Journal for Geometry and Graphics
Volume 1 (1997), No. 1, 1–3.

A Simple Proof for the Theorems of Pascal and Pappus

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Abstract. The theorem of Pascal concerning a hexagon inscribed in a conic is very useful in many geometrical constructions and ought to be included in a normal course on descriptive geometry. Though the citation of this theorem is possible in a short lecture, its proof is very often omitted due to a lack of time and because students at technical universities have no basic knowledge in projective geometry. As far as we know, this discipline is not contained in the curricula of technical universities. However, a lecture without proofs is incomplete and satisfies neither lecturers nor students. Therefore the author presents a proof of Pascal's theorem which does not require any knowledge of projective geometry. The conic is seen as the contour of a quadric Φ, and some pairs of lines define conical surfaces Γ_1, Γ_2. Then the intersections between these three quadrics Φ, Γ_1, Γ_2 lead to three collinear Pascal's points. When the quadric Φ is replaced by a conical surface Γ_3 the analysis of intersections between the three surfaces leads to an immediate proof of Pappus theorem.

There are mainly two methods used in the proofs of Pascal's theorem. One consists in proving the theorem at first for a circle utilizing the particular properties of circles and then transforming this circle into a conic [2]. The second method consists in applying some projective operations to the conic presupposing that the reader is familiar with the basic constructions of projective geometry [1]. The author presents a proof for arbitrary conics – without any projective constructions.

Let a *conic* φ be given with arbitrary points $1, \ldots, 6 \in \varphi$ (Fig. 1). Let us consider the hexagon we obtain from joining these points in natural order. The pairs of sides with a common vertex like $12, 23$ or $23, 34$ are called *adjacent*. The pairs of sides separated by one vertex like $12, 45$ or $23, 56$ are called *opposite*. Let us also introduce the notion of so-called *half-adjacent* sides for those which are separated by one side. At the hexagon $1 \ldots 6$ for example

ISSN 1433-8157/\$ 2.50

the sides $12, 34$ or $23, 45$ or $34, 56$ are half-adjacent. We choose two pairs of half-adjacent sides in which each side appears only once, e.g. $12, 34$ and $45, 61$. Let us assume that the straight lines $12, 34$ and $45, 61$ are generators of *conical surfaces* Γ_1, Γ_2, respectively. The vertices of these surfaces are denoted by W_1 and W_2.

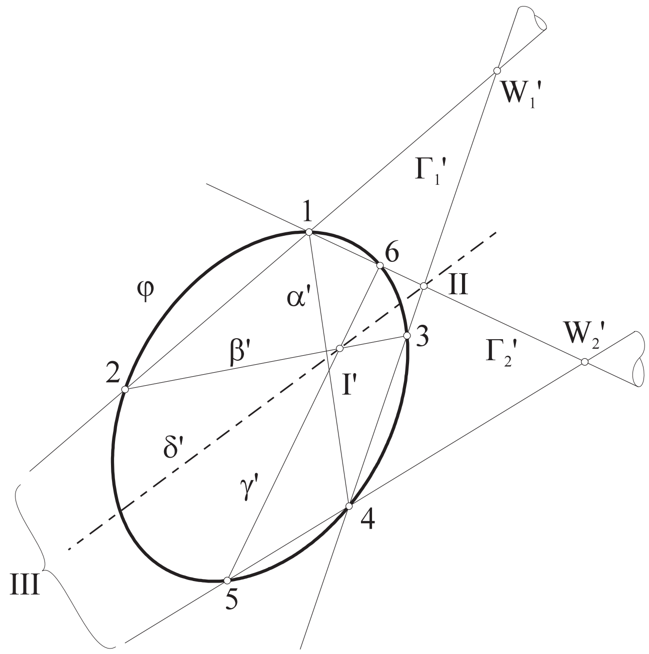

Figure 1: PASCAL's theorem

Let us interpret Fig. 1 as follows: Consider the conic φ as intersection of a *quadric* Φ with a plane π of symmetry. The lines $14, 23, 56$ in π are seen as projections α', β', γ' of conics $\alpha, \beta, \gamma \subset \Phi$. The conical surfaces Γ_1, Γ_2 shall intersect the quadric Φ in pairs of conics. Therefore they are also symmetric with respect to π, and the three surfaces Γ_1, Γ_2 and Φ share the conic α. There must be an *additional conic* δ such that $\Gamma_1 \cap \Gamma_2$ is the union of α and δ.

