Journal for Geometry and Graphics
Volume 1 (1997), No. 2, 119–133.

Affine and Projective Generalization of Wallace Lines

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Abstract. If one draws in a plane from a point X the perpendiculars onto the sides AB, BC, CA of a triangle ABC and if the feet of these perpendiculars $P \in AB$, $Q \in BC$, $R \in CA$ lie on a line — the Wallace line of X — then X lies on the circumcircle of the triangle ABC. We introduce two generalizations: If the affine feet P, Q, R lie on the affine Wallace line of X with respect to a center Z or if the projective feet P, Q, R lie on the projective Wallace line of X with respect to a center Z and an axis f then X lies on a conic.
Keywords: Wallace line, geometry of the triangle, collinear points
MSC 1994: 51M05, 51N10, 51N15

Introduction

If one draws from a point X in the Euclidean plane the perpendiculars to the sides AB, BC, CA of a triangle ABC and if the feet of these perpendiculars $P \in AB$, $Q \in BC$, $R \in CA$ lie on a line — the *Euclidean Wallace line*[1] of X for the triangle ABC — then X lies on the circumcircle of the triangle ABC (Fig. 1).

In the following we give an affine and a projective generalization of this statement (and of its converse, which also holds). Both generalizations give rise to further considerations which we omit for the time being. The generalizations show the following: If in the real affine plane, for a triangle ABC, the *affine feet* $P \in AB$, $Q \in BC$, $R \in CA$ of a given point X lie on a common line — namely the *affine* Wallace *line* of X with respect to the center Z — then X lies on a conic k (dependent on Z) in a two-parameter family of conics (Theorem 1). If Z is the orthocenter H, then k is the circumcircle of the triangle ABC. If, in the real projective plane, for a triangle ABC the *projective feet* $P \in AB, Q \in BC, R \in CA$ of a point X lie nontrivially on a common line — on the *projective Wallace line* of X with respect to an axis f — then X also lies on a conic (dependent on f) in a two-parameter family of conics (Theorem 5). Both families of conics contain pencils of conics. We distinguish, in particular, three pencils of

[1] after William Wallace 1797, see [7], p. 143, [1], p. 1234, [2], p. 45, [5], p. 855. This same line also appears in the literature as the Simson line.

ISSN 1433-8157/\$ 2.50

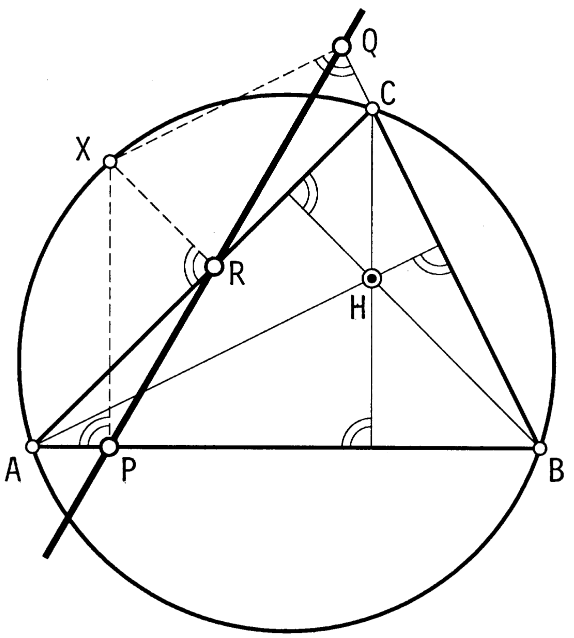

Figure 1: Euclidean Wallace line of X for the triangle ABC

conics within the affine family of conics. The conics of both families pass through the vertices of the triangle ABC. The projective and the affine Wallace lines corresponding to the points X of a conic envelop (as do the Euclidean Wallace lines, see, for example, [1], p. 1237f., [5], p. 637) a rational curve of fourth order and third class which is tangent with the sides of the triangle ABC.

The Euclidean Wallace lines and their neighborhood have been investigated repeatedly. Results obtained until about 1910 have been sketched by Simon [7], p. 143f. and Berkhan and Meyer [1], p. 1234f. In [4], p. 17f., Martini gives a summary of more recent results and previous generalizations. See also Shively [6], Coxeter andGreitzer [2] and Kratz [3]. Weiss [8] considers the Euclidean Wallace lines in the context of closure theorems.

Fig. 1 shows how to find for a point X of the Euclidean plane the feet P, Q, R: Consider the orthocenter H of the triangle ABC, draw through point X the parallels to the connecting lines (altitudes) HC, HA, HB and determine consecutively their points of intersection P, Q, R with the sides AB, BC, CA of the triangle. This construction allows the following affine and projective generalization.

1 Affine generalization

1.1 Affine Wallace lines

In the real affine plane replace the orthocenter H by an arbitrary point — the *center* Z — which determines three pairwise different transversals ZA, ZB, ZC, none of which is parallel to one of the triangle sides AB, BC, CA. Then draw through a point X of the affine plane the *parallels* to ZC, ZA, ZB, determine consecutively their intersection points P, Q, R with the sides AB, BC, CA of the triangle and determine the locus of the points X for which the *affine feet* P, Q, R lie on a common line. The affine generalization requires three pairwise different transversals ZA, ZB, ZC, which are not parallel to any side of the triangle. Therefore the center Z lies neither on AB, BC, CA nor on the parallel to a side of the triangle ABC through its opposite vertex. If Z assumes one of these special positions, the construction and the problem of the collinear position of the affine feet P, Q, R degenerate in an obvious way.

