Circumference of essentially 4-connected planar triangulations

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Abstract. A 3-connected graph $G$ is essentially 4-connected if, for any 3-cut $S \subseteq V(G)$ of $G$, at most one component of $G - S$ contains at least two vertices. We prove that every essentially 4-connected maximal planar graph $G$ on $n$ vertices contains a cycle of length at least $\frac{2}{3}(n + 4)$; moreover, this bound is sharp.

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1 Introduction and Preliminaries

We consider finite, simple, and undirected graphs. The circumference $\text{circ}(G)$ of a graph $G$ is the length of a longest cycle of $G$. A cycle $C$ of $G$ is an outer independent cycle of $G$ if the set $V(G) \setminus V(C)$ is independent. (Note that an outer independent cycle is sometimes called a dominating cycle \cite{3}, although this is in contrast to the more commonly used definition of a dominating subgraph $H$ of $G$, where $V(H)$ dominates $V(G)$ in the usual sense.) A set $S \subseteq V(G)$ ($S \subseteq E(G)$) is a $k$-cut (a $k$-edge-cut) of $G$ if $|S| = k$ and $G - S$ is disconnected. A 3-cut (a 3-edge-cut) $S$ of a 3-connected (3-edge-connected) graph $G$ is trivial if at most one component of $G - S$ contains at least two vertices and the graph $G$ is essentially 4-connected (essentially 4-edge-connected) if every 3-cut (3-edge-cut) of $G$ is trivial. A 3-edge-connected graph $G$ is cyclically 4-edge-connected if for every 3-edge-cut $S$ of $G$, at most one component of $G - S$ contains a cycle.

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It is well-known that for (3-connected) cubic graphs different from the triangular prism $K_3 \times K_2$ (which is essentially 4-connected only) these three notions coincide (see e.g. [6] and [16]). Obviously, the line graph $H = L(G)$ of a 3-connected graph $G$ is 4-connected if and only if $G$ is essentially 4-edge-connected. These two observations are reasons for the quite great interest in studying all these three concepts of connectedness of graphs intensively.

Zhan [17] proved that every 4-edge-connected graph has a Hamiltonian line graph. Broersma [3] conjectured that every essentially 4-edge-connected graph has a Hamiltonian line graph and showed that this is equivalent to the conjecture of Thomassen [14] stating that every 4-connected line graph is Hamiltonian (which is known to be equivalent to the conjecture by Matthews and Sumner [12] stating that every 4-connected claw-free graph is Hamiltonian, as shown by Ryjáček [13]). Among others, the subclass of essentially 4-edge-connected cubic graphs is interesting due to a conjecture of Fleischner and Jackson [6] stating that every essentially 4-edge-connected cubic graph has an outer independent cycle which is equivalent to the previous three conjectures.

Regarding to the existence of long cycles in essentially 4-connected graphs we mention the following

**Conjecture 1 (Bondy, see [8])** There exists a constant $c$, $0 < c < 1$, such that for every essentially 4-connected cubic graph on $n$ vertices, $\text{circ}(G) \geq cn$.

Note that the conjecture of Fleischner and Jackson implies Conjecture 1 with $c = \frac{2}{3}$. Bondy’s conjecture was later extended to all cyclically 4-edge-connected graphs (see [6]). Mácajová and Mazák [11] constructed essentially 4-connected cubic graphs on $n = 8m$ vertices with circumference $7m + 2$. We remark that the conjecture of Fleischner and Jackson and, therefore, also Bondy’s Conjecture with $c = \frac{3}{4}$ (this is the result of Grünbaum and Malkevitch [7]) are true for planar graphs, which can be seen easily by the forthcoming Lemma 1. Many results concerning the circumference of essentially 4-connected planar graphs $G$ can be found in the literature.

For the class of essentially 4-connected cubic planar graphs, Tutte [15] showed that it contains a non-Hamiltonian graph, Aldred, Bau, Holton, and McKay [1] found a smallest non-Hamiltonian graph on 42 vertices, and Van Cleemput and Zamfirescu [16] constructed a non-Hamiltonian graph on $n$ vertices for all even $n \geq 42$. As already mentioned, Grünbaum and Malkevitch [7] proved that $\text{circ}(G) \geq \frac{3}{4}n$ for any essentially 4-connected cubic planar graph $G$ on $n$ vertices and Zhang [18] (using the theory of Tutte paths) improved this lower bound on the circumference by 1. Recently, in [10], an infinite family of essentially 4-connected cubic planar graphs on $n$ vertices with circumference $\frac{3n}{4}n$ was constructed.

In [9], Jackson and Wormald extended the problem to find lower bounds on the circumference to the class of arbitrary essentially 4-connected planar graphs. Their result $\text{circ}(G) \geq \frac{2n+14}{3}$ was improved in [5] to $\text{circ}(G) \geq \frac{2}{3}(n + 2)$ for every essentially 4-connected planar graph $G$ on $n$ vertices. On the other side, there are infinitely many essentially 4-connected maximal planar graphs $G$ with $\text{circ}(G) = \frac{2}{3}(n + 4)$ ([9]). To see this, let $G'$ be a 4-connected maximal planar graph on $n' \geq 6$ vertices and let $G$ be obtained from $G'$ by inserting a new vertex into each face of $G'$ and connecting it with all three boundary vertices of that face. Then $G$ is an essentially 4-connected maximal planar graph on $n = 3n' - 4$ vertices and, since $G'$ is Hamiltonian, it is easy to see that $\text{circ}(G) = 2n' = \frac{2}{3}(n + 4)$. It is still open whether there is an essentially 4-connected planar graph $G$ that satisfies $\text{circ}(G) < \frac{2}{3}(n + 4)$. Indeed, we pose the following (to our knowledge so far unstated) Conjecture 2, which has been the driving force in that area for over a decade.

