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The Development of the Oloid

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Abstract. Let two unit circles k_A, k_B in perpendicular planes be given such that each circle contains the center of the other. Then the convex hull of these circles is called *Oloid*. In the following some geometric properties of the Oloid are treated analytically. It is proved that the development of the bounding torse Ψ leads to elementary functions only. Therefore it is possible to express the rolling of the Oloid on a fixed tangent plane τ explicitly. Under this staggering motion, which is related to the well-known spatial Turbula-motion, also an ellipsoid Φ of revolution inscribed in the Oloid is rolling on τ. We give parameter equations of the curve of contact in τ as well as of its counterpart on Φ.
The surface area of the Oloid is proved to equal the area of the unit sphere. Also the volume of the Oloid is computed.

Keywords: Oloid, Turbula-motion, development of torses.
MSC 1994: 51N05, 53A05, 53A17

1 Introduction

Let k_A, k_B be two unit circles in perpendicular planes Π_1, Π_2 such that k_A passes through the center M_B of k_B and k_B passes through the center M_A of k_A (see Fig. 1)[1], The *torse* (developable) Ψ connecting k_A and k_B is the enveloping surface of all planes τ that touch k_A and k_B simultaneously. If any tangent plane τ contacts k_A at A and k_B at B, then the line AB is a generator of Ψ. In this case the tangent line of k_A at A must intersect the tangent line of k_B at B in a finite or infinite point T on the line 12 of intersection between Π_1 and Π_2 (see Fig. 2; the triangle ABT can also be found in Fig. 5 and Fig. 6).

[1]All figures in this paper are orthogonal views. But only in Fig. 5 and Fig. 6 the superscript "n" is used to indicate that geometric objects have been projected orthogonally into a plane.

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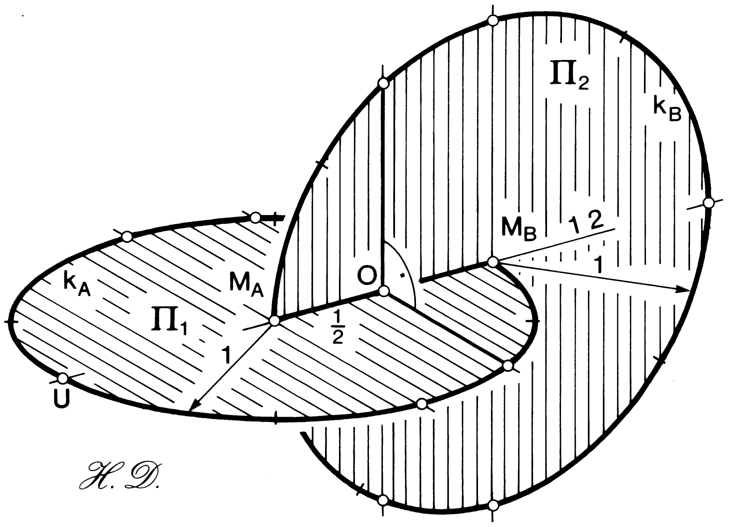

Figure 1: Circles k_A, k_B defining the Oloid

We choose the planes Π_1, Π_2 as coordinate planes and the midpoint O of $M_A M_B$ as the origin of a cartesian coordinate system. Then we may set up the equations of k_A, k_B as

$$(1) \qquad \begin{aligned} k_A:\ & x^2 + (y + \tfrac{1}{2})^2 = 1 \quad \text{and} \quad z = 0 \\ k_B:\ & (y - \tfrac{1}{2})^2 + z^2 = 1 \quad \text{and} \quad x = 0\,. \end{aligned}$$

We parametrize the torse Ψ by the arc-length t of k_A with the starting point $t = 0$ at U on