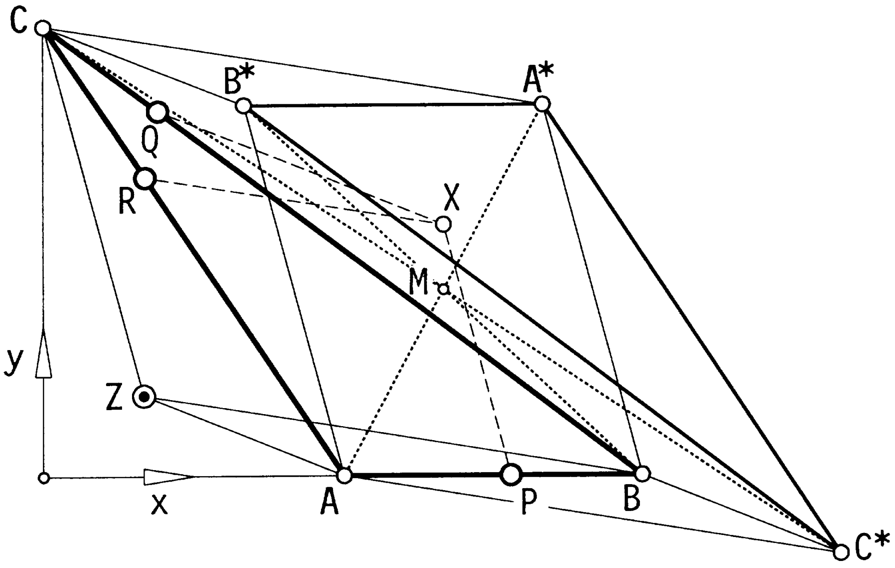

Figure 2: The affine feet P, Q, R

In order to investigate conveniently Euclidean distinguished triangles ABC and Euclidean distinguished positions of the center Z we use a *Cartesian xy*-coordinate system (Fig. 2), in which the vertices of the triangle ABC and the center Z have the coordinates[2]

(1) $$A = (a, 0),\ B = (b, 0),\ C = (0, c),\ Z = (p, q) \quad (\text{w.l.o.g. } b \neq 0,\ c > 0).$$

In this coordinate system the following lines have the given equations:

line	equation	line	equation
AB	$y = 0$	AB-parallel through C	$y - c = 0$
BC	$c(x - b) + by = 0$	BC-parallel through A	$c(x - a) + by = 0$
CA	$c(x - a) + ay = 0$	CA-parallel through B	$c(x - b) + ay = 0$

(2)

lines with equation		intersection point
ZB-parallel through A	$q(x - a) - (p - b)y = 0$	$C^* = (a + b - p, -q)$
ZA-parallel through B	$q(x - b) - (p - a)y = 0$	
ZC-parallel through B	$(c - q)(x - b) + py = 0$	$A^* = (b - p, c - q)$
ZB-parallel through C	$qx - (p - b)(y - c) = 0$	
ZA-parallel through C	$qx - (p - a)(y - c) = 0$	$B^* = (a - p, c - q)$
ZC-parallel through A	$(c - q)(x - a) + py = 0$	

Thus, the coordinates (p, q) of the center Z satisfy the conditions:

(3) $$\begin{aligned} q \neq 0, \quad c(p - b) + bq \neq 0, \quad c(p - a) + aq \neq 0, \\ q - c \neq 0, \quad c(p - a) + bq \neq 0, \quad c(p - a) + aq \neq 0. \end{aligned}$$

[2]For the derivation of affine statements the affine coordinate system $\{A = (0, 0); B = (1, 0), C = (0, 1)\}$ suffices. However, the replacement of the orthocenter H by a center Z results in an additional *Euclidean generalization* of the Euclidean Wallace lines and also provides new Euclidean statements.

As coordinates of the affine feet $P=(x_P,y_P)$, $Q=(x_Q,y_Q)$, $R=(x_R,y_R)$ of a moving point $X=(\xi,\eta)$ one calculates:

$$
\begin{aligned}
x_P &= \xi - \frac{p}{q-c}\eta, & y_P &= 0,\\
x_Q &= \frac{bq\xi - b(p-a)(\eta-c)}{bq+c(p-a)}, & y_Q &= \frac{-cq(\xi-b)+c(p-a)\eta}{bq+c(p-a)},\\
x_R &= \frac{aq\xi - a(p-b)(\eta-c)}{aq+c(p-b)}, & y_R &= \frac{-cq(\xi-a)+c(p-b)\eta}{aq+c(p-b)}.
\end{aligned} \tag{4}
$$

The affine feet P,Q,R lie on a common line if and only if

$$
\det\begin{pmatrix} x_P & y_P & 1\\ x_Q & y_Q & 1\\ x_R & y_R & 1\end{pmatrix} = 0. \tag{5}
$$

Taking (4) into consideration, (5) reads explicitly:

$$
\begin{gathered}
-qc\xi^2 + 2\Lambda\xi\eta + \Omega\eta^2 + q(a+b)c\xi + [\Omega(q-c)+\Lambda(p-a-b)]\eta - qabc = 0\\
\text{with}\quad \Lambda := \frac{pqc}{q-c},\quad \Omega := \frac{qab-(p-a)(p-b)c}{q-c}.
\end{gathered} \tag{6}
$$

From (6) one derives, with the aid of equations (2):

Theorem 1: *If in a triangle ABC in the real affine plane with sides AB, BC, CA the affine feet $P \in AB$, $Q \in BC$, $R \in CA$ of a point X with respect to a center $Z=(p,q)$ lie on a line (the affine Wallace line of X for the triangle ABC with respect to Z), then X lies on the conic $k(p,q)$ given by (6).*
The set of centers Z determines a two-parameter family of conics $k(p,q)$ with the parameters p,q. Each conic $k(p,q)$ passes through the vertices of the triangle ABC and through the points A^*, B^*, C^* (see Fig. 2) of the parallelograms ZAB^*C, ZBC^*A, ZCA^*B.
Consequently, the quadrangles AC^*A^*C, BA^*B^*A, CB^*C^*B are parallelograms inscribed in the conic $k(p,q)$ having the common midpoint $M=(x_M,y_M)$ with

$$
x_M = \frac{1}{2}(a+b-p),\quad y_M = \frac{1}{2}(c-q); \tag{7}
$$

$k(p,q)$ is for every center Z a central conic with the midpoint M.[3] *The triangles ABC and $A^*B^*C^*$ are mirror images of each other with respect to M.*

From (7) one obtains the following complement to Theorem 1.