**Conjecture 2** For every essentially 4-connected planar graph on $n$ vertices, $\text{circ}(G) \geq \frac{2}{3}(n + 4)$. 
By the forthcoming Theorem 1, Conjecture 2 is shown to be true for essentially 4-connected maximal planar graphs.

We remark that \( G - S \) has exactly two components for every 3-connected planar graph \( G \) and every 3-cut \( S \) of \( G \). Thus, in this case, \( G \) is essentially 4-connected if and only if \( S \) forms the neighborhood of a vertex of degree 3 of \( G \) for every 3-cut \( S \) of \( G \). This property will be used frequently in the proof of Theorem 1.

A cycle \( C \) of \( G \) is a good cycle of \( G \) if \( C \) is outer independent and \( \deg_C(x) = 3 \) for all \( x \in V(G) \setminus V(C) \). An edge \( xy \) of a good cycle \( C \) is extendable if \( x \) and \( y \) have a common neighbor \( z \in V(G) \setminus V(C) \). In this case, the cycle \( C' \) of \( G \), obtained from \( C \) by replacing the edge \( xy \) with the path \( (x, z, y) \) is again good (and longer than \( C \)). The forthcoming Lemma 1 is an essential tool in the proof of Theorem 1 (an implicit proof for cubic essentially 4-connected planar graphs can be found in [7], the general case is proved in [4]).

**Lemma 1** Every essentially 4-connected planar graph on \( n \geq 11 \) vertices contains a good cycle.

**Theorem 1** For every essentially 4-connected maximal planar graph \( G \) on \( n \geq 8 \) vertices,

\[
\text{circ}(G) \geq \frac{2}{3}(n + 4).
\]

2 Proof of Theorem 1

Suppose \( n \geq 11 \), as for \( n \in \{8, 9, 10\} \), Theorem 1 follows from the fact that \( G \) is Hamiltonian ([2]). Using Lemma 1, let \( C = [v_1, v_2, \ldots, v_k] \) \((\text{indices of vertices of } C \text{ are taken modulo } k \text{ in the whole paper})\) be a longest good cycle of length \( k \) of \( G \) \((\text{i.e., circ}(G) \geq k)\) and let \( H = G[V(C)] \) be the graph obtained from \( G \) by removing all vertices of degree 3 which do not belong to \( C \). Obviously, \( H \) is maximal planar and \( C \) is a Hamiltonian cycle of \( H \). A face \( \varphi \) of \( H \) is an empty face of \( H \) if \( \varphi \) is also a face of \( G \), otherwise \( \varphi \) is a non-empty face of \( H \). Denote by \( F_e(H) \) the set of empty faces of \( H \) and let \( f_e(H) = |F_e(H)| \). Note that every face of \( G \) has at least two \((\text{of three})\) vertices on \( C \). The three neighbors of a vertex of \( V(G) \setminus V(C) \) induce a separating 3-cycle of \( G \) creating the boundary of a non-empty face of \( H \), which has no edge in common with \( C \) because otherwise such an edge would be an extendable edge of \( C \) in \( G \).

Let \( H_1 \) and \( H_2 \) be the spanning subgraphs of \( H \) consisting of the cycle \( C \) and of its chords lying in the interior and in the exterior of \( C \), respectively. Note that \( E(H_1) \cap E(H_2) = E(C) \) and \( H_1 \) and \( H_2 \) are maximal outerplanar graphs, both having \( k \)-gonal outer face and \( k - 2 \) triangular faces. Let \( T_i \) be the weak dual of \( H_i \), \( i \in \{1, 2\} \), which is the graph having all triangular faces of \( H_i \) as vertex set such that two vertices of \( T_i \) are adjacent if the triangular faces share an edge in \( H_i \). Obviously, \( T_i \) is a tree of maximum degree at most three.

A face \( \varphi \) of \( H \) is a \( j \)-face if exactly \( j \) of its three incident edges belong to \( E(C) \). Since \( n \geq 11 \), there is no 3-face in \( H \) and each face of \( H \) is a \( j \)-face with \( j \in \{0, 1, 2\} \). Denote by \( f_j(H_i) \) the number of empty \( j \)-faces of \( H_i \). Since \( C \) does not contain any extendable edge, the following claim is obvious.

**Claim 1** Each face of \( H \) incident with an edge of any longest good cycle \((\text{in particular, each } 1 \text{- or } 2 \text{-face})\) is empty.

An edge \( e \) of \( C \) incident with a \( j \)-face \( \varphi \) and an \( \ell \)-face \( \psi \), where \( j, \ell \in \{1, 2\} \), is a \((j, \ell)\)-edge. Let \( \varphi \) be a 2-face of \( H_i \). The sequence \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \), \( r \geq 2 \), is the \( \varphi \)-branch if \( \varphi_2, \ldots, \varphi_{r-1} \) are 1-faces of \( H_i \), \( \varphi_r \) is a 0-face of \( H_i \), and \( \varphi_j, \varphi_{j+1} (1 \leq j \leq r - 1) \) are adjacent \((\text{i.e. } B_\varphi \text{ is a}) \)
minimal path in \( T_i \) with end vertices of degree 1 and 3). The rim \( R(B_\varphi) \) of the \( \varphi \)-branch \( B_\varphi \) is the subgraph of \( C \) induced by all edges of \( C \) that are incident with an element of \( B_\varphi \). Hence, it is easy to see:

**Claim 2** The rim of a \( \varphi \)-branch \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) is a path of length \( r \).

