x,t,s,\xi)\xi \geq \lambda_0 |\xi|^p \tag{1.1}$$

$$|a(x,t,s,\xi)| \leq \alpha(h(x,t) + |s|^{p-1} + |\xi|^{p-1}) \tag{1.2}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)][\xi - \eta] > 0. \tag{1.3}$$

We look for weak solutions, i.e., for function $u$ such that $u \in L^1(0,T;W_0^{1,1}(\Omega))$, $a(x,t,u,Du) \in L^1(Q)$ and

$$-\int_Q u\frac{\partial \phi}{\partial t} dxdt + \int_Q a(x,t,u,Du)D\phi dxdt = \int_Q \phi df + \int_\Omega \phi(x,0)du_0(x), \tag{1.4}$$

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for every $\phi \in C^\infty(\bar{Q})$ which is zero in a neighborhood of $\Gamma \cup (\Omega \times \{T\})$.

We recall the following results(see [1], [2] or [3]).

**Theorem 1.1.** *Assume* (1.1)–(1.3) *hold,* $p > 2 - 1/(N+1)$, $f \in M(Q), u_0 \in M(\Omega)$. *Then there exists a solution* $u$ *of problem* (P) *such that*

$$u \in L^q(0,T; W_0^{1,q}(\Omega)), \quad \forall q < p - \frac{N}{N+1} = q_0.$$

**Remark 1.2.** We observe that $q_0 > 1$ if and only if $p > 2 - \frac{1}{N+1}$.

In order to obtain $u \in L^{q_0}(0,T; W_0^{1,q_0}(\Omega))$, a sufficient condition is given by the following theorem.

**Theorem 1.3.** *Assume* (1.1)–(1.3) *hold,* $p > 2 - \frac{1}{N+1}$, $f \in L^1(0,T; L^1 \log L^1(\Omega))$, $u_0 \in L^1 \log L^1(\Omega)$, *where* $L^1 \log L^1(\Omega)$ *is the Orlicz space generated by the function* $s \log(1+s)$. *Then there exists a solution* $u$ *of problem* (P) *such that*

$$u \in L^{q_0}(0,T; W_0^{1,q_0}(\Omega)).$$

**Remark 1.4.** The results given in Theorem 1.1 and Theorem 1.3 are only proved in the case $p > 2 - \frac{1}{N+1}$. However, in general, if $1 < p \le 2 - \frac{1}{N+1}$, $f$ and $u_0$ are only bounded measures(or even integrable functions), a weak solution in the sense of (1.4) doesn't exist. Recently, existence of so-called renormalized solutions has been proved for problems of type (P) and of the corresponding elliptic problem for every $p > 1$ (see [4],[5]).

This paper extends the analogous result of [6] to parabolic equations. Here we give the improvement of the summability of the solution $u$ to problem (P) only under the assumptions of Theorem 1.1.

## 2. Regularity results

In this section, we give an improvement of the summability of the solution $u$ to problem (P). We begin by recalling the well-known Gagliardo-Nirenberg embedding theorem (see [8]).

**Lemma 2.1.** *Let* $v$ *be a function in* $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$, *with* $q \ge 1$, $\rho \ge 1$. *Then there exists a positive constant* $C_1$ *depending on* $N$, $q$, *and* $\rho$, *such that*

$$\|v\|_{L^\gamma(\Omega)} \le C_1 \|Dv\|^\theta_{L^q(\Omega)} \|v\|^{1-\theta}_{L^\rho(\Omega)}, \tag{2.1}$$

*for every* $\theta$ *and* $\gamma$ *satisfying*

$$0 \le \theta \le 1,\ 1 \le \gamma < +\infty,\ \frac{1}{\gamma} = \theta(\frac{1}{q} - \frac{1}{N}) + \frac{1-\theta}{\rho}. \tag{2.2}$$

We state the main result of this paper.