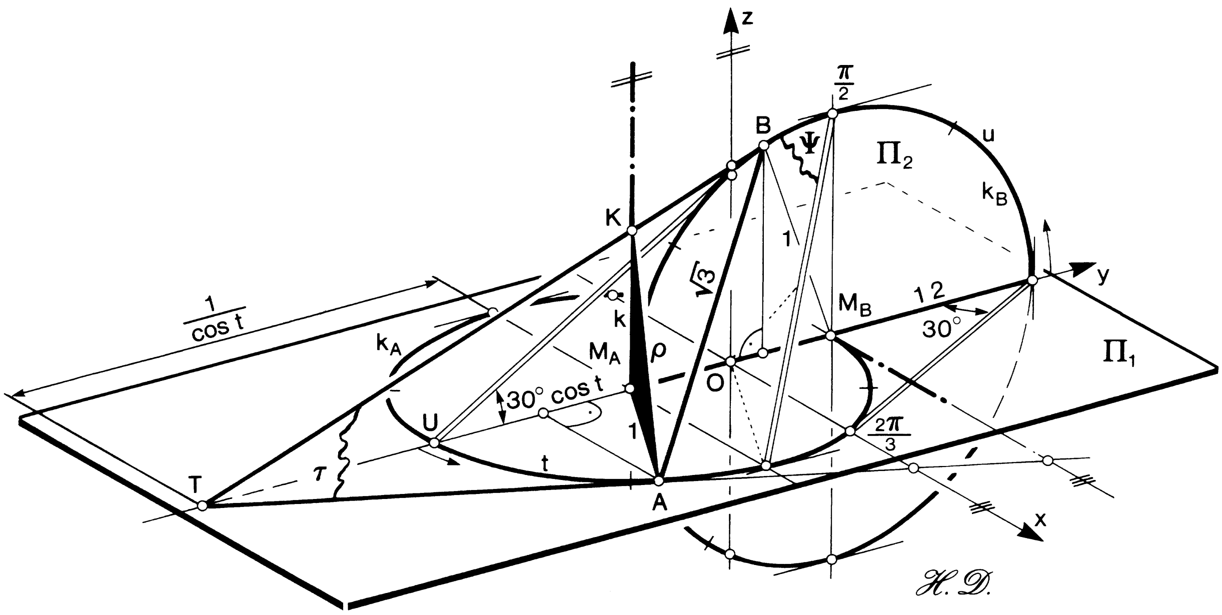

Figure 2: Coordinate system and notation

the negative y-axis. Then we obtain the coordinates

$$(2) \qquad A = \left(\sin t\,,\ -\frac{1}{2} - \cos t\,,\ 0\right).$$

Since the point T on the y-axis is conjugate to A with respect to k_A, we get

$$T = \left(0,\ -\frac{2+\cos t}{2\cos t}\,,\ 0\right). \tag{3}$$

In the same way conjugacy between T and B with respect to k_B implies

$$B = \left(0,\ \frac{1}{2} - \frac{\cos t}{1+\cos t}\,,\ \pm\frac{\sqrt{1+2\cos t}}{1+\cos t}\right). \tag{4}$$

The upper sign of the z-coordinate corresponds to the upper half of Ψ.[2]

From (2) and (4) we compute the squared length of the line segment AB as

$$\overline{AB}^2 = \sin^2 t + \left(1+\cos t - \frac{\cos t}{1+\cos t}\right)^2 + \frac{1+2\cos t}{(1+\cos t)^2} = \sin^2 t + (1+\cos t)^2 - 2\cos t + 1\,,$$

which results in

Theorem 1: *All line segments AB of the torse Ψ are of equal length*

$$\overline{AB} = \sqrt{3}\,. \tag{5}$$

This surprising result has already been proved in [7]. But probably also P. Schatz *was aware of this result when he took out a patent for the Oloid (cf. [8]) in 1933 (see also [9], Figures 155, 156 and p. 122).*

Let u denote the arc-length of k_B, starting on the positive y-axis. Then $A \in k_A$ and $B \in k_B$ are points of the same generator of Ψ if and only if the parameters t of A and u of B obey the involutive relation

$$\cos u = -\frac{\cos t}{1+\cos t} \quad \text{or} \quad \cos^2\frac{t}{2}\,\cos^2\frac{u}{2} = \frac{1}{4}\,. \tag{6}$$

For real generators of Ψ the condition $1 + 2\cos t \geq 0$ is necessary. By the restriction

$$-\frac{2\pi}{3} < t < \frac{2\pi}{3} \quad \text{and} \quad -\frac{2\pi}{3} < u < \frac{2\pi}{3} \tag{7}$$

we avoid vanishing denominators. It has to be noted that for Ψ the parametrization by t becomes singular at $t = \pm 2\pi/3$.

In the following we restrict each generator of Ψ to the line segment AB. Thus we obtain just the boundary of the *convex hull* of k_A and k_B.

2 Development of the Torse Ψ

When Ψ is developed into a plane τ, then the circles k_A, k_B are isometrically transformed into planar curves k_A^d, k_B^d, respectively. It is well-known from Differential Geometry (see e.g. [11], p. 209 or [12], p. 72) that at corresponding points $A \in k_A \subset \Psi$ and $A^d \in k_A^d \subset \tau$ the geodesic curvatures are equal. This can be expressed in a more geometric way as follows (cf. [4], p. 295): When τ is specified as the tangent plane of Ψ along the generator AB, then the curvature

[2] In the generalization presented in [5] the circles k_A, k_B are replaced by congruent ellipses with a common axis.

center K of k_A^d at $A^d = A$ is located on the curvature axis of k_A at A, which is the axis of revolution of (the curvature circle) k_A (see Fig. 2, compare Fig. 3). Since $K = (-\frac{1}{2}, 0, \pm k)$[3] is aligned with T and B, we get for the squared curvature radius

$$\rho^2 = \overline{AK}^2 = 1 + k^2 = \frac{2 + 2\cos t}{1 + 2\cos t}.$$

Hence the curvature κ of k_A^d reads

$$\frac{1}{\rho} = \kappa(t) = \sqrt{\frac{1 + 2\cos t}{2(1 + \cos t)}}. \tag{8}$$

This is the so-called natural equation of k_A^d with arc-length t.[4]

In order to deduce an explicit representation of k_A^d, we choose τ as the tangent plane at the point $U \in k_A$ with minimal y-coordinate. In τ we introduce a cartesian coordinate system with origin $U^d = U$ and axes I and II (see Fig. 3). We define the first coordinate-axis I parallel to the tangent vector of k_A at U. Then due to Riccati's formula (see e.g. [10], p. 44) we get

$$\begin{aligned} I_A(t) &= I_0 + \int_0^t \cos\alpha(t)\,dt \\ II_A(t) &= II_0 + \int_0^t \sin\alpha(t)\,dt \end{aligned} \qquad \text{for} \quad \alpha(t) := \alpha_0 + \int_0^t \kappa(t)\,dt \tag{9}$$

with the specifications $\alpha_0 = I_0 = II_0 = 0$. By integration of (8) we obtain

$$\alpha(t) = 2\arcsin\frac{\sqrt{6}\,\sin t}{3\sqrt{1 + \cos t}} - \arcsin\frac{\sqrt{3}\,\tan\frac{t}{2}}{3}. \tag{10}$$

Theorem 2: *In the cartesian coordinate system (I, II) (see Fig. 4) the arc length parametrization of the development k_A^d of the circle k_A reads*

$$\begin{aligned} I_A(t) &= \frac{2\sqrt{3}}{9}\left[\sqrt{2(1 + 2\cos t)(1 - \cos t)} + \arccos\frac{\sqrt{2}\cos t}{\sqrt{1 + \cos t}}\right] \\ II_A(t) &= \frac{\sqrt{3}}{9}\left[4(1 - \cos t) + \ln\frac{2}{1 + \cos t}\right]. \end{aligned} \tag{11}$$