**Claim 3** Let \( \varphi = [v_1, v_2, v_3] \) be a 2-face of \( H_i \), let \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r), \ r \geq 2, \) be the \( \varphi \)-branch of \( H_i \), and let \( v_0v_2 \in E(H_{3-i}) \). If

(a) \( R(B_\varphi) = (v_1, v_2, \ldots, v_{r+1}) \) is the rim of \( B_\varphi \) or

(b) \( R(B_\varphi) = (v_0, v_1, \ldots, v_r) \) is the rim of \( B_\varphi \) and \( v_{-1}v_2 \in E(H_{3-i}) \), or

(c) \( R(B_\varphi) = (v_{3-r}, \ldots, v_2, v_3) \) is the rim of \( B_\varphi \) and \( v_{-1}v_2 \in E(H_{3-i}) \),

then \( \varphi_r \) is empty.

![Fig. 1. A longest good cycle (cyan) sharing an edge with \( \varphi_r \).](image-url)
(Fig. 1(b)), or by the path \((v_{-1}, v_2, v_1, v_3, ..., v_{r-1}, v_0, v_r)\), for \(s \leq r - 1\) (Fig. 1(c)), is a longest good cycle of \(G\) and contains the edge \(v_0v_r\) incident with \(\varphi_r\), thus \(\varphi_r\) is empty (by Claim 1).

(c) If \(r \leq 3\), then \(\varphi_r\) is empty by (a) or (b). If \(r \geq 4\), then \(v_0v_3, v_{-1}v_3 \in E(H_i)\), thus \(\{v_{-1}, v_2, v_3\}\) is a non-trivial 3-cut, a contradiction. □

These tools will be used continuously in the following; we continue with the proof of Theorem 1. Hereby, we consider two cases. In the first case, both subgraphs \(H_1\) and \(H_2\) have some 0-faces. By using a customized discharging method, we distribute some weights from edges to faces to prove that sufficiently many faces are empty (each empty face will finally contain weight at most \(\frac{2}{3}\)). In the second case, there are only empty faces on one side of \(C\), so that all vertices not in \(C\) are located on the other side of \(C\). We have to prove that there are some additional empty faces on this side.

**CASE 1.** Let \(H_1\) and \(H_2\) both contain at least two 0-faces or one non-empty 0-face. For every edge \(e\) of \(C\) we define the weight \(w_0(e) = 1\). Obviously, \(\sum_{e \in E(C)} w_0(e) = |E(C)| = k\).

**First redistribution of weights.**

Each edge of \(C\) sends weight to both incident faces as follows

- **Rule R1.** A \((1,1)\)-edge sends \(\frac{1}{2}\) to both incident 1-faces.
- **Rule R2.** A \((1,2)\)-edge sends \(\frac{2}{3}\) to the incident 1-face and \(\frac{1}{3}\) to the incident 2-face.
- **Rule R3.** A \((2,2)\)-edge sends \(\frac{1}{3}\) to both incident 2-faces.

The edges of \(C\) completely redistribute their weights to incident 1- and 2-faces. For an empty face \(\varphi\), let \(w_1(\varphi)\) be the total weight obtained by \(\varphi\) (in first redistribution). Obviously, for an empty face \(\varphi\), it is

\[
w_1(\varphi) = \begin{cases} 
1, & \text{if } \varphi \text{ is a 2-face incident with two } (2,2)\text{-edges}, \\
\frac{5}{6}, & \text{if } \varphi \text{ is a 2-face incident with a } (1,2)\text{-edge and a } (2,2)\text{-edge}, \\
\frac{3}{6}, & \text{if } \varphi \text{ is a 2-face incident with two } (1,2)\text{-edges}, \\
\frac{2}{3}, & \text{if } \varphi \text{ is a 1-face incident with a } (1,2)\text{-edge}, \\
\frac{1}{2}, & \text{if } \varphi \text{ is a 1-face incident with a } (1,1)\text{-edge}, \\
0, & \text{if } \varphi \text{ is a 0-face}.
\end{cases}
\]

Moreover, \(\sum_{\varphi \in F_e(H)} w_1(\varphi) = |E(C)| = k\).