**Theorem 2.2.** *The solution* $u$ *of problem* (P) *given by* Theorem 1.1 *is such that for every* $\beta > \frac{1}{p-1}$,

$$\frac{u}{[\log(2+|u|)]^\beta} \in L^{q_0}(0,T; W_0^{1,q_0}(\Omega)), \tag{2.3}$$

**Proof.** We follow the method of [1], [2] or [3]. Let $\{f_n\} \subset L^\infty(Q)$, $\{u_{0n}\} \subset L^\infty(\Omega)$, such that

$$f_n \longrightarrow f \text{ weak* in } M(Q), \tag{2.4}$$

$$u_{0n} \longrightarrow u_0 \text{ weak* in } M(\Omega). \tag{2.5}$$

Moreover, there exists a positive constant $C_2$ independent of $n$, such that for every $n$

$$\|f_n\|_{L^1(Q)} \le C_2, \tag{2.6}$$

$$\|u_{0n}\|_{L^1(\Omega)} \le C_2. \tag{2.7}$$

We consider the approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - div(a(x,t,u_n,Du_n)) = f_n & \text{in } Q, \\ u_n(x,t) = 0 & \text{on } \Gamma, \\ u_n(x,0) = u_{0n} & \text{in } \Omega. \end{cases} \tag{P$_n$}$$

For any given positive integer $n$, by [9], problem $(\mathrm{P}_n)$ admits a solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$ and satisfies

$$\langle u_{nt}, \psi\rangle + \int_Q a(x,t,u_n,Du_n)D\psi dxdt = \langle f_n, \psi\rangle, \quad \forall\psi \in L^p(0,T;W_0^{1,p}(\Omega)). \tag{2.8}$$

For any given constant $k > 0$, we define the cut function $T_k : R \to R$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\,\mathrm{sign}(s) & \text{if } |s| > k. \end{cases} \tag{2.9}$$

and let $\Theta_k(\sigma)$ be

$$\Theta_k(\sigma) = \int_0^\sigma T_k(s)ds. \tag{2.10}$$

For any given $\tau \in (0,T)$, we take $\psi = T_1(u_n)\chi_{(0,\tau)}$ in (2.8), where $\chi_{(0,\tau)}$ denotes the characteristic function of set $(0,\tau)$ in $[0,T]$, we easily obtain

$$\int_\Omega \Theta_1(u_n(x,\tau))dx - \int_\Omega \Theta_1(u_{0n})dx \le \int_Q |f_n||T_1(u_n)|dxdt \le \|f_n\|_{L^1(Q)}. \tag{2.11}$$

Since we have $|s| - 1/2 \le \Theta_1(s) \le |s|$ for every $s \in R$, we obtain for every $\tau \in (0,T)$,

$$\int_\Omega |u_n(x,\tau)|dx \le \|f_n\|_{L^1(Q)} + 1/2meas\Omega + \|u_{0n}\|_{L^1(\Omega)}. \tag{2.12}$$

By (2.6), (2.12) yields

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C_3, \tag{2.13}$$

where $C_3$ is a positive constant independent of $n$.

For any given $k \geq 0$, let us define the function $\phi_k(s) = \min\{(|s|-k)^+, 1\}\,\mathrm{sign}(s)$ and the set $B_k = \{(x,t) \in Q : k \leq |u_n(x,t)| < k+1\}$. We take $\psi_k = \phi_k(u_n)$ in (2.8), by (1.1), we obtain

$$\int_\Omega \psi_k(u_n(x,T))dx - \int_\Omega \psi_k(u_{0n})dx + \lambda_0 \int_{B_k} |Du_n|^p dxdt \leq \|f_n\|_{L^1(Q)}. \tag{2.14}$$

where we have defined $\psi_k(\sigma) = \int_0^\sigma \phi_k(s)ds$. Since $0 \leq \psi_k(\sigma) \leq |\sigma|$, by (2.6),

$$\int_{B_k} |Du_n|^p dxdt \leq C_4 \tag{2.15}$$

From now we denote by $C_4, C_5, \cdots$ various positive constants which only depend on the known data of the problem, but independent of $n$, $u_n$, $k$.

For any given $\beta > \frac{1}{p-1}$, let us define a function $\phi(s)$ as

$$\phi(s) = \frac{s}{[\log(2+|s|)]^\beta}, \quad \forall s \in R. \tag{2.16}$$

(2.13) and (2.16) imply that

$$\|\phi(u_n)\|_{L^\infty(0,T;L^1(\Omega))} \leq C_5. \tag{2.17}$$

By (2.15) and $\beta > \frac{1}{p-1}$, we obtain

$$\begin{aligned} &\int_Q \frac{|Du_n|^p}{(2+|u_n|)[\log(2+|u_n|)]^{(p-1)\beta}} dxdt \\ &= \sum_{k=0}^\infty \int_{B_k} \frac{|Du_n|^p}{(2+|u_n|)[\log(2+|u_n|)]^{(p-1)\beta}} dxdt \\ &\leq C_4 \sum_{k=0}^\infty \frac{1}{(2+k)[\log(2+k)]^{(p-1)\beta}} \leq C_6. \end{aligned} \tag{2.18}$$

