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The Single Screw Reciprocal to the General Plane-Symmetric Six-Screw Linkage

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Abstract. The degree of mobility of a given linkage depends upon the order of its screw system, whereby a loop with six joint freedoms may be related to its reciprocal screw system, a single screw axis with an associated pitch. It has been postulated that this reciprocal screw can provide an alternative means of identifying its parent linkage, and an investigation in this respect has been recently carried out for the line-symmetric six-screw linkage. Here we undertake a similar enquiry for the plane-symmetric six-screw loop and so determine the essential characteristics of its reciprocal screw.

Key Words: Kinematics, overconstrained linkages.
MSC 1994: 53A17

1 Introduction

A mechanical *linkage* is an assemblage of rigid bodies interconnected by joints about or along which relative motion is possible. The linkage is modelled mathematically as a *kinematic chain* which has been studied extensively in diverse ways for various purposes. One area of investigation is that of existence criteria for *overconstrained* chains, those which are mobile in defiance of general criteria for their movability. Such chains possess singular geometrical properties which result in additional freedom of movement. Multiloop chains have been examined piecemeal, but single loops have received much systematic attention.

In general, a closed, spatial kinematic chain requires seven joint freedoms for (internal) mobility, but many exceptions to this rule have been found. For fewer than five joint freedoms it is believed that all cases are known. Several solutions of overconstrained loops with five or six joint freedoms have also been established, but a comprehensive analysis is difficult because of the large numbers of governing equations and variables involved. It may be necessary to develop new mathematical means for studying these chains, and this paper is intended as a contribution in that vein.

The relative motion capability between two rigid bodies can be expressed in terms of a *screw motor*, and that of a kinematic loop as a series of such entities, thereby forming a *screw system*. For a loop with one *degree of mobility*, the *order* of its screw system is one

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fewer than the number of its joint freedoms. Now, to any screw system there is a *reciprocal* system such that the sum of the orders of the two systems is six. Consequently, as the number of joint freedoms in a mobile loop increases, the order of the relevant reciprocal screw system decreases. In particular, a mobile (overconstrained), single-loop linkage with six joint freedoms has a reciprocal screw system of order one. It is plausible that each family of six-bar linkages may be characterised by a unique reciprocal screw system, and it could be that this system of order one provides an easier means of analysis or definition than does the linkage itself.