**Second redistribution of weights.**

The weight of 2-faces of \(H\) exceeding \(\frac{2}{3}\) will be redistributed to 1-faces and empty 0-faces of \(H\) by the following rules. Let \(\varphi\) be a 2-face of \(H_i\) with \(w_1(\varphi) > \frac{2}{3}\) (i.e. incident with at least one \((2,2)\)-edge) and let \(B_\varphi = (\varphi, \varphi_2, ..., \varphi_r)\), \(r \geq 2\), be the \(\varphi\)-branch. Moreover, let \(\alpha\) be a 2-face of \(H_{3-i}\) adjacent to \(\varphi\) and let \(\alpha_2\) be the face of \(H_{3-i}\) adjacent to \(\alpha\).
**Rule R4.** $\varphi$ sends $w_1(\varphi) - \frac{2}{3}$ to $\varphi_r$ if $\varphi_r$ is empty and $r \leq 3$.
**Rule R5.** $\varphi$ sends $\frac{1}{6}$ to $\varphi_j$ if $\varphi_j$ ($2 \leq j \leq r - 1$) is a 1-face incident with a (1,1)-edge.
**Rule R6.** $\varphi$ sends $\frac{1}{6}$ to $\varphi_r$ if $\varphi_r$ is empty and $r \geq 4$.
**Rule R7.** $\varphi$ sends $\frac{1}{6}$ to $\alpha_2$ if $\alpha$ is incident with a (1,2)-edge and $\alpha_2$ is an empty 0-face.
**Rule R8.** $\varphi$ sends $\frac{1}{6}$ to $\beta_2$, where $\beta$ is a 2-face of $H_{3-i}$ having exactly one common vertex with $\varphi$ and incident with two (1,2)-edges and $\beta_2$ is an empty 0-face of $H_{3-i}$ adjacent to $\beta$.

![Redistribution rules R4–8](image)

For an empty face $\varphi$, let $w_2(\varphi)$ be the total weight obtained by $\varphi$ (after second redistribution). Obviously, $\sum_{\varphi \in \mathcal{F}_e(H)} w_2(\varphi) = |E(C)| = k$ (as non-empty faces do not obtain any weight). In the following, we will show that the weight $w_2(\varphi)$ of each (empty) face $\varphi$ does not exceed $\frac{2}{3}$ which will mean $k = \sum_{\varphi \in \mathcal{F}_e(H)} w_2(\varphi) \leq \frac{2}{3}f_e(H)$. The maximal planar graph $G$ has exactly $2n - 4$ faces. Each of $f_e(H) \geq \frac{3}{2}k$ empty faces of $H$ is a face of $G$ as well, and each of $n - k$ (pairwise non-adjacent) vertices of $G$ not belonging to $C$ (whose removal has created a non-empty face of $H$) is incident with three (“private”) faces of $G$. Hence $2n - 4 = |F(G)| = f_e(H) + 3(n - k) \geq \frac{3}{2}k + 3n - 3k$ and finally $k \geq \frac{3}{4}(n + 4)$ will follow.
Weight of a 2-face.

Let \( \varphi = [v_1, v_2, v_3] \) be a 2-face of \( H_i \) and let \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \), \( r \geq 2 \), be the \( \varphi \)-branch. As already mentioned, \( \frac{2}{3} \leq w_1(\varphi) \leq 1 \). We check that the weight of \( \varphi \) exceeding \( \frac{2}{3} \) will be shifted in the second redistribution.

1. Let \( \varphi \) be incident with two \((2,2)\)-edges (note that \( w_1(\varphi) = 1 \)). Denote \( \alpha = [v_0, v_1, v_2] \) and \( \beta = [v_2, v_3, v_4] \) the 2-faces of \( H_{3-i} \) adjacent to \( \varphi \). Let \( \alpha_2 \) and \( \beta_2 \) be the face of \( H_{3-i} \) adjacent to \( \alpha \) and \( \beta \), respectively. Each of the faces \( \varphi_2 \), \( \alpha_2 \), and \( \beta_2 \) is either a 1-face or empty 0-face (by Claim 3a).

1.1. Let \( \alpha_2 \) and \( \beta_2 \) be 0-faces (possibly \( \alpha_2 = \beta_2 \)).

1.1.1. If edges \( v_0v_1 \) and \( v_3v_4 \) of \( C \) do not belong to the rim \( R(B_\varphi) \) of \( B_\varphi \), then \( r = 2 \), thus \( \varphi \) sends \( \frac{1}{3} \) to empty 0-face \( \varphi_2 \) (by R4).

1.1.2. If \( v_0v_1 \) belongs to the rim \( R(B_\varphi) \) and \( v_3v_4 \) does not belong to \( R(B_\varphi) \), then \( \varphi_2 = [v_0, v_1, v_2] \) is a 1-face and \( \varphi_r \) is empty (by Claim 3a). Thus \( \varphi \) sends weight \( \geq \frac{1}{3} \) to \( \varphi_r \) (by R4 or R6) and \( \frac{1}{3} \) to \( \alpha_2 \) (by R7). (Similarly if \( v_0v_1 \) does not belong to \( R(B_\varphi) \) and \( v_3v_4 \) belongs to \( R(B_\varphi) \).)

1.1.3. If edges \( v_0v_1 \) and \( v_3v_4 \) belong to the rim \( R(B_\varphi) \), then both are \((1,2)\)-edges. Thus \( \varphi \) sends \( \frac{1}{6} \) to \( \alpha_2 \) and \( \frac{1}{6} \) to \( \beta_2 \) (by R7).

1.2. Let \( \alpha_2 = [v_{-1}, v_0, v_2] \) be a 1-face and \( \beta_2 \) be a 0-face. (Similarly if \( \alpha_2 \) is a 0-face and \( \beta_2 \) is a 1-face.)

1.2.1. If \( v_3v_4 \) does not belong to the rim \( R(B_\varphi) \), then \( r \leq 3 \) and \( \varphi_r \) is empty (by proof of Claim 3c). Thus \( \varphi \) sends \( \frac{1}{3} \) to \( \varphi_r \) (by R4).