Setting $g(u_n) = (2+|u_n|)[\log(2+|u_n|)]^{(p-1)\beta}$, Hölder's inequality and (2.18) yield

$$\begin{aligned} \int_Q |D\phi(u_n)|^{q_0} dxdt &\leq \int_Q |Du_n|^{q_0} |\phi'(u_n)|^{q_0} dxdt \\ &= \int_Q \frac{|Du_n|^{q_0}}{g(u_n)^{q_0/p}} |\phi'(u_n)|^{q_0} g(u_n)^{q_0/p} dxdt \\ &\leq (\int_Q \frac{|Du_n|^p}{g(u_n)} dxdt)^{\frac{q_0}{p}} (\int_Q |\phi'(u_n)|^{\frac{pq_0}{p-q_0}} g(u_n)^{\frac{q_0}{p-q_0}} dxdt)^{1-\frac{q_0}{p}} \\ &\leq C_6^{\frac{q_0}{p}} (\int_Q |\phi'(u_n)|^{\frac{pq_0}{p-q_0}} g(u_n)^{\frac{q_0}{p-q_0}} dxdt)^{1-\frac{q_0}{p}} \\ &\leq C_7 (\int_Q (1+|\phi(u_n)|)^{\frac{q_0}{p-q_0}} dxdt)^{1-\frac{q_0}{p}} \\ &\leq C_8 + C_9 (\int_0^T \|\phi(u_n(t))\|_{L^{\frac{q_0}{p-q_0}}(\Omega)}^{\frac{q_0}{p-q_0}} dt)^{1-\frac{q_0}{p}}. \end{aligned} \tag{2.19}$$

In fact, since $\forall s \in R$, $(2+|s|)\log(2+|s|) > 1+|s|$, then $|\phi'(s)| \le (1+\beta)[\log(2+|s|)]^{-\beta}$. Thus we have

$$|\phi'(s)|^p g(s) \le (1+\beta)^p[\log(2+|s|)]^{-\beta}(2+|s|) \le 3(1+\beta)^p[\log 2)]^{-\beta}(1+|\phi(s)|).$$

The above inequality yields the first of these two last inequalities in (2.19).

Taking $v = \phi(u_n(t))$ for a. e. $t \in (0,T)$, $q = q_0$, $\rho = 1$, $\gamma = \frac{q_0}{p-q_0}$ in Lemma 2.1, we have

$$\|\phi(u_n(t))\|_{L^{\frac{q_0}{p-q_0}}(\Omega)} \le C_1\|D\phi(u_n(t))\|^{\theta}_{L^{q_0}(\Omega)}\|\phi(u_n(t))\|^{\theta}_{L^1(\Omega)}, \text{ a. e. } t \in (0,T), \tag{2.20}$$

and

$$\frac{p-q_0}{q_0} = \theta(\frac{1}{q_0} - \frac{1}{N}) + 1 - \theta,\ 0 \le \theta \le 1. \tag{2.21}$$

(2.20), (2.17) and (2.19) imply that

$$\int_Q |D\phi(u_n)|^{q_0} dxdt \le C_8 + C_{10}(\int_0^T \|D\phi(u_n(t))\|^{\frac{\theta q_0}{p-q_0}}_{L^{q_0}(\Omega)} dt)^{1-\frac{q_0}{p}}. \tag{2.22}$$

If we choose $\frac{\theta}{p-q_0} = 1$, by (2.21), we must take $\theta = \frac{N}{N+1}$. Thus we have

$$\int_Q |D\phi(u_n)|^{q_0} dxdt \le C_8 + C_{10}(\int_Q |D\phi(u_n)|^{q_0} dxdt)^{1-\frac{q_0}{p}}. \tag{2.23}$$

Since $0 < 1 - \frac{q_0}{p} < 1$, (2.23) yields

$$\int_Q |D\psi(u_n)|^{q_0} dxdt \le C_{11}. \tag{2.24}$$

Poincaré's inequality and (2.24) imply that

$$\int_0^T \|\phi(u_n(t))\|^{q_0}_{W_0^{1,q_0}(\Omega)} dt \le C_{12}. \tag{2.25}$$

The method of [2] allow us to pass to the limit in the approximate equations and we can prove that there exists a solution $u$ to problem (P) such that (2.3) hold. This completes the proof of Theorem 2.2. □

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