Theorem 2: *In the real affine plane the center $Z=(p,q)$ is mapped by reflection with respect to the centroid S of the triangle ABC and with a subsequent dilatation out of S by*

[3]If one seeks (in the projectively extended plane) among the conics (6) the parabolas (identified by two coinciding points at infinity) it turns out that the centers Z necessarily lie on one of the sides AB, BC, CA of the triangle. However, these points Z do not provide three pairwise different transversals ZA, ZB, ZC and hence cannot be counted among the centers Z! The midpoint M of a central conic (6) is easily determined as the pole of the line at infinity.

the factor $\frac{1}{2}$ *onto the midpoint* $M = (x_M, y_M)$ *of the conic* $k(p,q)$ *(6). The orthocenter* H *in particular has the circumcenter* U *of the triangle* ABC *as image. If the centers* Z *lie on a line*

$$\alpha p + \beta q + \gamma = 0, \tag{8}$$

then the midpoints M *of the conics* $k(p,q)$ *corresponding to the centers* Z *according to (6) lie on the parallel line*

$$2\alpha x_M + 2\beta y_M - [\alpha(a+b) + \beta c + \gamma] = 0. \tag{9}$$

The lines (8) (9) are coincident if and only if they contain the centroid S. *This holds in particular for the Euler line of the triangle* ABC, *which contains its centroid* S, *orthocenter* H *and circumcenter* U. *Here the following holds:*

$$S = \left(\frac{a+b}{3}, \frac{c}{3}\right), \quad H = \left(0, -\frac{ab}{c}\right), \quad U = \left(\frac{a+b}{2}, \frac{ab+c^2}{2c}\right).$$

From Theorem 2 follows: If the centers Z lie on the parallel through C to the median s_A ($A \in s_A$) of the triangle ABC, then the midpoints M according to (9) lie on the s_A-parallel through the midpoint of the side AB. The same holds for the medians s_B ($B \in s_B$) and s_C ($C \in s_C$) of the triangle ABC.

If for a point $X = (\xi, \eta)$ the *midpoint* of the line segment ZX and two of the affine feet $P \in AB$, $Q \in BC$, $R \in CA$ are assumed to lie on a common line, for the position $X = (\xi, \eta)$ one again gets the conic $k(p,q)$. As further complement for Theorem 1 thus the following holds:

Theorem 3: *The affine Wallace line of a point* $X \in k(p,q)$ *for the triangle* ABC *with respect to the center* Z *bisects the segment* XZ.

For the case that Z is the orthocenter H of the triangle ABC, the reader is referred to [7], p. 143f., [2], p. 50, [6], p. 52 with respect to Theorem 3.

1.2 Pencils of conics in the family of conics $k(p,q)$

We show that the parameters p, q of the two-parameter family of conics $k(p,q)$ can be coupled in such a way that the result is a pencil of conics. If one assigns to (6) the representation

$$\begin{gathered} -qc(q-c)(\xi-a)(\xi-b) + \eta\,[2pqc\xi + \Phi(\eta+q-c) + pqc(p-a-b)] = 0 \\ \text{with} \quad \Phi := qab - (p-a)(p-b)c, \end{gathered} \tag{10}$$

it turns out that (10) represents an equation quadratic in p and q and that p can be cancelled if $q - c$ is a function linear in p without a constant term,

$$\mu(q-c) = \lambda p. \tag{11}$$

As expected, the center $Z = (p,q)$ lies neither on the AB-parallel through C nor on the sides CA, CB of the triangle. By (3) we thus obtain in (11)

$$\begin{gathered} q - c \neq 0, \text{ hence } (\mu : \lambda) \neq (1:0), \quad a(q-c) \neq -cp, \text{ hence } (\mu:\lambda) \neq (a:-c), \\ b(q-c) \neq -cp, \text{ hence } (\mu:\lambda) \neq (b:-c). \end{gathered}$$