Proof: The integrals in the left column of (9) could not be immediately solved with the use of common computer-algebra-systems. We succeeded as follows: The integral for $II_A(t)$ can be transformed into

$$\begin{aligned} II_A(t) &= \int_0^t \sin\left(2\arcsin\frac{\sqrt{6}\,\sin t}{3\sqrt{1 + \cos t}} - \arcsin\frac{\sqrt{3}\,\tan\frac{t}{2}}{3}\right)dt = \\ &= \int_0^t \left[\frac{4\sqrt{6}\,\sin\frac{t}{2}\,(1 + 2\cos t)}{9\sqrt{1 + \cos t}} - \frac{\sqrt{3}\,\tan\frac{t}{2}\,(4\cos t - 1)}{9}\right]dt, \end{aligned}$$

and this gives rise to the second equation in (11). From

$$\frac{dII_A}{dt} = \frac{\sqrt{3}}{9}\sin t\left(4 + \frac{1}{1 + \cos t}\right) = \sin\alpha \tag{12}$$

[3]The sign of the z-coordinate is equal to that of B in (4).

[4]Note $\dot\rho(0) = 0$, but $\ddot\rho(0) = \frac{1}{18}\sqrt{3} \neq 0$. This proves that at U^d there is exactly a four-point contact between k_A^d and its curvature circle (see Fig. 4 or Fig. 5).

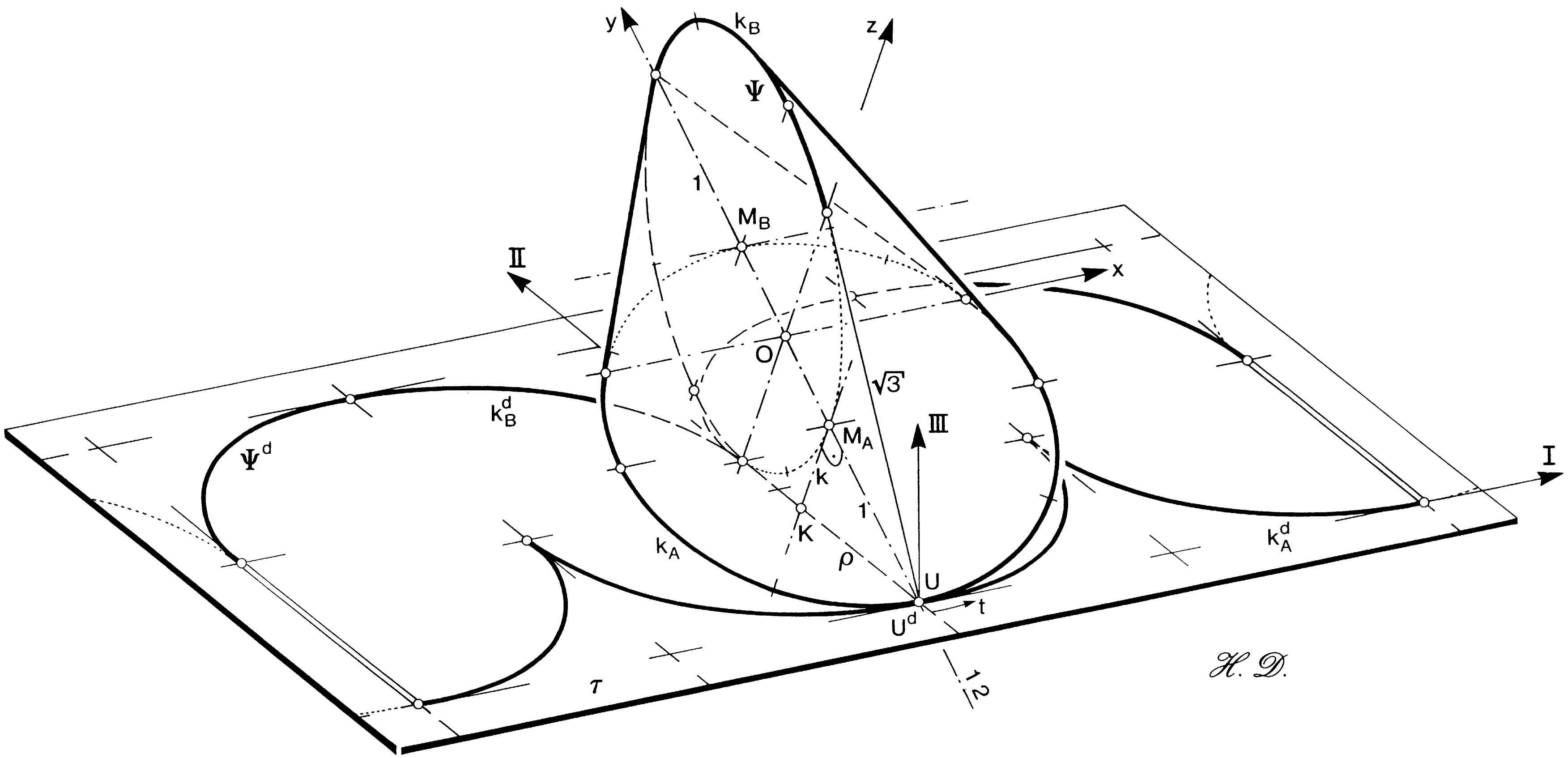

Figure 3: Axonometric view of the Oloid and its development into τ

due to (9) we obtain

$$\frac{dI_A}{dt} = \cos\alpha = \sqrt{1 - \left(\frac{dII_A}{dt}\right)^2} = \frac{\sqrt{6}(1+2\cos t)^{\frac{3}{2}}}{9\sqrt{1+\cos t}}. \tag{13}$$