Let us take into consideration a point $\mathrm{I} \in \beta \cap \gamma$. This is a common point of the conical surfaces $\Gamma_1 \supset \beta$ and $\Gamma_2 \supset \gamma$. Therefore in the case $\mathrm{I} \notin \alpha$ point I must belong to δ. But this holds also when point I happens to be located on α. For, in this case I is a point of contact between each two of the three surfaces Γ_1, Γ_2, Φ and therefore a double point of the reducible curve $\Gamma_1 \cap \Gamma_2$ of intersection. Thus *we conclude* $\mathrm{I}' \in \delta'$.

This conclusion is the main element of PASCAL's theorem proving. For, we construct δ' as a line passing through the points $\mathrm{II} = 16 \cap 34$ and $\mathrm{III} = 12 \cap 45$. Hence the points I', II and III must be collinear. Obviously, these are the points of intersection between opposite sides of the hexagon 123456. This means that the theorem concerning a hexagon inscribed in a conic is true.

It is possible that two pairs of vertices in the hexagon 123456 are coinciding, e.g. $1 = 2$ and $3 = 4$. Then the conics α and β are coinciding too, and consequently the conical surface Γ_1 contacts the quadric Φ along α. Then I is a point of contact between Γ_1, Γ_2 and Φ which again implies $\mathrm{I} \in \delta$. The graphical construction is a bit simpler in this case, but similar to the general case. The projection of the conic δ, i.e. the line δ', *coincides with the straight line of* PASCAL.

Now let us replace the conic φ by two different *straight lines* a, b (Fig. 2). Then instead

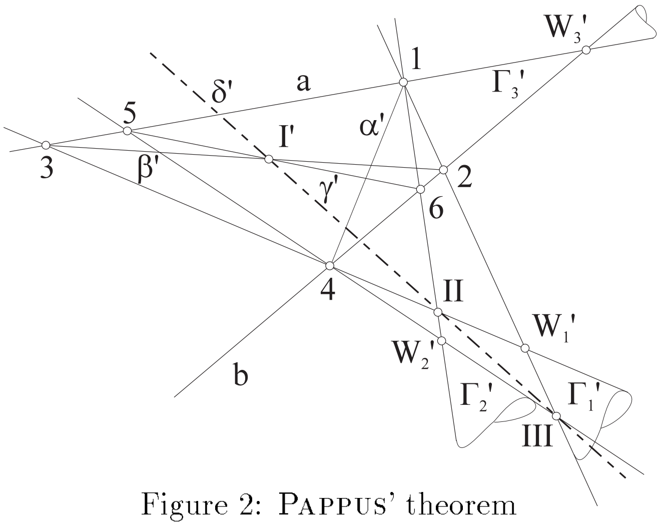

Figure 2: PAPPUS' theorem

of a quadric Φ we get a conical surface Γ_3 with the outline consisting of a, b. Let us choose the vertices of a hexagon 123456 on the straight lines a and b in a way that no side of the hexagon is lying on the lines a, b. Repeating the reasoning of the first part of the paper, let us introduce two auxiliary conical surfaces Γ_1, Γ_2 with the pairs $12, 34$ and $45, 61$ of half-adjacent sides as generators. Again, we consider the lines 14, 23 and 56 as projections of conics α, β, γ belonging to the surfaces $\Gamma_1, \Gamma_2, \Gamma_3$. Then each point $\mathrm{I} \in \beta \cap \gamma$ must belong to the conic δ which completes the intersection $\Gamma_1 \cap \Gamma_2$ together with α. The projection of δ is a line δ' joining the points $\mathrm{II} = 34 \cap 16$ and $\mathrm{III} = 12 \cap 45$. Hence we conclude that the points I', II and III are collinear. However, these are the points of intersection between opposite sides of the hexagon with vertices alternating on the lines a and b. In this way we proved a property which is stated in the well-known theorem of PAPPUS.

References

[1] N.F. CHETVEROUKHIN: *Projective Geometry* (in Russian). Proswieszczenje, Moscow 1969, p. 165.

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Received November 26, 1996; final form May 12, 1997