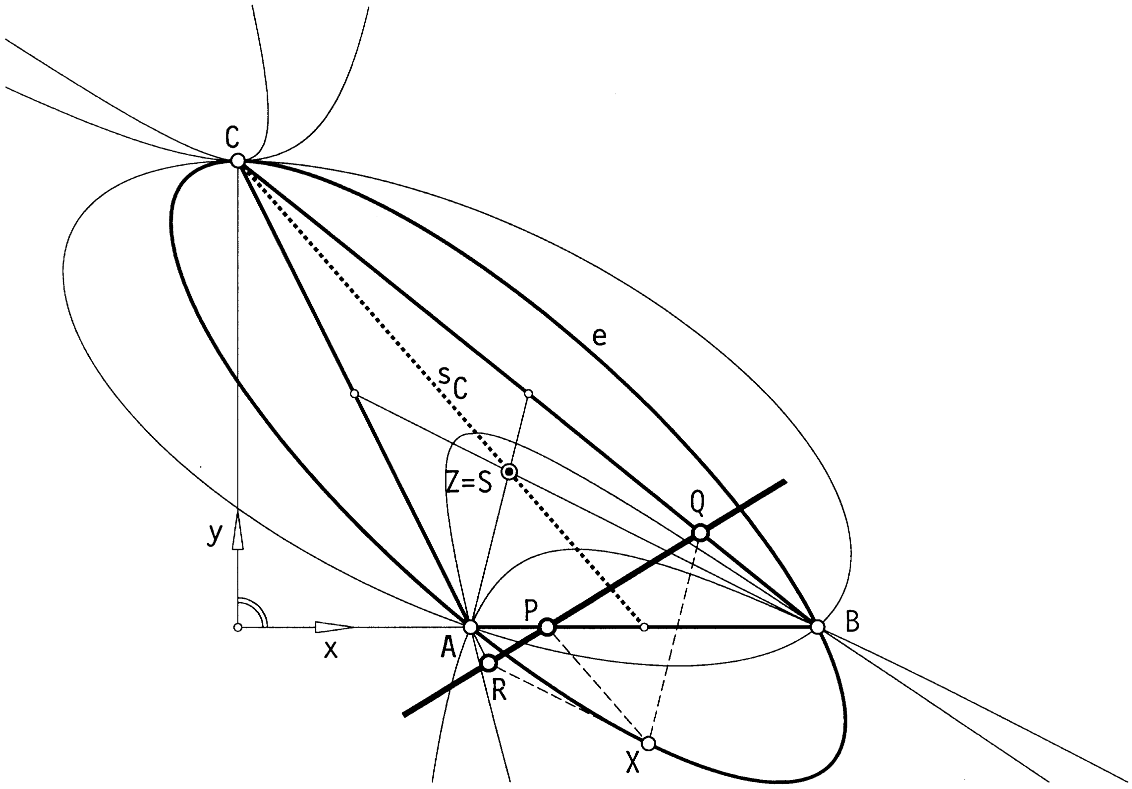

Figure 3: The transversal is a median

If (11) holds, then the centers Z lie on a transversal through a vertex of the triangle ABC that passes through C (but not on CA, CB and the AB-parallel through C), and the midpoints $M = (x_M, y_M)$ of the conics $k(p,q)$ corresponding to the centers Z, according to (9), lie on the parallel line

$$2\lambda x_M - 2\mu y_M - \lambda(a+b) = 0. \tag{12}$$

Using (11) in (10), one obtains for fixed (μ, λ) the following pencil of conics with q as pencil parameter

$$\begin{aligned}\{\lambda^2[ab\eta - c(\xi-a)(\xi-b)] + 2\lambda\mu c\xi\eta - \mu^2 c\eta(\eta-c)\}\, q + \\ + [\lambda^2 ab + \lambda\mu(a+b)c + \mu^2c^2]\,\eta(\eta-c) = 0.\end{aligned} \tag{13}$$

Keeping in mind that it is irrelevant which vertex of the triangle carries the label C, one then obtains

Theorem 4: *If in Theorem 1 the centers Z lie on a transversal through a vertex of a triangle ABC which is neither parallel to the opposite side of the vertex nor a triangle side, then the conics of the two-parameter family of conics $k(p,q)$ corresponding to the centers Z lie on a pencil of conics, which for the vertex C has the equation (13) with q as pencil parameter.*

As expected, each center Z determines three pairwise different transversals ZA, ZB, ZC. The conic $k(p,q)$ (p,q given) corresponding to $Z = (p,q)$ thus lies on at least three pencils of conics. We now consider as locus of the centers Z *distinguished transversals through a vertex* and the pencils of conics determined by them. Here we leave the simple proofs of some of the properties cited to the reader.

1.2.1 The transversal is a median

For $(\lambda : \mu) = (-2c : a + b)$ (see (11)), the median s_C is the transversal through a vertex of the triangle ABC (see Fig. 3). Then the pencil of conics (13) is a tangential pencil with the basic points A, B, C. All conics of the pencil are tangent at C with the AB-parallel through C, have s_C as diameter and consequently intersect s_C with a tangent parallel to AB.

The conic $k(p, q)$ for $p = (a + b)/3$, $q = c/3$, corresponding to the centroid $S = (p, q)$ of the triangle ABC is an ellipse e with midpoint S. The ellipse e lies in every pencil of conics determined by the medians s_A, s_B, s_C as locus of centers Z; the tangent to the ellipse e at A, B, C is parallel to BC, CA, AB respectively; hence e is (according to [5], p. 637) a Steiner ellipse.

1.2.2 The transversal is an altitude

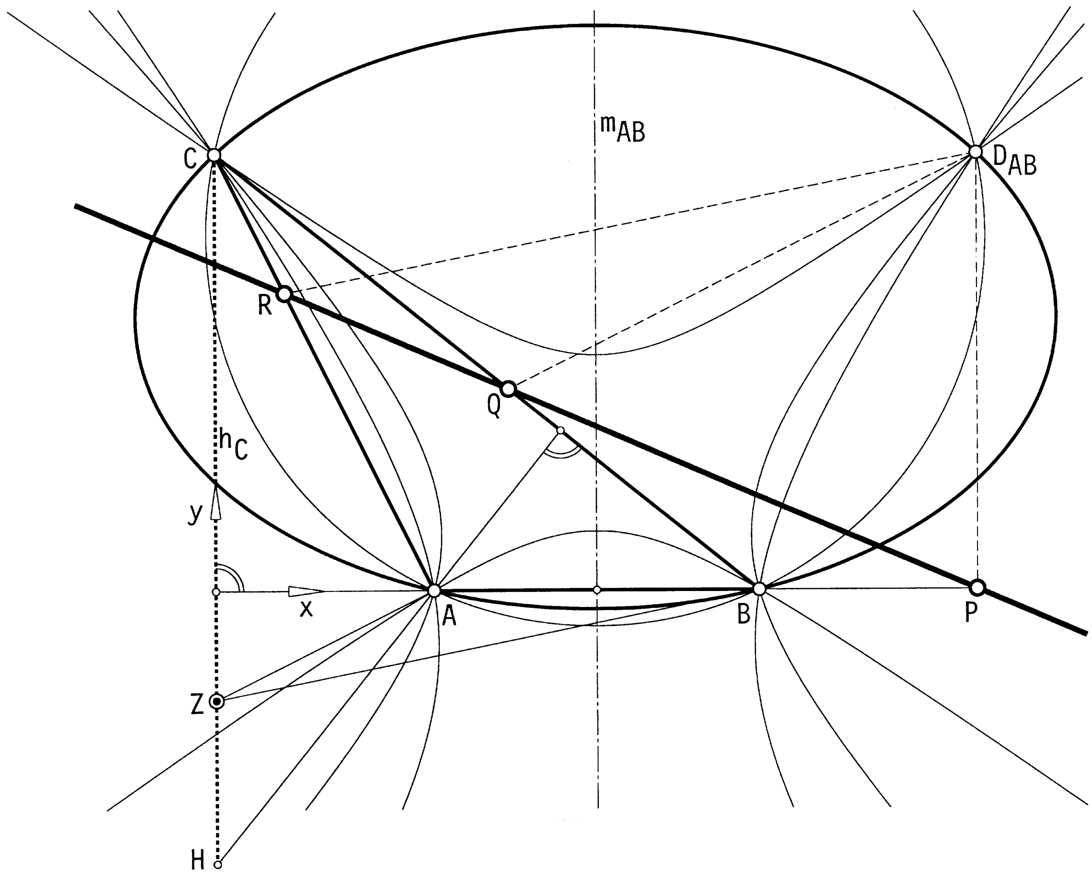

Figure 4: The transversal is an altitude

For $(\lambda : \mu) = (1 : 0)$ (see (11)), the transversal through a vertex is the altitude h_C $(p = 0)$ of the triangle ABC (see Fig. 4). The pencil of conics (13) corresponding to h_C has the equation:

$$[ab\eta - c(\xi - a)(\xi - b)]\, q + ab\eta(\eta - c) = 0. \tag{14}$$

The midpoints of the conics (14) lie (see Theorem 2) on the midperpendicular m_{AB} of AB. The pencil of conics (14) has the vertices A, B, C and the reflection point D_{AB} $(a + b, c)$ of the vertex C on m_{AB} as basic points. All conics (14) are symmetric to m_{AB}. The point of reflection D_{AB} and the corresponding points of reflection D_{BC} und D_{CA} lie on the circumcircle of the triangle ABC (which belongs to each of the three pencils of conics that are assigned to the altitudes). The circumcircle corresponds to the orthocenter H as center Z.

If Z lies on h_C, then the affine Wallace lines of the point $X = (\xi, \eta) = D_{AB}$ always pass through the foot P of the perpendicular from D to AB. The same holds for h_A with respect to D_{BC} and for h_B with respect to D_{CA}.

The conics of the pencil of conics (14) are the only conics $k(p,q)$ which contain the point D_{AB}.

1.2.3 The transversal is parallel to an altitude

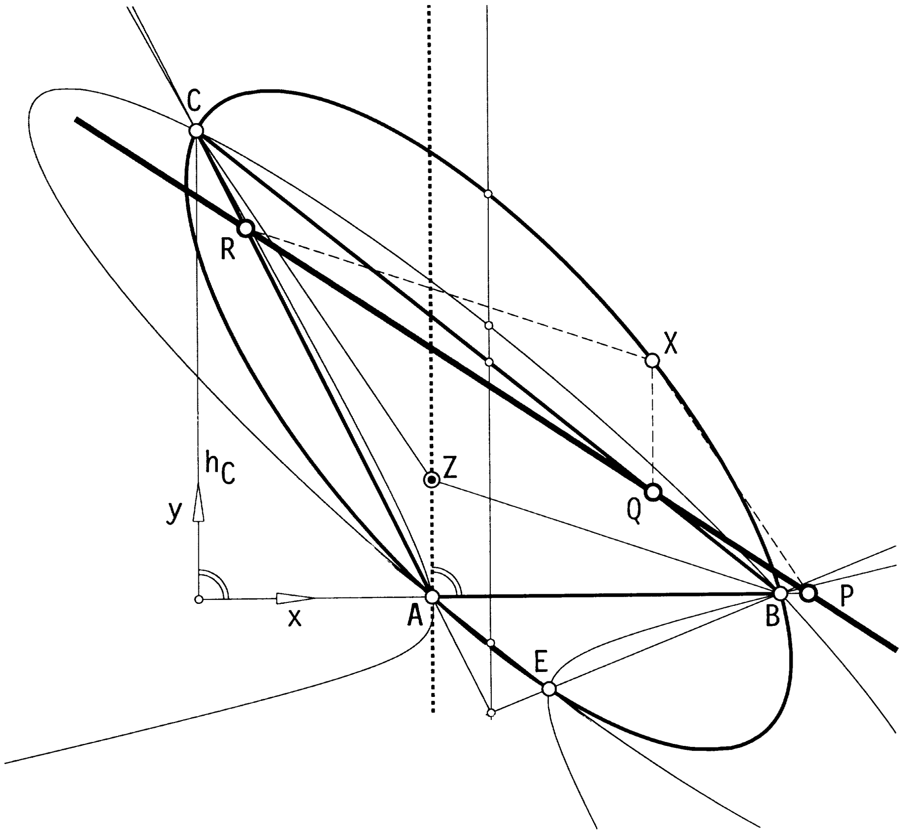

Figure 5: The transversal is parallel to an altitude

For $p = a$ the transversal is the h_C-parallel through A (see Fig. 5). For $p = a$, the family of conics (6) contains the pencil of conics with the equation

$$\omega a\eta[2c\xi + b(\eta - c)] + ab(\eta - c) - c\xi(\xi - a - b) = 0 \tag{15}$$

and with the pencil parameter $\omega = 1/(q-c)$. According to (7), the midpoints $M = (x_M, y_M)$ of the conics (15) lie on the h_C-parallel $x_M = b/2$. The basic points of the pencil of conics (15) are the vertices A, B, C and the point

$$E\left(b - a,\ \frac{c}{b}(2a - b)\right).$$

The quadrangle of basic points $ABCE$ is a trapezoid with the parallel sides BC and AE. Sides AC and BE intersect on the h_C-parallel $x_M = b/2$. The trapezoid $ABCE$ and the conics (15) lie skew-symmetrically with respect to the connecting line of the midpoints of the parallel sides BC and AE.

2 Projective generalization

2.1 Projective Wallace lines

The affine generalization of the Wallace lines uses the *parallels* to the transversals ZA, ZB, ZC through a point X of the affine plane. If the affine plane is extended projectively then (in the affine generalization) the transversals ZA, ZB, ZC must intersect the line at infinity f. Let the intersection points at infinity be labeled consecutively A_1, B_1, C_1. Then the connection lines A_1X, B_1X, C_1X need to be considered (see Fig. 6). For the projective generalization in the projective plane select the following:

1. a *center* Z, which determines three pairwise different *transversals* ZA, ZB, ZC, and dually
2. an *axis* ("line at infinity") f, which has three pairwise different *side points of intersection* $A_0 := f \cap BC$, $B_0 := f \cap CA$, $C_0 := f \cap AB$.
   Further, let the following hold:
3. Center Z and axis f do not meet, i.e. $Z \notin f$,
4. the point triples (Z, A, A_0), (Z, B, B_0), (Z, C, C_0) are not collinear or (equivalently) the line triples (f, BC, ZA), (f, CA, ZB), (f, AB, ZC) are not concurrent.