Then the integration can be carried out using the substitution $\bar{t} := \tan\frac{t}{2}$. The first quarter of the developed curve k_A^d ends at

$$\left(I_A\left(\tfrac{2\pi}{3}\right), II_A\left(\tfrac{2\pi}{3}\right)\right) = \left(\frac{2\pi\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}(3+\ln 2)\right) \approx (1.2092, 1.4215). \quad \square$$

There is an analogous representation of the developed image k_B^d of the circle k_B in terms of its arc-length u. The curves k_A^d and k_B^d are congruent since halfturns about the axes $x \pm z = y = 0$ interchange k_A and k_B while the Oloid is transformed into itself. However, based on (11) and due to (5) the curve k_B^d can also be parametrized in the form

$$\begin{aligned} I_B(t) &= I_A(t) + \sqrt{3}\cos e_{AB}, \\ II_B(t) &= II_A(t) + \sqrt{3}\sin e_{AB}. \end{aligned} \tag{14}$$

Here the angle $e_{AB} = \alpha + \gamma$ (see Fig. 4) defines the direction of the developed generator A^dB^d. Angle α has already been computed in (12) and (13). γ is the angle made by the generator AB of Ψ and the tangent vector

$$\mathfrak{v}_A = (\cos t, \sin t, 0) \tag{15}$$

of k_A at A. The dot product of $\mathfrak{v}_A$ and the vector $\overrightarrow{AB}$ according to (2) and (4) gives

$$\sqrt{3}\,\cos\gamma = \mathfrak{v}_A \cdot \overrightarrow{AB} = -\sin t\cos t + \sin t + \sin t\cos t - \frac{\sin t\cos t}{1+\cos t} = \frac{\sin t}{1+\cos t}.$$

Elementary trigonometry leads to

$$\sin\gamma = \sqrt{\frac{2(1+2\cos t)}{3(1+\cos t)}} \tag{16}$$

and finally to

$$\begin{aligned} \sin e_{AB} &= \frac{7+7\cos t+4\cos^2 t}{9(1+\cos t)} \\ \cos e_{AB} &= -\frac{2\sqrt{2}(2+\cos t)\sqrt{(1-\cos t)(1+2\cos t)}}{9(1+\cos t)}. \end{aligned} \tag{17}$$

We substitute these formulas in (14). Then due to (11) we obtain

Theorem 3: *In the cartesian coordinate system (I, II) of τ (see Fig. 4 or Fig. 3) the development k_B^d of the circle k_B has the parametrization with respect to the arc-length t of k_A as follows:*

$$\begin{aligned} I_B(t) &= \frac{2\sqrt{3}}{9}\left[\arccos\frac{\sqrt{2}\cos t}{\sqrt{1+\cos t}} - \frac{\sqrt{2(1-\cos t)(1+2\cos t)}}{(1+\cos t)}\right] \\ II_B(t) &= \frac{\sqrt{3}}{9}\left[\ln\frac{2}{1+\cos t} + \frac{11+7\cos t}{1+\cos t}\right]. \end{aligned} \tag{18}$$