1.2.2. If \( v_3v_4 \) belongs to the rim \( R(B_\varphi) \) and \( v_0v_1 \) does not belong to \( R(B_\varphi) \), then \( \varphi_2 = [v_1, v_2, v_4] \) is a 1-face and \( \varphi_r \) is empty (by Claim 3a). Thus \( \varphi \) sends weight \( \geq \frac{1}{3} \) to \( \varphi_r \) (by R4 or R6) and \( \frac{1}{3} \) to \( \beta_2 \) (by R7).

1.2.3. Let edges \( v_3v_4 \) and \( v_0v_1 \) belong to the rim \( R(B_\varphi) \), then both are \((1,2)\)-edges. If \( v_0v_1 \) and \( v_3v_4 \) are incident with \( \varphi_2 \) and \( \varphi_3 \), then \( \{v_0, v_2, v_4\} \) is a non-trivial 3-cut, a contradiction. If \( \varphi_2 = [v_0, v_1, v_3] \) and \( \varphi_3 = [v_{-1}, v_0, v_3] \), then \( \{v_{-1}, v_2, v_3\} \) is a non-trivial 3-cut, a contradiction as well. Thus \( \varphi_2 = [v_1, v_3, v_4] \) and \( \varphi_3 = [v_1, v_4, v_5] \).

1.2.3.1. If \( v_{-1}v_0 \) does not belong to the rim \( R(B_\varphi) \), then \( \varphi_r \) is empty (by Claim 3b). Thus \( \varphi \) sends \( \frac{1}{3} \) to \( \varphi_r \) (by R6) and \( \frac{1}{3} \) to \( \beta_2 \) (by R7).

1.2.3.2. If \( v_{-1}v_0 \) belongs to the rim \( R(B_\varphi) \), then \( v_{-1}v_0 \) is a \((1,1)\)-edge. Thus \( \varphi \) sends \( \frac{1}{6} \) to \( \varphi_j \), a 1-face of \( B_\varphi \) incident with \( v_{-1}v_0 \) (by R5) and \( \frac{1}{6} \) to \( \beta_2 \) (by R7).

1.3. Let \( \alpha_2 = [v_{-1}, v_0, v_2] \) and \( \beta_2 = [v_2, v_4, v_5] \) be 1-faces.

1.3.1. If \( v_3v_4 \) does not belong to the rim \( R(B_\varphi) \), then \( r \leq 3 \) and \( \varphi_r \) is empty (by proof of Claim 3c). Thus \( \varphi \) sends \( \frac{1}{3} \) to \( \varphi_r \) (by R4). (Similarly if \( v_0v_1 \) does not belong to \( R(B_\varphi) \).)

1.3.2. Let edges \( v_0v_1 \) and \( v_3v_4 \) belong to the rim \( R(B_\varphi) \), then both are \((1,2)\)-edges. If \( v_0v_1 \) and \( v_3v_4 \) are incident with \( \varphi_2 \) and \( \varphi_3 \), then \( \{v_0, v_2, v_4\} \) is a non-trivial 3-cut, a contradiction. If \( \varphi_2 = [v_0, v_1, v_3] \) and \( \varphi_3 = [v_{-1}, v_0, v_3] \), then \( \{v_{-1}, v_2, v_3\} \) is a non-trivial 3-cut, a contradiction as well. (Similarly if \( \varphi_2 = [v_1, v_3, v_4] \) and \( \varphi_3 = [v_1, v_4, v_5] \).)

2. Let \( \varphi \) be incident with \((2,2)\)-edge \( v_1v_2 \) and \((1,2)\)-edge \( v_2v_3 \) (note that \( w_1(\varphi) = \frac{5}{6} \)). Denote \( \alpha = [v_0, v_1, v_2] \) the 2-face of \( H_{3-i} \) adjacent to \( \varphi \) and let \( \alpha_2 \) be the face of \( H_{3-i} \) adjacent to \( \alpha \). Each of the faces \( \varphi_2 \) and \( \alpha_2 \) is either a 1-face or empty 0-face (by Claim 3a).

2.1. Let \( \alpha_2 \) be 0-face.

2.1.1. If \( v_0v_1 \) does not belong to the rim \( R(B_\varphi) \), then \( \varphi_r \) is empty (by Claim 3a). Thus \( \varphi \) sends \( \frac{1}{6} \) to \( \varphi_r \) (by R4 or R6).

2.1.2. If \( v_0v_1 \) belongs to the rim \( R(B_\varphi) \), then \( v_0v_1 \) is a \((1,2)\)-edge. Thus \( \varphi \) sends \( \frac{1}{6} \) to \( \alpha_2 \) (by R7).
2.2. Let $\alpha_2$ be a 1-face incident with $v_1v_2$ (i.e. $\alpha_2 = [v_1, v_0, v_2]$).

2.2.1. If $v_3v_4$ does not belong to the rim $R(B_\varphi)$, then $r \leq 3$ and $\varphi_r$ is empty (by proof of Claim 3c). Thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_r$ (by R4).

2.2.2. If $v_3v_4$ belongs to the rim $R(B_\varphi)$ and $v_0v_1$ does not belong to $R(B_\varphi)$, then $\varphi_2 = [v_1, v_3, v_4]$ is a 1-face and $\varphi_r$ is empty (by Claim 3a). Thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_r$ (by R4 or R6).