Then draw through a point X of the projective plane the connection lines XA_1, XB_1, XC_1, determine on the sides AB, BC, CA the *projective feet*

$$P := XC_1 \cap AB, \quad Q := XA_1 \cap BC, \quad R := XB_1 \cap CA \tag{16}$$

and calculate the locus of the points X for which the projective feet P, Q, R are collinear. Denote their connection line by PQR.

Remarks:

1. Condition (d) corresponds to the affine condition requiring that the transversals are not parallel to any side of the triangle (for example, ZC *not parallel to* AB, projectively: Z, C *and the point at infinity of* AB *not collinear*).
2. If in condition (d), Z, C, C_0 are collinear, for example, then $P := XC_1 \cap AB$ is the fixed point C_0 for all points X and the lines PQR lie in the pencil of lines at C_0.

Now let us base the analytical treatment on a projective $(x_0 : x_1 : x_2)$ coordinate system with the fundamental points $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$ and the center $Z = (1 : 1 : 1)$ as unit point. In this coordinate system, in which the points

$$Z_A := ZA \cap BC, \quad Z_B := ZB \cap CA, \quad Z_C := ZC \cap AB \tag{17}$$

have the coordinates $Z_A = (0 : 1 : 1)$, $Z_B = (1 : 0 : 1), gZ_C = (1 : 1 : 0)$, let

$$ax_0 + bx_1 + cx_2 = 0 \tag{18}$$

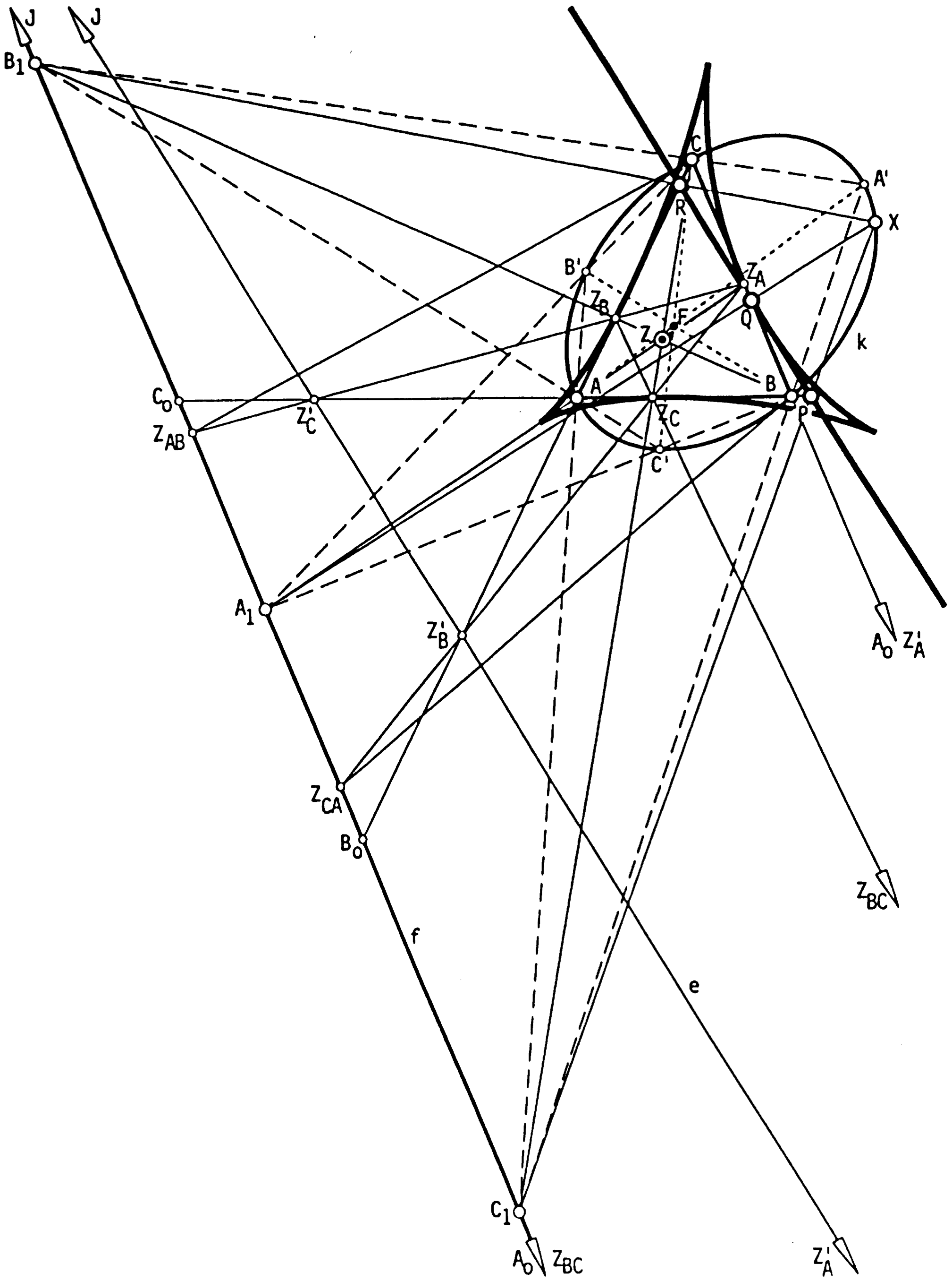

Figure 6: Projective Wallace lines

be the equation of the *axis* f. Therefore, all projective results depend only on the position of the axis f (fixed by $(a : b : c)$) to the triangle ABC and center Z. Because of $Z \notin f$ we always have

$$a + b + c \neq 0. \tag{19}$$

The points of intersection of the axis f with the sides ABC have the coordinates

$$A_0 = (0 : c : -b), \quad B_0 = (c : 0 : -a), \quad C_0 = (b : -a : 0). \tag{20}$$