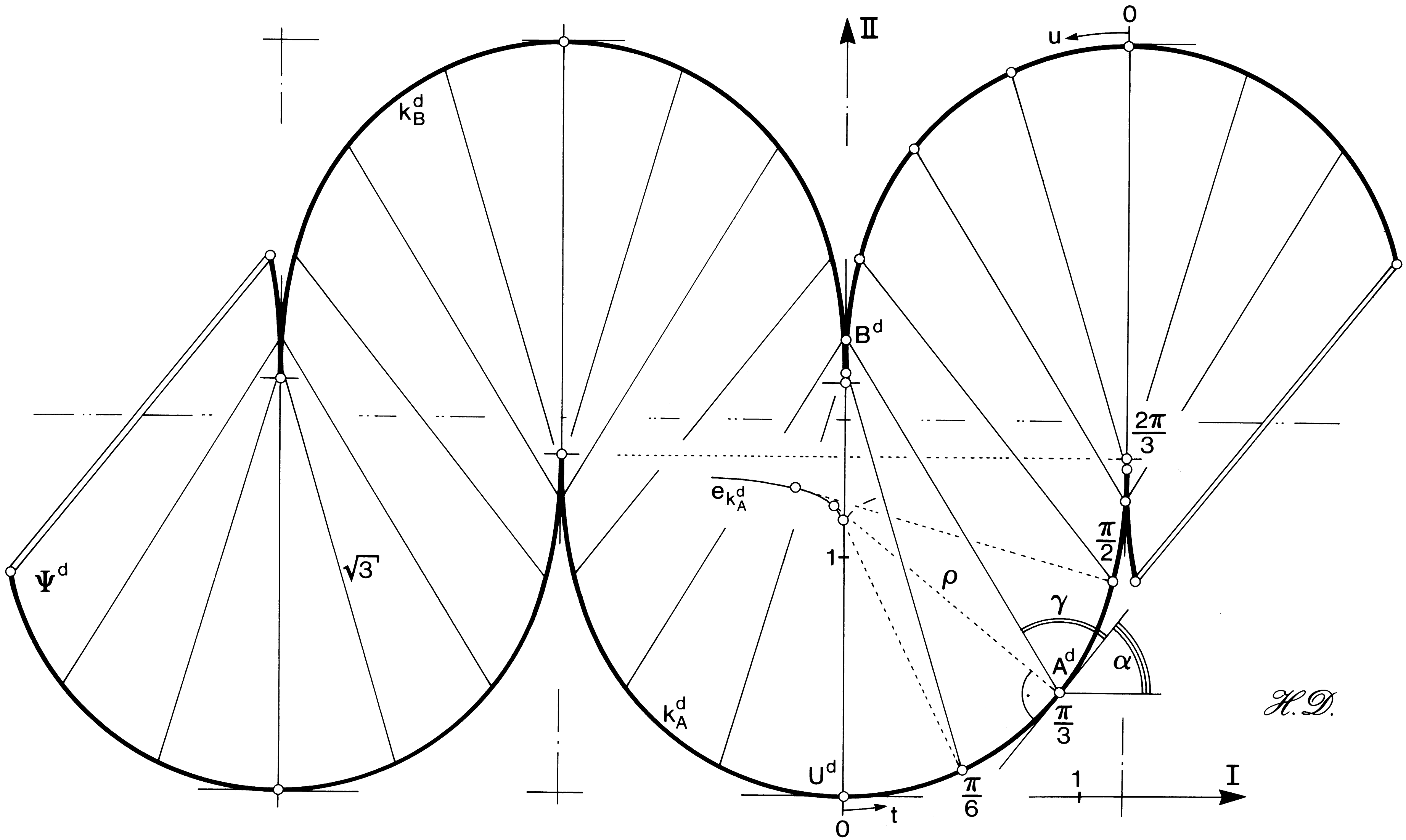

Figure 4: The development of the Oloid with the images k_A^d of k_A and k_B^d of k_B together with the evolute $e_{k_A^d}$ of k_A^d

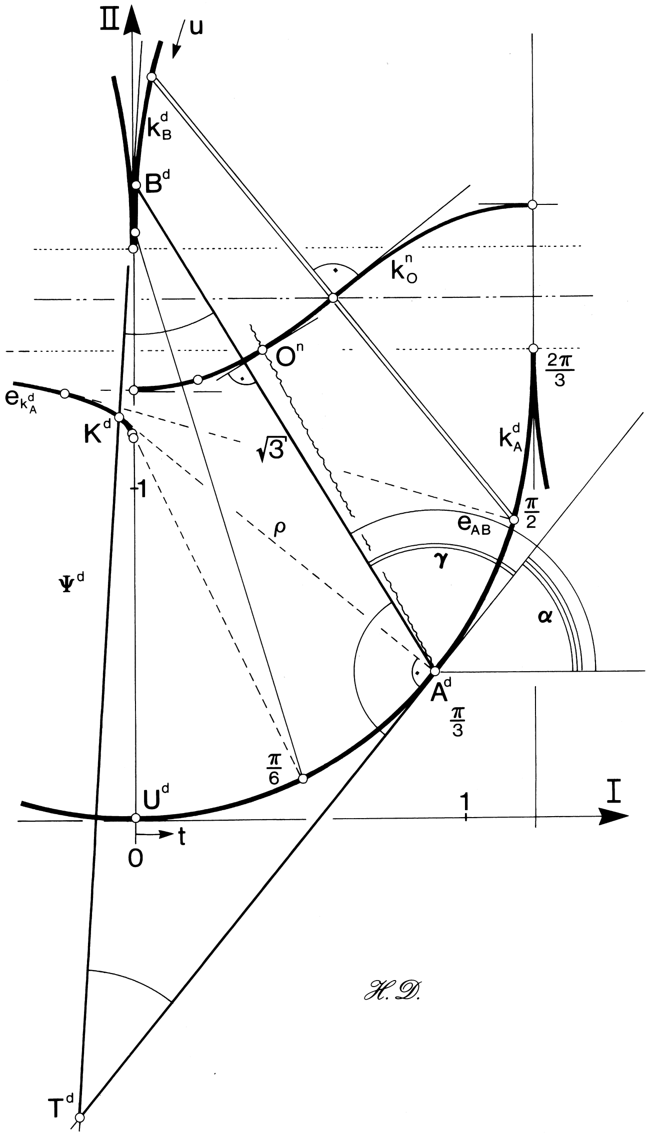

Figure 5: Detail of Fig. 4 with the image k_O^n of the center curve k_O under orthogonal projection into τ

In a similar way also the evolute $e_{k_A^d}$ *of* k_A^d *(see Fig. 4 or Fig. 5) can be computed. The parameter representation*

$$\begin{aligned} I_K(t) &= I_A(t) - \rho \sin\alpha \\ II_K(t) &= II_A(t) + \rho \cos\alpha \end{aligned}$$

of $e_{k_A^d}$ *makes use of the curvature radius* ρ *according to (8). From (12) and (13) we obtain*

$$\begin{aligned} \rho \sin\alpha &= \frac{(5 + 4\cos t)\sqrt{6(1-\cos t)}}{9\sqrt{1+2\cos t}} \\ \rho \cos\alpha &= \frac{2\sqrt{3}}{9}(1 + 2\cos t) \end{aligned}$$

and finally as parametrization of the evolute $e_{k_A^d}$ of k_A^d

$$(19)\qquad \begin{aligned} I_K(t) &= \frac{2\sqrt{3}}{9}\arccos\frac{\sqrt{2}\,\cos t}{\sqrt{1+\cos t}} - \frac{\sqrt{2(1-\cos t)}}{\sqrt{3(1+2\cos t)}} \\ II_K(t) &= \frac{\sqrt{3}}{9}\left[6+\ln\frac{2}{1+\cos t}\right]. \end{aligned}$$

The evolute $e_{k_A^d}$ obviously (see Fig. 4) does not pass through the cusps of k_A^d; the curvature radius ρ tends to infinity. This reveals that these cuspidal points are not ordinary. For k_B^d the Taylor*-series expansion of the parameter representation (18) at $t = 0$ is*

$$I_B(t) = \frac{1}{360}\,t^5 + O(t^7), \qquad II_B(t) = \sqrt{3} + \frac{\sqrt{3}}{12}\,t^2 + O(t^4).$$

Therefore the singularities of k_A^d and k_B^d are of order 2 and class 3 (German: Rückkehrflachpunkte).