2.2.3. Let edges $v_3v_4$ and $v_0v_1$ belong to the rim $R(B_\varphi)$. If $v_1v_0$ does not belong to $R(B_\varphi)$, then $\varphi_r$ is empty (by Claim 3b). Thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_r$ (by R6). Otherwise $v_1v_0$ belongs to $R(B_\varphi)$, thus it is a $(1,1)$-edge incident with a 1-face $\varphi_j$ of $B_\varphi$. Hence $\varphi$ sends $\frac{1}{6}$ to $\varphi_j$ (by R5).

2.3. Let $\alpha_2$ be a 1-face incident with $v_2v_3$ (i.e. $\alpha_2 = [v_0, v_2, v_3]$). Since $v_0v_3 \in E(H_{3-1})$, $\varphi_2$ cannot be the 1-face $[v_0, v_1, v_2]$ in $H_r$.

2.3.1. If $v_3v_4$ does not belong to the rim $R(B_\varphi)$, then $r = 2$, thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_2$ (by R4).

2.3.2. If $v_3v_4$ belongs to the rim $R(B_\varphi)$, then $r \geq 3$ and $\varphi_2 = [v_1, v_3, v_4]$.

2.3.2.1. If $v_3v_4$ is incident with a 1-face of $H_{3-1}$ (i.e., $v_3v_4$ is a $(1,1)$-edge), then $\varphi$ sends $\frac{1}{6}$ to $\varphi_2$ (by R5).

2.3.2.2. Let $v_3v_4$ be incident with a 2-face $\beta$ of $H_{3-1}$ (necessarily, $\beta = [v_3, v_4, v_5]$). If $r = 3$, then $\varphi_3$ is empty (by Claim 3a), thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_3$ (by R4). If $r = 4$, then $\varphi_3 = [v_1, v_4, v_5]$ (as $\{v_0, v_3, v_4\}$ is a non-trivial 3-cut if $\varphi_3 = [v_0, v_1, v_4]$) and $\varphi_4$ is empty (by Claim 3a), thus $\varphi$ sends $\frac{1}{6}$ to $\varphi_4$ (by R6). Finally, let $r \geq 5$. Necessarily $\varphi_3 = [v_1, v_4, v_5]$ (as for $r = 4$) and $\varphi_4 = [v_1, v_5, v_6]$ (as $\{v_0, v_3, v_5\}$ is a non-trivial 3-cut if $\varphi_4 = [v_0, v_1, v_5]$) are 1-faces of $B_\varphi$. If $v_3v_4$ is a $(1,1)$-edge, then $\varphi$ sends $\frac{1}{6}$ to $\varphi_4$ (by R5). Otherwise $v_3v_5$ is a $(1,2)$-edge, thus it does not belong to $\beta$-branch (in $H_{3-1}$) and therefore $\beta_2$ is a 0-face, which is, moreover, empty (as the cycle obtained from $C$ by replacing the path $(v_0, \ldots, v_3)$ by the path $(v_0, v_2, v_1, v_4, v_3, v_5)$ is a longest good cycle of $G$ and contains the edge $v_3v_5$ incident with $\beta_2$ (Claim 1)). Hence $\varphi$ sends $\frac{1}{6}$ to $\beta_2$ (by R8).

Weight of a 1-face.

To estimate the weight of a 1-face, we use the following simple observation:

Claim 4 Each 1-face of $H$ belongs to at most one branch.

Let $\psi$ be a 1-face incident with an edge $e$ of $C$. If $e$ is a $(1,2)$-edge, then $\psi$ obtains weight $\frac{2}{3}$ from $e$ (by R2) only. Otherwise $e$ is a $(1,1)$-edge, thus $\psi$ obtains $\frac{1}{3}$ from $e$ (by R1). Furthermore, in this case, $\psi$ can get $\frac{1}{6}$ from a 2-face $\varphi$ (by R5) if $\varphi$ belongs to the $\varphi$-branch. Hence $w_2(\psi) \leq \frac{2}{3}$.

Weight of an empty 0-face.

Each empty 0-face $\omega$ belongs to at most two branches (in Case 1). Let $\varphi$ be a 2-face of $H_i$ with the $\varphi$-branch $B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r)$ such that $\varphi_r = \omega$, and let $e$ be the edge incident with $\varphi_r$ and $\varphi_{r-1}$ (where $\varphi_{r-1} = \varphi$ for $r = 2$).

If $\varphi$ is adjacent to two 2-faces, then $\omega$ gets through $e$ the weight $\frac{1}{4}$ (by R4) for $r \leq 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. If $\varphi$ is adjacent to one 2-face, then $\omega$ gets through $e$ the weight $\frac{1}{6}$ (by R4) and additionally $\frac{1}{6}$ (by R7) for $r = 2$ or the weight at most $\frac{1}{6}$ (by R4) for $r = 3$ or the weight $\frac{1}{6}$ (by R6) for $r \geq 4$. Finally, if $\varphi$ is adjacent to no 2-face, then $\omega$ gets through $e$ the weight $\frac{1}{6}$ (by R6) for $r \geq 4$ or the weight at most $2 \times \frac{1}{6}$ (by R8) for $r \leq 3$.

We showed that $w_2(\varphi) \leq \frac{2}{3}$ for each empty face $\varphi$ and completed the Case 1. Thus, we can assume that in $H_i$ are only empty faces and among them, at most one face is a 0-face. To complete the proof, we have to show that there are some empty faces in $H_{3-1}$ as well.
**CASE 2.** Let \( H_i \) contain no 0-face or exactly one 0-face which is additionally empty.