Since according to (d) the point triples (Z, A, A_0), (Z, B, B_0), (Z, C, C_0) are not collinear, we have

$$u := b + c \neq 0, \quad v := c + a \neq 0, \quad w := a + b \neq 0. \tag{21}$$

Because of (21) the axis f contains none of the points Z_A, Z_B, Z_C. Thus we have

$$Z_A \neq A_1, \quad Z_B \neq B_1, \quad Z_C \neq C_1.$$

According to (b) the points A_0, B_0, C_0 are pairwise different. From this follows that the axis f is not incident with any vertex of the triangle. Consequently we obtain

$$a \neq 0, \; b \neq 0, \; c \neq 0. \tag{22}$$

The points A_1, B_1, C_1 necessary for the construction of the projective feet (16) have the coordinates

$$A_1 = (-u : a : a), \; B_1 = (b : -v : b), \quad C_1 = (c : c : -w). \tag{23}$$

Using (23) one finds, with respect to a running point $X = (\xi_0 : \xi_1 : \xi_2)$ as coordinates of the *projective feet* P, Q, R

$$\begin{gathered} P = (w\xi_0 + c\xi_2 : w\xi_1 + c\xi_2 : 0), \quad Q = (0 : a\xi_0 + u\xi_1 : a\xi_0 + u\xi_2), \\ R = (v\xi_0 + b\xi_1 : 0 : v\xi_2 + b\xi_1).^4 \end{gathered} \tag{24}$$

The projective feet P, Q, R lie on a common line if and only if the determinant formed by their coordinate triples vanishes:

$$\det \begin{pmatrix} 0 & u\xi_1 + a\xi_0 & u\xi_2 + a\xi_0 \\ v\xi_0 + b\xi_1 & 0 & v\xi_2 + b\xi_1 \\ w\xi_0 + c\xi_2 & w\xi_1 + c\xi_2 & 0 \end{pmatrix} = 0. \tag{25}$$

From (25) follows, after simple computation

Theorem 5: *If in a triangle ABC of the real projective plane (Fig. 6) with the sides AB, BC, CA the projective feet $P \in AB$, $Q \in BC$, $R \in CA$ of a point X with respect to a center Z and an axis f $(ax_0 + bx_1 + cx_2 = 0)$ lie on a common line (the projective Wallace line of X for the triangle ABC with center Z and axis f), then X lies on the curve of third order (25), which decomposes into the axis f and the conic k obeying*

$$(a+b)\xi_0\xi_1 + (b+c)\xi_1\xi_2 + (c+a)\xi_2\xi_0 = 0. \tag{26}$$

k contains the vertices A, B, C and is called the circumconic of the triangle ABC with center Z and axis f. This means that to the two-parameter family of axes $f(a : b : c)$ (18) (dependent on the homogeneous parameters $a : b : c$) corresponds bijectively the two-parameter family of conics $k(a : b : c)$ (26).

[4] For $w = 0$, $c \neq 0$ we always have $P = Z_C$; the lines PQR then lie in the pencil of lines around P. For $u = 0$, $a \neq 0$ the lines PQR lie in the pencil of lines around Q, for $v = 0$, $b \neq 0$ in the pencil of lines around R. Because of (21) these pencils of lines do not occur.

Remarks:

1. The circumconic belonging to the *Euclidean Wallace lines* is the circumcircle of the triangle ABC with the orthocenter H as center Z and the line at infinity (of the projectively extended Euclidean plane) as axis f.
2. If one draws from a point X in the Euclidean plane the perpendiculars to the sides AB, BC, CA of a triangle ABC and if the feet $P \in AB$, $Q \in BC$, $R \in CA$ lie on a common line (the *Euclidean Wallace line* of X), then X lies on the circumcircle of the triangle ABC. The converse also holds: If in the Euclidean plane a point X lies on the circumcircle of a triangle ABC, then its feet $P \in AB$, $Q \in BC$, $R \in CA$ are collinear. The projective generalization (and the affine one containing it) also has the following converse: If a point $X = (\xi_0 : \xi_1 : \xi_2)$ lies on the circumconic k (26) of a triangle ABC with center Z and axis f, then the projective feet (24) lie on a common line.
3. The dualization of the projective generalization leads to the concept of the *projective Wallace point* of a line x for the trilateral abc with axis z and center F.
4. We used the center Z as unit point of the projective coordinate system and the axis f as an arbitrary line. Equally we can use f as unit line of the projective coordinate system and Z as an arbitrary point. Both possibilities produce a two-parameter family of circumconics of the triangle ABC.

2.2 Pole and polar with respect to a circumconic

The polar of a point (pole) $P = (\xi_0' : \xi_1' : \xi_2')$ with respect to a circumconic k has the equation

(27) $$(w\xi_1' + v\xi_2')x_0 + (u\xi_2' + w\xi_0')x_1 + (v\xi_0' + u\xi_1')x_2 = 0.$$

For $P \in k$ (27) provides the tangent of k in P. Table 2 shows special poles and their polars with respect to k:

pole	coordinates	polar	equation
center Z	$(1:1:1)$	polar z	$(w+v)x_0 + (u+w)x_1 + (v+u)x_2 = 0$
pole E	$(au:bv:cw)$	unit line e	$x_0 + x_1 + x_2 = 0$
pole F	$(bcu:cav:abw)$[5]	axis f	$ax_0 + bx_1 + cx_2 = 0$
P_A	$(-u:v:w)$	side BC	$x_0 = 0$
P_B	$(u:-v:w)$	side CA	$x_1 = 0$
P_C	$(u:v:-w)$	side AB	$x_2 = 0$
A	$(1:0:0)$	k-tangent in A	$wx_1 + vx_2 = 0$
B	$(0:1:0)$	k-tangent in B	$ux_2 + wx_0 = 0$
C	$(0:0:1)$	k-tangent in C	$vx_0 + ux_1 = 0$

Table 2: Special poles and their polars

Using the table one obtains:

1. The lines AP_A, BP_B, CP_C intersect at the point $N = (u : v : w)$.
2. The k-tangent in A contains the intersection Z_{BC} of the axis f with the line $Z_B Z_C$. There are analogous results for B and C. Here the following holds: $Z_{BC} = (b - c : -v : w)$, $Z_{CA} = (u : c - a : -w)$, $Z_{AB} = (-u : v : a - b)$, $Z_B Z_C \ \ldots \ -x_0 + x_1 + x_2 = 0$, $Z_C Z_A \ \ldots \ x_0 - x_1 + x_2 = 0$, $Z_A Z_B \ \ldots \ x_0 + x_1 - x_2 = 0$.