3 Motions Related to the Oloid

According to Fig. 3 we assume that the Oloid is rolling on the upper side of τ. In the following we therefore choose for point B in (4) the negative z-coordinate. For the sake of brevity we substitute

$$(20)\qquad s := \sin t \ \text{ and } \ c := \cos t \ \text{ with } \ -\tfrac{1}{2} < c \le 1\,, \quad -1 \le s \le 1\,, \quad s^2 + c^2 = 1\,.$$

In order to describe the rolling of Ψ on τ we introduce a moving frame of Ψ with origin $A \in k_A$. The first vector of this frame is the tangent vector $\mathfrak{v}_A$ according to (15). The second vector $\mathfrak{w}_A$ perpendicular to $\mathfrak{v}_A$ is specified in the tangent plane τ. We define

$$(21)\qquad \mathfrak{w}_A := \frac{1}{\sin\gamma}\left(\frac{1}{\sqrt{3}}\,\overrightarrow{AB} - \mathfrak{v}_A\cos\gamma\right) = \frac{1}{\sqrt{2(1+c)}}\left(-s\sqrt{1+2c}\,,\ c\sqrt{1+2c}\,,\ -1\right).$$

The vector

$$(22)\qquad \mathfrak{n}_A := \mathfrak{v}_A \times \mathfrak{w}_A = \frac{1}{\sqrt{2(1+c)}}\left(-s\,,\ c\,, \sqrt{1+2c}\right)$$

perpendicular to τ completes this cartesian frame. $\mathfrak{n}_A$ is pointing to the interior of Ψ.

While the Oloid is rolling on the fixed plane τ, the frame $(A;\mathfrak{v}_A,\mathfrak{w}_A,\mathfrak{n}_A)$ shall be moving along Ψ in such a way, that A is the running point of contact between k_A and τ. This implies that $A \in k_A$ is always coincident with the corresponding point $A^d \in k_A^d$. Therefore the elements of the moving frame get the following coordinates with respect to the cartesian coordinate system $(U^d; I, II, III)$ attached to τ:

$$(23)\qquad \begin{aligned} A &= (I_A(t), II_A(t), 0)\,, & \mathfrak{v}_A &= (\cos\alpha, \sin\alpha, 0)\,, \\ \mathfrak{w}_A &= (-\sin\alpha, \cos\alpha, 0)\,, & \mathfrak{n}_A &= (0,0,1)\,. \end{aligned}$$

Let (x,y,z) denote the coordinates of any point P attached to the Oloid. The required representation of the motion consists of a matrix equation which allows to compute the instantaneous coordinates (I, II, III) of point P with respect to the fixed plane τ, in dependence of the motion parameter t. In order to obtain this equation we firstly compute the coordinates (ξ, η, ζ) of P with respect to the moving frame $(A; \mathfrak{v}_A, \mathfrak{w}_A, \mathfrak{n}_A)$. Though P is attached to Ψ, these coordinates are dependent on t. From (2), (15), (21) and (22) we get

$$(24)\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -\frac{1}{2} - c \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2(1+c)}} \begin{pmatrix} c\sqrt{2(1+c)} & -s\sqrt{1+2c} & -s \\ s\sqrt{2(1+c)} & c\sqrt{1+2c} & c \\ 0 & -1 & \sqrt{1+2c} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Secondly, according to (23) the motion of the moving frame with respect to τ (see Fig. 3) reads

$$\begin{pmatrix} I \\ II \\ III \end{pmatrix} = \begin{pmatrix} I_A(t) \\ II_A(t) \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Now we eliminate (ξ, η, ζ) from these two matrix equations with orthogonal 3×3-matrices. After substituting (11), (12) and (13) we obtain by straight-forward calculation

Theorem 4: *Based on the cartesian coordinate systems (x, y, z) in the moving space and (I, II, III) in the fixed space, the rolling of the Oloid on the tangent plane τ can be represented as*

$$\begin{pmatrix} I \\ II \\ III \end{pmatrix} = \frac{\sqrt{3}}{9} \begin{pmatrix} \dfrac{cs\sqrt{1+2c}}{2(1+c)\sqrt{2(1+c)}} + 2\arccos\dfrac{c\sqrt{2}}{\sqrt{1+c}} \\ \dfrac{15+13c-c^2}{2(1+c)} + \ln\dfrac{2}{1+c} \\ \dfrac{3\sqrt{3}\,(2+c)}{2\sqrt{2(1+c)}} \end{pmatrix} + \begin{pmatrix} a_{ij} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \textit{where}$$

$$\begin{pmatrix} a_{ij} \end{pmatrix} = \frac{\sqrt{3}}{9} \begin{pmatrix} \dfrac{(5+c)\sqrt{1+2c}}{\sqrt{2(1+c)}} & \dfrac{(2+c)s\sqrt{1+2c}}{(1+c)\sqrt{2(1+c)}} & \dfrac{(5+4c)s}{(1+c)\sqrt{2(1+c)}} \\ \dfrac{(c-1)s}{1+c} & \dfrac{5+5c-c^2}{1+c} & -\dfrac{(1+2c)\sqrt{1+2c}}{1+c} \\ -\dfrac{3s\sqrt{3}}{\sqrt{2(1+c)}} & \dfrac{3c\sqrt{3}}{\sqrt{2(1+c)}} & \dfrac{3\sqrt{3(1+2c)}}{\sqrt{2(1+c)}} \end{pmatrix}.$$

Here c and s stand for $\cos t$ and $\sin t$, respectively, while the motion parameter t obeys (7).