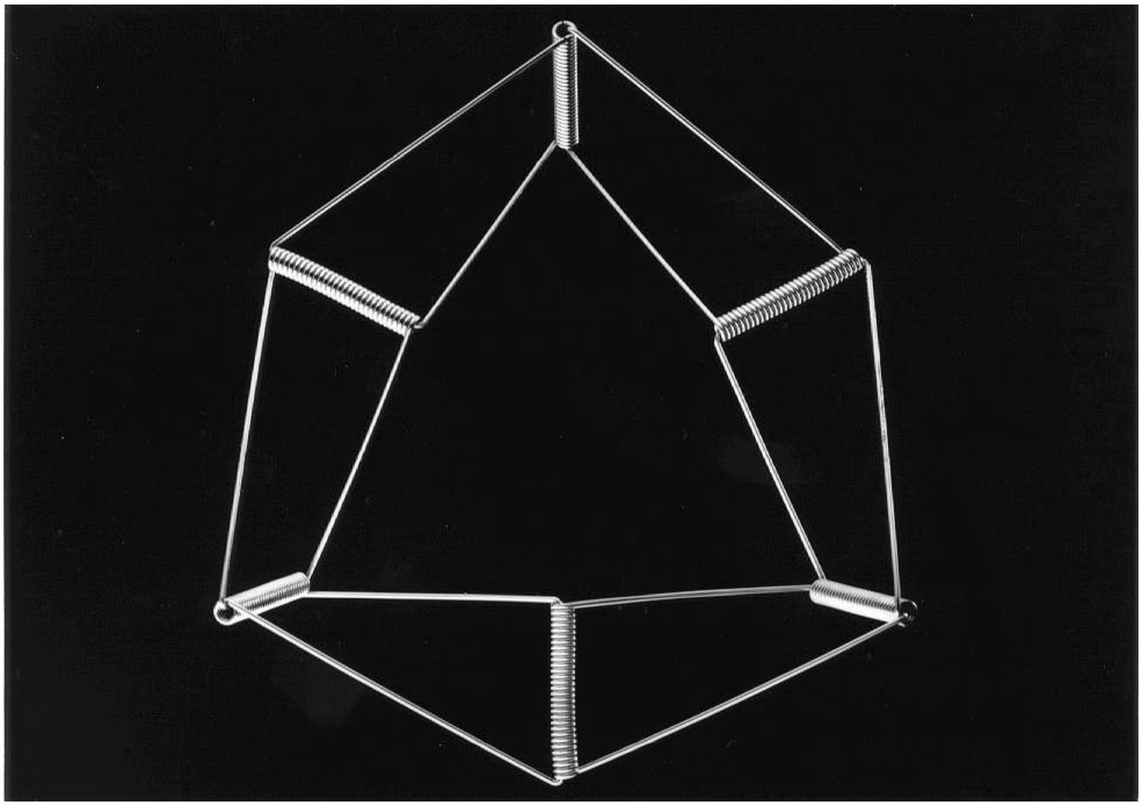

Figure 1:

The reciprocal system of the generalised line-symmetric six-bar linkage has been recently identified and described in Ref. [4]. The next most accessible six-screw linkage is the generalised plane-symmetric one which is the subject of this paper. (A commercial example of a very particular case of this chain is illustrated in Fig. 1.) We determine its reciprocal screw system, obtaining thereby a surprising result for the *pitch* of the screw, and we report on its salient geometrical features.

2 The general plane-symmetric chain

In Fig. 2 we make the xOy plane of reference the plane of symmetry for our six-bar loop in its most general form [8], namely, that in which there are two symmetrically disposed pairs of *helical joints* (3,5 and 6,2) and two unpaired (degenerate) *revolute joints* (1 and 4) with zero offsets lying in the plane of symmetry. Opposed links and articulations will have corresponding parameters of position and orientation. We choose to define each helical axis

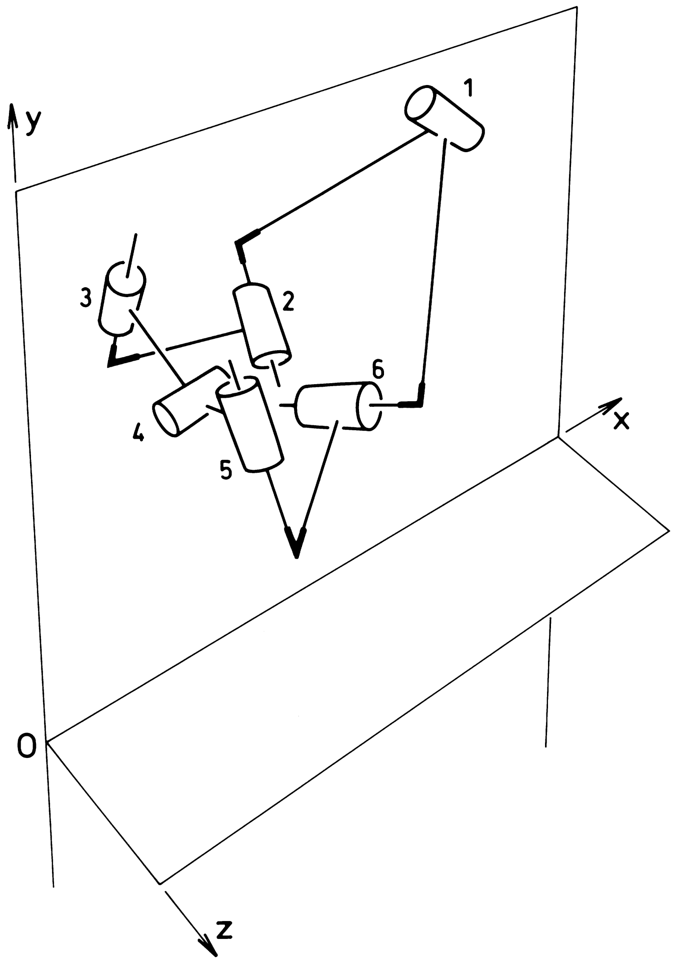

Figure 2:

by a point (x_i, y_i, z_i) which it contains and its direction cosines l_i, m_i, n_i, and the paired joints have assigned screw-pitches h_i. In turn, the direction cosines may be identified with the orientation

$$\widehat{\boldsymbol{\omega}}_i = (l_i, m_i, n_i)$$

of an *instantaneous screw axis* (ISA) located by position vector

$$\mathbf{r}_i = (x_i, y_i, z_i).$$

The *motion screw* effected by the joint is represented concisely by a screw motor expressed as

$$\$_i = (\widehat{\boldsymbol{\omega}}_i,\ h_i\widehat{\boldsymbol{\omega}}_i + \mathbf{r}_i \times \widehat{\boldsymbol{\omega}}_i).$$

The net motion experienced around the whole chain is, then,

$$(\mathbf{0} =)\ \mathbf{S} = \sum_{i=1}^{6} \omega_i \$_i\,,$$

where ω_i is the magnitude of angular velocity about joint i. For the last equation to be satisfied, the six screw motors must form a linearly dependent set. If we put

$$h_i\boldsymbol{\omega}_i + \mathbf{r}_i \times \boldsymbol{\omega}_i = \mathbf{u}_i\,, \text{ where } \quad \boldsymbol{\omega}_i = \omega_i\widehat{\boldsymbol{\omega}}_i\,,$$

we see that

$$\widehat{\boldsymbol{\omega}}_i \cdot \mathbf{u}_i = \omega_i h_i \quad \text{and} \quad \widehat{\boldsymbol{\omega}}_i \times \mathbf{u}_i = \omega_i \mathbf{p}_i \,,$$

$\mathbf{p}_i$ being the perpendicular position vector for screw axis i.

Although a linkage possessing plane-symmetric parameters does not necessarily exhibit plane-symmetric configurations, we may assume that, for the loop under examination here, the *closure mode* selected guarantees the manifestation of plane-symmetric positions only. Hence, the following vectors can be adopted.

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} \quad \mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \quad \mathbf{r}_3 = \begin{pmatrix} x_3 \\ y_3 \\ 0 \end{pmatrix} \quad \mathbf{r}_4 = \begin{pmatrix} x_4 \\ y_4 \\ 0 \end{pmatrix} \quad \mathbf{r}_5 = \begin{pmatrix} x_3 \\ y_3 \\ 0 \end{pmatrix} \quad \mathbf{r}_6 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}$$

$$\widehat{\boldsymbol{\omega}}_1 = \begin{pmatrix} l_1 \\ m_1 \\ 0 \end{pmatrix} \quad \widehat{\boldsymbol{\omega}}_2 = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} \quad \widehat{\boldsymbol{\omega}}_3 = \begin{pmatrix} l_3 \\ m_3 \\ n_3 \end{pmatrix}$$

$$\widehat{\boldsymbol{\omega}}_4 = \begin{pmatrix} l_4 \\ m_4 \\ 0 \end{pmatrix} \quad \widehat{\boldsymbol{\omega}}_5 = \begin{pmatrix} l_3 \\ m_3 \\ -n_3 \end{pmatrix} \quad \widehat{\boldsymbol{\omega}}_6 = \begin{pmatrix} l_2 \\ m_2 \\ -n_2 \end{pmatrix}.$$

As well, we may take the respective screw-pitches to be $0, h_2, h_3, 0, -h_3, -h_2$. In consequence, the six screw motors are as follows.