Obviously, if \( H_i \) contains no 0-face, then it contains two 2-faces \( \alpha_1 \) and \( \alpha_2 \) (since \( T_i \) is a path and 2-faces of \( H_i \) are leaves of \( T_i \)). Note that, (only) in this case, the branches in \( H_i \) are not defined.

Remember that \( H = G[V(G)] \) has \( k \geq 7 \) vertices (as otherwise \( G \) with at most \( k + 2 \leq 8 \) vertices is Hamiltonian). If \( H_i \) contains exactly one 0-face, then it contains three 2-faces \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) (since \( T_i \) is a subdivision of \( K_{1,3} \) and 2-faces of \( H_i \) are leaves of \( T_i \)). We assume that \( H_{3-i} \) contains at least two 0-faces as otherwise all but at most one faces of \( H_{3-i} \) are empty and \( G \) has \( n \leq |V(H)| + 1 = k + 1 \) vertices and Theorem 1 follows immediately (with \( n \geq 11 \)).

**Distribution of points.**

To estimate the number of empty 0- and 1-faces in \( H_{3-i} \), each 2-face \( \alpha_j \) of \( H_i \) (\( j \in \{1,2\} \) if \( H_i \) contains no 0-face and \( j \in \{1,2,3\} \) if \( H_i \) contains one 0-face, respectively) will distribute 1 or 2 points to faces of \( H_{3-i} \). Let \( \alpha_j \) be adjacent to the faces \( \varphi \) and \( \psi \) of \( H_{3-i} \).

**Rule P1.** If \( \varphi \) and \( \psi \) are 2-faces of \( H_{3-i} \) with branches \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) and \( B_\psi = (\psi, \psi_2, \ldots, \psi_t) \), then \( \varphi_r \) and \( \psi_t \) will each receive 1 point (or 2 points if \( \varphi_r = \psi_t \)) from \( \alpha_j \).

**Rule P2.** If \( \varphi \) and \( \psi \) are 1-faces of \( H_{3-i} \), then \( \varphi \) and \( \psi \) will each receive 1 point from \( \alpha_j \).

**Rule P3.** If \( \varphi \) is a 2-faces of \( H_{3-i} \) with \( \varphi \)-branch \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) and \( \psi \) is a 1-face of \( H_{3-i} \) not belonging to \( B_\varphi \), then \( \varphi \) and \( \psi \) will each receive 1 point from \( \alpha_j \).

**Rule P4.** If \( \varphi \) is a 2-faces of \( H_{3-i} \) with \( \varphi \)-branch \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) and \( \psi \) is a 1-face of \( H_{3-i} \) belonging to \( B_\varphi \), then only \( \psi \) will receive 1 point from \( \alpha_j \).

For a face \( \varphi \) of \( H_{3-i} \), let \( p(\varphi) \) be the total number of points carried by \( \varphi \) (in the distribution of points).

**Claim 5** \( f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq \sum_{\varphi \in F_\alpha(H_{3-i})} p(\varphi) \).

**Proof.** We have to prove that each 1-face of \( H_{3-i} \) gets at most 1 point and that each 0-face of \( H_{3-i} \) gets points only if it is empty and it gets at most 2 points. Consequently, Claim 5 follows by simple counting.

Let \( \beta \) be a 1-face of \( H_{3-i} \). Since \( \beta \) can only get points if it is adjacent to some \( \alpha_j \) and there can only be one such face then \( p(\beta) \leq 1 \).

Let \( \beta \) be a 0-face of \( H_{3-i} \). Since \( \beta \) can only get points if it belongs to a branch and it belongs to at most two branches (as there are at least two 0-faces in \( H_{3-i} \)), then \( p(\beta) \leq 2 \). Assume first that \( \beta \) gets a point by P1. Then there is \( \alpha_j \) incident with two \( (2,2) \)-edges and adjacent 2-faces \( \varphi \) and \( \psi \) of \( H_{3-i} \). Let \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) with \( \varphi_r = \beta \) be the branch which ends in \( \beta \). By Claim 3a, \( \varphi_r = \beta \) is an empty 0-face.

Thus, assume that \( \beta \) gets a point by P3. Then there is \( \alpha_j \) incident with a \( (1,2) \)-edge with adjacent 1-face \( \psi \) in \( H_{3-i} \) and a \( (2,2) \)-edge with adjacent 2-face \( \varphi \) such that \( \psi \) does not belong to the branch \( B_\varphi = (\varphi, \varphi_2, \ldots, \varphi_r) \) with \( \varphi_r = \beta \). Since the common edge of \( \alpha_j \) and \( \psi \) does not belong to the rim \( R(B_\varphi) \), again by Claim 3a, \( \varphi_r = \beta \) is an empty 0-face.

**Claim 6** \( f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4 \).

**Proof.** If \( \sum_{\varphi \in F_\alpha(H_{3-i})} p(\varphi) \geq 4 \), then \( f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4 \) (by Claim 5). Assume \( \sum_{\varphi \in F_\alpha(H_{3-i})} p(\varphi) \leq 3 \).
1. Let $H_i$ contains exactly one 0-face. As there are three 2-faces $\alpha_1, \alpha_2, \alpha_3$ in $H_i$ (note, that $T_i$ is a subdivided 3-star in this case), then $\sum_{\varphi \in F_r(H_{3-i})} p(\varphi) = 3$. Furthermore, only P4 was applied to each $\alpha_j$ ($j \in \{1, 2, 3\}$) hence there are three 1-faces with 1 point and they belong to three different branches.