The first vector on the right side of this matrix equation represents the path k_O of the Oloid's center O under this rolling motion. In particular, the third coordinate of this vector gives the oriented distance

$$(25)\qquad r := \frac{2+c}{2\sqrt{2(1+c)}}$$

between O and the tangent plane for each t. Due to the introduced moving frame, this distance r equals the dot product $\mathfrak{n}_A \cdot \overrightarrow{AO}$. One can verify that for each t the velocity vector

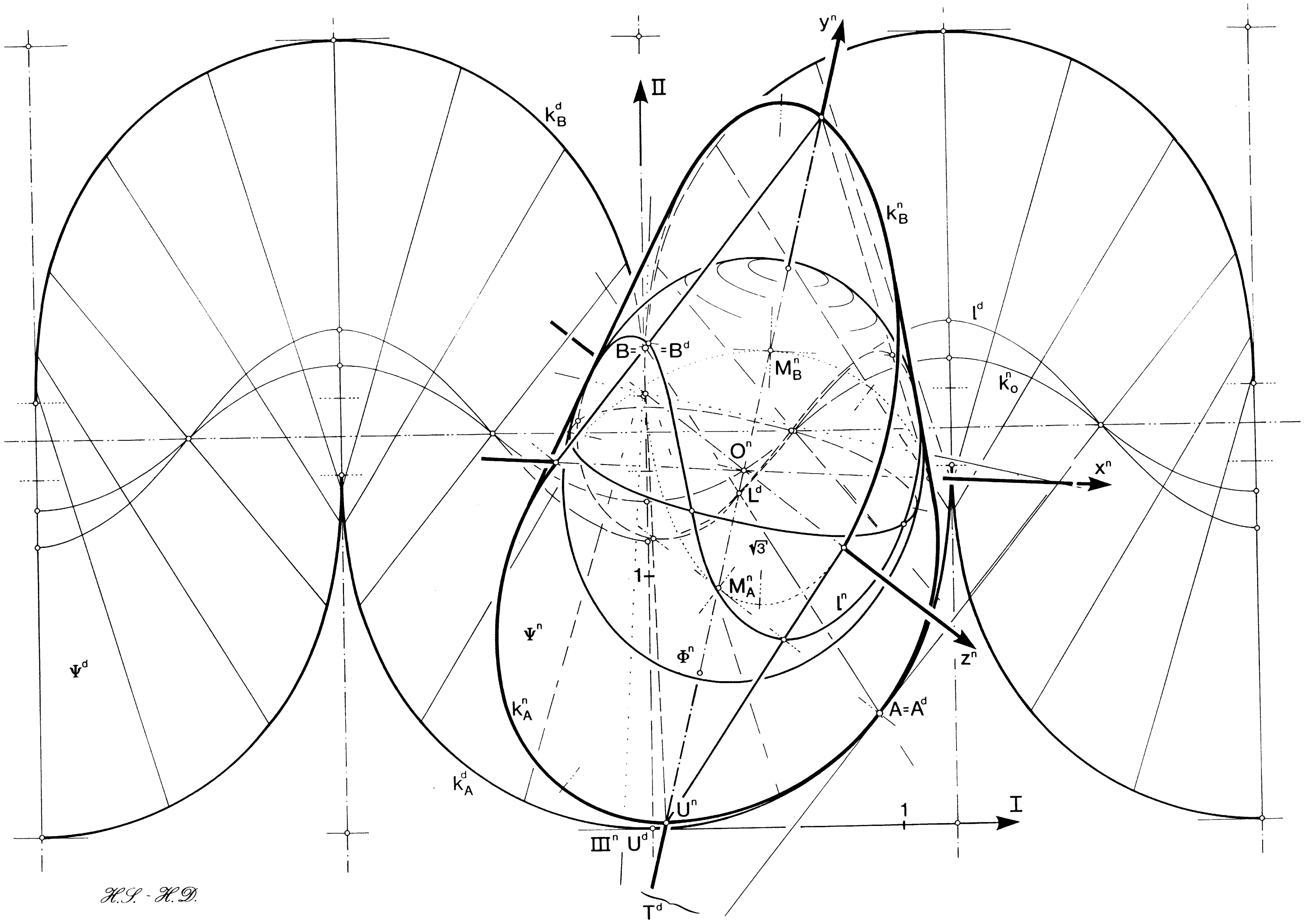

Figure 6: The Oloid and the inscribed ellipsoid Φ are rolling on τ while the curve l of contact between Ψ and Φ traces l^d

of the center curve k_O is perpendicular to the axis A^dB^d of the instantaneous rotation[5] (see orthogonal view k_O^n of k_O in Fig. 5 or Fig. 6).

The rolling of the Oloid is truly staggering. It is related to the Turbula motion (see [13] or [7][6] and the references there) which is used for shaking liquids. It turns out that the Turbula motion is inverse to the motion of the moving frame in (24).

The circles k_A and k_B can also be seen as singular surfaces $\widehat{k}_A, \widehat{k}_B$ of 2nd class. The coordinates $(u_0 : u_1 : u_2 : u_3)$ of their tangent planes

$$u_0 + u_1 x + u_2 y + u_3 z = 0$$

match the "tangential equations"

$$\widehat{k}_A\colon\ 4u_0^2 - 4u_0u_2 - 4u_1^2 - 3u_2^2 = 0\,, \quad \widehat{k}_B\colon\ 4u_0^2 + 4u_0u_2 - 3u_2^2 - 4u_3^2 = 0\,.$$

Then due to a standard theorem of Projective Geometry the torse Ψ is not only tangent to k_A and k_B but to all surfaces of 2nd class included in the range which is spanned by $\widehat{k}_A$ and $\widehat{k}_B$. Among these surfaces there is an ellipsoid Φ of revolution[7] obeying the equation

$$(26) \qquad \Phi\colon\ 6x^2 + 4y^2 + 6z^2 = 3 \quad \text{or} \quad \widehat{\Phi}\colon\ \tfrac{1}{2}(\widehat{k}_A + \widehat{k}_B) = 4u_0^2 - 2u_1^2 - 3u_2^2 - 2u_3^2 = 0$$

with focal points M_A, M_B and semi-axes $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}}$ (cf. [13], p. 31). The curve l of contact between Φ and the torse Ψ is located on cylinders which are the images of k_A and k_B, respectively, in the polarity with respect to Φ. Therefore this curve has the representations

$$(27) \qquad \begin{aligned} &l\colon\ 3x^2 + \left(y - \tfrac{1}{2}\right)^2 = 3z^2 + \left(y + \tfrac{1}{2}\right)^2 = 1 \quad \text{or} \\ &x = \frac{s}{2+c}\,, \quad y = -\frac{3c}{2(2+c)}\,, \quad z = \frac{\pm\sqrt{1+2c}}{2+c}\,. \end{aligned}$$