$$\$_1 = \begin{pmatrix} l_1 \\ m_1 \\ 0 \\ 0 \\ 0 \\ m_1x_1 - l_1y_1 \end{pmatrix} = \begin{pmatrix} L_1 \\ M_1 \\ 0 \\ 0 \\ 0 \\ R_1 \end{pmatrix} \qquad \$_2 = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \\ h_2l_2 + n_2y_2 \\ h_2m_2 - n_2x_2 \\ h_2n_2 + m_2x_2 - l_2y_2 \end{pmatrix} = \begin{pmatrix} L_2 \\ M_2 \\ N_2 \\ P_2 \\ Q_2 \\ R_2 \end{pmatrix}$$

$$\$_3 = \begin{pmatrix} l_3 \\ m_3 \\ n_3 \\ h_3l_3 + n_3y_3 \\ h_3m_3 - n_3x_3 \\ h_3n_3 + m_3x_3 - l_3y_3 \end{pmatrix} = \begin{pmatrix} L_3 \\ M_3 \\ N_3 \\ P_3 \\ Q_3 \\ R_3 \end{pmatrix} \qquad \$_4 = \begin{pmatrix} l_4 \\ m_4 \\ 0 \\ 0 \\ 0 \\ m_4x_4 - l_4y_4 \end{pmatrix} = \begin{pmatrix} L_4 \\ M_4 \\ 0 \\ 0 \\ 0 \\ R_4 \end{pmatrix}$$

$$\$_5 = \begin{pmatrix} l_3 \\ m_3 \\ -n_3 \\ -h_3l_3 - n_3y_3 \\ -h_3m_3 + n_3x_3 \\ h_3n_3 + m_3x_3 - l_3y_3 \end{pmatrix} = \begin{pmatrix} L_3 \\ M_3 \\ -N_3 \\ -P_3 \\ -Q_3 \\ R_3 \end{pmatrix} \qquad \$_6 = \begin{pmatrix} l_2 \\ m_2 \\ -n_2 \\ -h_2l_2 - n_2y_2 \\ -h_2m_2 + n_2x_2 \\ h_2n_2 + m_2x_2 - l_2y_2 \end{pmatrix} = \begin{pmatrix} L_2 \\ M_2 \\ -N_2 \\ -P_2 \\ -Q_2 \\ R_2 \end{pmatrix}.$$

Digressing briefly, we draw attention to the fact that, in a loop which functions plane-symmetrically, and for the representation employed here, two symmetrically disposed motion screws experience the same rate of angular displacement at any time, and so they may be combined by addition to produce a resultant screw in the plane of symmetry. Thus, screws 2 and 6 yield the motor

$$(l_2, m_2, 0, 0, 0, h_2n_2 + m_2x_2 - l_2y_2)$$

and screws 3 and 5 the motor

$$(l_3, m_3, 0, 0, 0, h_3n_3 + m_3x_3 - l_3y_3)\,.$$

Both of these resultant screws have zero pitch and are of the same form as $\mathbf{\$}_1$ and $\mathbf{\$}_4$, all effectively three-dimensional entities. That is, the four screws so determined are linearly dependent, belonging to the same three-dimensional "simplified screw system" [8], thereby establishing mobility of the chain.

Taking now all six of the linkage's motors obtained above, let us define the matrix

$$\mathcal{S} = \mathbf{\$}_1\mathbf{\$}_2\mathbf{\$}_3\mathbf{\$}_4\mathbf{\$}_5\mathbf{\$}_6 .$$

Then the screws will be linearly dependent, and so belong to the same five-system, when

$$|\mathcal{S}| = 0 .$$

But it is clear that we can write

$$\mathcal{S}' = \mathcal{S}\mathcal{C},$$

where $\mathcal{C}$ is a (non-singular) matrix composed of elementary column operators, and

$$\mathcal{S}' = \begin{pmatrix} L_1 & 0 & 0 & L_4 & L_3 & L_2 \\ M_1 & 0 & 0 & M_4 & M_3 & M_2 \\ 0 & 2N_2 & 2N_3 & 0 & 0 & 0 \\ 0 & 2P_2 & 2P_3 & 0 & 0 & 0 \\ 0 & 2Q_2 & 2Q_3 & 0 & 0 & 0 \\ R_1 & 0 & 0 & R_4 & R_3 & R_2 \end{pmatrix} .$$

Because $\mathcal{S}'$ is singular so too is $\mathcal{S}$. Thus, the mobility of the loop is alternatively proved and the rank of the matrix cannot exceed 5. This maximum value applies when the linkage has mobility 1, the case of interest here.