Since $|V(H)| = k \geq 7$, there is $j \in \{1, 2, 3\}$ such that $\alpha_j$ is adjacent to a 1-face $\delta$ of $H_i$. Let $\varphi$ be the adjacent 2-face of $\alpha_j$ in $H_{3-i}$ and $B_{\varphi} = (\varphi, \varphi_2, \ldots, \varphi_r)$ be its branch.

1.1. If $r \geq 4$, then $\varphi_2$ and $\varphi_3$ are 1-faces of the same branch. Thus, at most one among $\varphi_2$ and $\varphi_3$ has a point and $f_1(H_{3-i}) \geq 4$.

1.2. If $r = 3$, then $\delta$ and $\varphi$ are not adjacent (i.e. $\delta \neq \varphi_2$, since $H$ has no multiple edges) and $\varphi_3$ is an empty 0-face (by Claim 3b), hence $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2. Let $H_i$ contains no 0-face. Since $\sum_{\varphi \in F_r(H_{3-i})} p(\varphi) \leq 3$, there is $j \in \{1, 2\}$ such that P4 was applied to $\alpha_j$. Let $\delta$ be the 1-face of $H_i$ adjacent to $\alpha_j$ (since $|V(H)| = k \geq 7$), let $\varphi$ be the 2-face and 1-face of $H_{3-i}$ adjacent with $\alpha_j$, respectively, and let $B_{\varphi} = (\varphi, \varphi_2, \ldots, \varphi_r)$ be the branch of $\varphi$. We may assume $\alpha_j = [v_1, v_2, v_3]$ and $\varphi = [v_2, v_3, v_4]$.

2.1. Let $r \leq 4$.

2.1.1 If $\delta = [v_0, v_1, v_3]$, then $v_0v_1$ does not belong to the rim $R(B_{\varphi})$ (otherwise $\varphi_2 = [v_1, v_2, v_4]$, $\varphi = [v_0, v_1, v_4]$ and $v_0, v_3, v_4$ is a non-trivial 3-cut, a contradiction) and $\varphi_3$ is an empty 0-face (by Claim 3b). By P1–4, there is a face in $H_{3-i}$ other than $\varphi$ with a point, thus $f_1(H_{3-i}) + 2f_0(H_{3-i}) \geq 4$.

2.1.2 If $\delta = [v_1, v_3, v_4]$, then $\varphi_2 = [v_2, v_4, v_5]$ (since $v_1v_4 \in E(H_i)$), $\varphi = [v_1, v_2, v_5]$, and $\{v_1, v_4, v_5\}$ is a non-trivial 3-cut, a contradiction.

2.2. Let $r = 5$. There are three 1-faces (in fact $\varphi_2, \varphi_3$, and $\varphi_4$) all belonging to the same branch $B_{\varphi}$. We may assume that P4 was applied to $\alpha_j$ and P2 was applied to $\alpha_{3-j}$, and all three 1-faces are adjacent to $\alpha_1$ or $\alpha_2$ (since otherwise there is another 1-face or empty 0-face and Claim 6 follows).

2.2.1. If $\alpha_{3-j} = [v_{-1}, v_0, v_1]$, then rim $R(B_{\varphi}) = (v_{-1}, \ldots, v_4)$, thus $\varphi_2 = [v_1, v_2, v_4]$ and $\delta = [v_1, v_3, v_4]$, a contradiction to the simplicity of $H$.

2.2.2. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_0, v_1, v_3]$, then rim $R(B_{\varphi}) = (v_1, \ldots, v_6)$ and $\varphi_5$ is an empty 0-face (by Claim 3b), thus $f_1(H_{3-i}) + f_0(H_{3-i}) \geq 4$.

2.2.3. If $\alpha_{3-j} = [v_4, v_5, v_6]$ and $\delta = [v_1, v_3, v_4]$, then rim $R(B_{\varphi}) = (v_1, \ldots, v_6)$. Hence $v_1v_6 \in E(H_{3-i})$ and consequently $\{v_1, v_4, v_5\}$ is a non-trivial 3-cut, a contradiction.

2.3. If $r \geq 6$, then there are at least four 1-faces in $B_{\varphi}$, thus $f_1(H_{3-i}) \geq 4$.

Remember that each $j$-face of $H_{3-i}$ is incident with $j$ (“private”) edges of $C$, hence $2f_2(H_{3-i}) + f_1(H_{3-i}) = k$. As each of the $k-2$ triangular faces of $H_i$ is empty, all non-empty faces of $H$ belong to $H_{3-i}$ and their number is $(k-2) - 2f_2(H_{3-i}) - f_1(H_{3-i}) - f_0(H_{3-i}) = (k-2) - \frac{1}{2}(k-f_1(H_{3-i})) - f_1(H_{3-i}) - f_0(H_{3-i}) = \frac{k}{2} - 2 - \frac{1}{2}(f_1(H_{3-i}) + 2f_0(H_{3-i})) \leq \frac{k}{2} - 4$ (by Claim 6). Finally, at most $\frac{k}{2} - 4$ vertices of $G$ lie outside the cycle $C$ (and exactly $k$ vertices on $C$), hence $n \leq k + \left(\frac{k}{2} - 4\right)$ and $k \geq \frac{3}{2}(n + 4)$ follows, which completes the proof of Theorem 1.
References