Together with the Oloid also the inscribed ellipsoid Φ is rolling on τ. In the fixed plane τ the point of contact with the rolling ellipsoid traces a curve l^d. The parameter representation

$$(28) \qquad l^d\colon\ I = \frac{2\sqrt{3}}{9}\arccos\frac{c\sqrt{2}}{\sqrt{1+c}}\,, \quad II = \frac{\sqrt{3}}{9}\left[\ln\frac{2}{1+c} + \frac{3(5+c)}{2+c}\right], \quad III = 0$$

of this isometric image of $l \subset \Psi$ is obtained by transforming the coordinates of l given in (27) (negative sign) under the matrix equation of Theorem 4.

Fig. 6 shows not only the fixed tangent plane τ with the developed curves k_A^d, k_B^d and l^d in true shape. In this figure also an orthogonal view of the Oloid with the inscribed ellipsoid Φ and the curve l of tangency is displayed.

[5] In general the instantaneous motion is a helical motion. However when a torse is rolling on a plane, the helical parameter must vanish (cf. [3], p. 161 or [6]).

[6] In this paper a very particular plane-symmetric six-bar loop is studied which is also displayed in [1], Figure 1. In each position of this loop and for each two opposite links Σ, Σ' there is a plane τ of symmetry. It turns out that relatively to Σ these planes τ are tangent to a torse of type Ψ. The Turbula motion is the motion of Σ relative to τ, when in τ the generator of the torse is kept fixed.

[7] In the cases treated in [5] k_A and k_B are ellipses, but Φ is a sphere. This implies that the center O of gravity has a constant distance to τ during the rolling motion.

4 Surface Area and Volume of the Oloid

As the development of a torse is locally an isometry, the computation of the area of Ψ can be carried out either in the 3-space or after the development into the plane τ. We prefer the latter and use a formula given in [2], p. 118, eq. (5): The area swept out by the line segment AB under a planar motion for $t_0 \le t \le t_1$ can be computed according to

$$S = \int_{t_0}^{t_1} \left\| \tfrac{1}{2}(\mathfrak{v}_A + \mathfrak{v}_B) \times \overrightarrow{AB} \right\| dt\,, \tag{29}$$

where $\mathfrak{v}_A, \mathfrak{v}_B$ are the velocity vectors of the endpoints. For vectors in $\mathbb{R}^2$ the norm in this formula can be cancelled which gives rise to an even oriented area.

In the coordinate system (I, II) of τ we obtain due to (14) and (11)

$$\overrightarrow{AB} = \left(\sqrt{3}\,\cos e_{AB}\,,\ \sqrt{3}\,\sin e_{AB}\right), \quad \mathfrak{v}_A = \left(\frac{dI_A}{dt}\,,\ \frac{dII_A}{dt}\right),$$

$$\mathfrak{v}_B = \left(\frac{dI_A}{dt} - \sqrt{3}\,\sin e_{AB}\,\frac{d\,e_{AB}}{dt}\,,\ \frac{dII}{dt} + \sqrt{3}\,\cos e_{AB}\,\frac{d\,e_{AB}}{dt}\right)$$

and according to (12) and (13)

$$\frac{dS}{dt} = \sqrt{3}\left(\frac{dI_A}{dt}\,\sin e_{AB} - \frac{dII_A}{dt}\,\cos e_{AB}\right) - \frac{3}{2}\,\frac{d\,e_{AB}}{dt} = \sqrt{3}\,\sin\gamma - \frac{3}{2}\,\frac{d\,e_{AB}}{dt}\,.$$

Eq. (16) and the derivation of (17) lead to

$$\frac{dS}{dt} = \frac{\sqrt{2(1+2\cos t)}}{\sqrt{1+\cos t}} - \frac{3\sqrt{2}\,\cos t}{2\sqrt{(1+\cos t)(1+2\cos t)}}$$

which finally results in

$$\frac{dS}{dt} = \frac{2+\cos t}{\sqrt{2(1+\cos t)(1+2\cos t)}}\,. \tag{30}$$

After integration we obtain up to a constant k

$$S(t) = \frac{1}{2}\left[\arcsin\frac{1-4\cos t}{3} - \arcsin\frac{1+5\cos t}{3(1+\cos t)}\right] + k$$

and for the complete torse Ψ

$$S_\Psi = 8\left[S\left(\frac{\pi}{2}\right) - S(0)\right] = 4\left[S\left(\frac{2\pi}{3}\right) - S(0)\right] = 4\pi\,. \tag{31}$$

Theorem 5: *The surface area of the Oloid equals that of the unit sphere.*

The computation of the Oloid's volume starts from (30): Each surface element of Ψ is the base of a volume element forming a pyramid with apex O. Its altitude r has already been computed in (25) as it equals the distance between O and the corresponding tangent plane. Thus we obtain

$$dV = \frac{r}{3}\,dS = \frac{(2+\cos t)^2}{12(1+\cos t)\sqrt{1+2\cos t}}\,dt\,. \tag{32}$$

A numerical integration gives

$$V_\Psi = 8\left[V\left(\frac{\pi}{2}\right) - V(0)\right] \approx 3.05241\,. \tag{33}$$

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