3 The reciprocal screw

Two screws $(\widehat{\boldsymbol{\omega}}_k, \mathbf{u}_k)$ and $(\widehat{\boldsymbol{\omega}}_l, \mathbf{u}_l)$ are said to be reciprocal if and only if

$$\widehat{\boldsymbol{\omega}}_k \cdot \mathbf{u}_l + \widehat{\boldsymbol{\omega}}_l \cdot \mathbf{u}_k = 0 .$$

Given a screw system, the reciprocal system is that consisting of all the screws which are reciprocal to every screw in the system provided. The sum of the orders of a pair of reciprocal systems is 6. Consequently, in the general case for which the screws of our six-bar belong to the same five-system, there is a single screw reciprocal to all screws of the linkage and we denote it by

$$\mathbf{\$}_r = (\Omega_x, \Omega_y, \Omega_z, U_x, U_y, U_z) .$$

The axis of this screw is also coincident with the central axis of any linear complex defined by the joint axes of the kinematic chain [1].

Following some minor operations on the columns of $\mathcal{S}'$, we can state the requirement of reciprocity by means of the equation

$$\begin{pmatrix} L_1 & M_1 & 0 & 0 & 0 & R_1 \\ 0 & 0 & N_2 & P_2 & Q_2 & 0 \\ 0 & 0 & N_3 & P_3 & Q_3 & 0 \\ L_4 & M_4 & 0 & 0 & 0 & R_4 \\ L_3 & M_3 & 0 & 0 & 0 & R_3 \\ L_2 & M_2 & 0 & 0 & 0 & R_2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

The equation may be immediately partitioned into the two separate ones,

$$\begin{pmatrix} L_1 & M_1 & R_1 \\ L_4 & M_4 & R_4 \\ L_3 & M_3 & R_3 \\ L_2 & M_2 & R_2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} N_2 & P_2 & Q_2 \\ N_3 & P_3 & Q_3 \end{pmatrix} \begin{pmatrix} U_z \\ \Omega_x \\ \Omega_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first of these equations expresses the fact that $\mathbf{\$}_r$ is reciprocal to each of the screws of the simplified screw system, only three of which can be independent. The second equation expresses reciprocity to the difference motors

$$\mathbf{\$}_2 - \mathbf{\$}_6, \ \mathbf{\$}_3 - \mathbf{\$}_5$$

of the pairs of opposed screws, the axes of which are normal to the plane of symmetry. In the first equation the rank of the 4×3 matrix is generally 3, so that the solution must be

$$U_x = U_y = 0 = \Omega_z \,.$$

Whatever the pitch h_r of the reciprocal screw, therefore, we may present the set of line coordinates of the central axis (to the various complexes which comprise the five-system) as

$$\hat{\mathbf{c}} = (\Omega_x, \Omega_y, 0, 0, 0, V_z), \ \text{where} \quad V_z = U_z - h_r\Omega_z = U_z$$

and we have been free to impose the condition that

$$\Omega_x^2 + \Omega_y^2 = 1 \,.$$

So $\hat{\mathbf{c}}$ lies in the plane of symmetry.

The second of the foregoing matrix equations can be used to determine the relationships between

$$\hat{\mathbf{\Omega}} = \begin{pmatrix} \Omega_x \\ \Omega_y \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} 0 \\ 0 \\ U_z \end{pmatrix}.$$

Writing the equation in the expanded form

$$\begin{aligned} P_2\Omega_x + Q_2\Omega_y &= -N_2U_z \\ P_3\Omega_x + Q_3\Omega_y &= -N_3U_z \,, \end{aligned}$$

we have that

$$\Omega_x = -\frac{D_P}{D_N}U_z \,, \quad \Omega_y = -\frac{D_Q}{D_N}U_z \,,$$

where

$$D_P = \begin{vmatrix} N_2 & Q_2 \\ N_3 & Q_3 \end{vmatrix}, \quad D_N = \begin{vmatrix} P_2 & Q_2 \\ P_3 & Q_3 \end{vmatrix}, \quad D_Q = \begin{vmatrix} P_2 & N_2 \\ P_3 & N_3 \end{vmatrix}.$$

Hence,

$$\mathbf{\$}_r = \hat{\mathbf{c}} = \frac{U_z}{D_N}(-D_P, -D_Q, 0, 0, 0, D_N) = \frac{1}{\sqrt{D_P^2 + D_Q^2}}(-D_P, -D_Q, 0, 0, 0, D_N)\,,$$

which is entirely independent of the locations and orientations of joint axes 1 and 4. It is immediate that

$$h_r = \hat{\mathbf{\Omega}} \cdot \mathbf{U} = 0$$

and

$$\mathbf{p}_r = \widehat{\mathbf{\Omega}} \times \mathbf{U} = \frac{U_z^2}{D_N}(-D_Q, D_P, 0) = \frac{D_N}{D_P^2 + D_Q^2}(-D_Q, D_P, 0) \,.$$

The pitches h_2, h_3 of the screws contributing to the five-system are arbitrary, and so the result of permanently zero pitch to be associated with the central axis is unexpected. This is especially so because it is known that a reciprocal screw of permanently zero pitch is characteristic of a six-revolute linkage for which the joint axes have a common transversal, typically BRICARD's trihedral linkage [5] which is not generally plane-symmetric. Here, of course, the central axis does not generally intersect any of the four screw axes inclined to the plane of symmetry. If, for example, it were to intersect screw axis 2, we should require that

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \mathbf{p}_r + r_2 \widehat{\mathbf{\Omega}},$$

where r_2 is some parameter of length to be determined. That is,

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{D_N}{D_P^2 + D_Q^2} \begin{pmatrix} -D_Q \\ D_P \end{pmatrix} + \frac{r_2}{\sqrt{D_P^2 + D_Q^2}} \begin{pmatrix} -D_P \\ -D_Q \end{pmatrix},$$

whence

$$y_2 D_P - x_2 D_Q = D_N \,.$$

Upon substitution and simplification this condition becomes

$$h_2 h_3 (l_2 m_3 - l_3 m_2) = h_2 \left\{ n_3 l_2 (x_3 - x_2) + m_2 n_3 (y_3 - y_2) \right\} .$$

The two possible implications are that $h_2 = 0$ or, as can be easily demonstrated, the central axis is orthogonal to axis 2. [This conclusion may be reached alternatively by means of a reciprocity relationship such as that given in Ref. [6].] The first possibility only arises for a special form of the linkage. The second can, conceivably, be realised at a particular configuration of the loop. Interestingly, if both circumstances were to obtain, it would entail a singular position of part-chain mobility owing to the coincidence of two screws of zero pitch; there could follow a degenerate motion of the loop in which the other four joints were permanently locked. Such a form of behaviour has been pointed out elsewhere, in Ref. [3], for instance.

In the (degenerate) case where all loop screws are of zero pitch, namely, BRICARD's [5] plane-symmetric linkage, the reciprocal screw is immediately determined by the points (x_2, y_2), (x_3, y_3), independently of any other parameters, and the central axis is a common transversal to the six joint axes. This result cannot be generally related to that for BRICARD's trihedral loop, because of the individual character of the former and the probability that the latter is itself a special case of a larger linkage grouping [2], [5].

4 Closing remarks

We have now established apparently distinguishing features of the screw system reciprocal to each of the line-symmetric and plane-symmetric families of six-screw linkages. We can add to these the fairly obvious result that, for the physically varied family of parallel-screw chains [7], of which a particular case is the SARRUS linkage, the single reciprocal screw is of infinite pitch and is directed normal to the planes defined by the rotary joint axes. It is by no means certain that this approach to the study of overconstrained loops will be of value, but it seems to be worth pursuing at this stage. One must be vigilant in doing so that broad inferences be not drawn from particular solutions of mobile loops. As the other known six-bars are not necessarily the most general representatives of their respective families, this caution must be highly respected.

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