[5]Because of $abc(a+b+c) \neq 0$ we always have $F \notin f$. For $a = b = c$ we have $F = Z$.

3. The polar z of the center Z, the axis f and the unit line e intersect each other in point $J = (b-c : c-a : a-b)$. Thus we obtain:
4. The center Z, the pole F of the axis f and the pole E of the unit line e lie on a common line.

2.3 Distinguished points of a circumconic

On a circumconic k of the triangle ABC, in addition to the vertices A, B, C, let the following triples of points (A', B', C') and (A'', B'', C'') be distinguished.

Case 1, the points A', B', C': The points of intersection (Fig. 6)

$$\begin{aligned} A' &:= B_1C \cap BC_1 = (-bc : vc : wb) \\ B' &:= C_1A \cap CA_1 = (uc : -ac : wa) \\ C' &:= A_1B \cap AB_1 = (ub : va : -ab) \end{aligned} \tag{28}$$

are also three distinguished points of a circumconic k. Since the points of intersection A_1, B_1, C_1 of the three pairs of opposite sides

$$(CB', C'B), \quad (AC', A'C), \quad (BA', B'A)$$

of the hexagon AC', BA', CB' inscribed in the circumconic k lie on the axis f the figure of the theorem of Pascal arises (see [5], p. 317); the axis f is the Pascal *line* of the hexagon $AC'BA'CB'$. The diagonals AA', BB', CC' of the hexagon $AC'BA'CB'$ intersect in the pole F of the axis f with respect to k.

Case 2, the points A'', B'', C'': On a circumconic k, starting with A, B, C, three further distinguished points A'', B'', C'' arise, if for B, C, Z_A one considers the fourth harmonic point $Z'_A = (0 : 1 : -1)$, for C, A, Z_B the fourth harmonic point $Z'_B = (1 : 0 : -1)$ and for A, B, Z_C the fourth harmonic point $Z'_C = (1 : -1 : 0)$. Then one obtains:

$$\begin{aligned} \{A, A''\} &= k \cap AZ'_A \quad \text{with} \quad A'' = (b+c : b-c : c-b), \\ \{B, B''\} &= k \cap BZ'_B \quad \text{with} \quad B'' = (a-c : a+c : c-a), \\ \{C, C''\} &= k \cap CZ'_C \quad \text{with} \quad C'' = (a-b : b-a : a+b); \end{aligned} \tag{29}$$

Z'_A, Z'_B, Z'_C lie on the unit line e. Further, the point triples (Z_A, Z_B, Z'_C), (Z_B, Z_C, Z'_A), (Z_C, Z_A, Z'_B) are all collinear. For $a = b$, the axes $f(a : b : c)$ lie in the pencil of lines around Z'_C, for $b = c$ in the pencil around Z'_A, for $c = a$ in the pencil around Z'_B.

2.4 Enveloping curve of the Wallace lines

If $X = (\xi_0 : \xi_1 : \xi_2)$ is a point of the circumconic k, then its projective Wallace line (use (24) and (26)) has the equation

$$\begin{aligned} (v\xi_0 + b\xi_1)(a\xi_0 + u\xi_1)(v\xi_0 + u\xi_1)x_2 - (v\xi_0 + b\xi_1)(av\xi_0 - bu\xi_1)\xi_0 x_1 + \\ + (a\xi_0 + u\xi_1)(av\xi_0 - bu\xi_1)\xi_1 x_0 = 0. \end{aligned} \tag{30}$$

From (30) follows that with a fixed point $\overline{X} = (x_0 : x_1 : x_2)$ exactly three projective Wallace lines meet, algebraically counted.

If X passes through the circumconic k, one obtains the envelope of the projective Wallace lines. Its determination, starting with (30) (one differentiates by using the inhomogeneous parameter $t := \xi_1/\xi_0$) brings us to

Theorem 6: *The envelope of the projective Wallace lines of the points X of a circumconic k for the triangle ABC with center Z and axis f is a rational curve of fourth order and third class (see Fig. 6) with the parameter representation following from (30):*

$$\begin{aligned} x_0(\xi_1:\xi_0) &= u(b\xi_1+v\xi_0)^2[bu^2\xi_1^2-2auv\xi_1\xi_0-av(v+w)\xi_0^2],\\ x_1(\xi_1:\xi_0) &= -v(u\xi_1+a\xi_0)^2[bu(u+w)\xi_1^2+2buv\xi_1\xi_0-av^2\xi_0^2],\\ x_2(\xi_1:\xi_0) &= (bu\xi_1-av\xi_0)^2(bu\xi_1^2+2uv\xi_1\xi_0+av\xi_0^2).\end{aligned} \tag{31}$$

The envelope (31) contacts the sides AB, BC, CA of the triangle ABC at the points$G_C = (b^2:a^2:0)$, $G_A=(0:c^2:b^2)$, $G_B=(c^2:0:a^2)$ determined by (see (31)) $bu\xi_1-av\xi_0=0$, $b\xi_1+v\xi_0=0$, $u\xi_1+a\xi_0=0$. The lines AG_A, BG_B, CG_C intersect in the point $G=(b^2c^2 : c^2a^2 : a^2b^2)$.

Remarks:

1. For the Euclidean Wallace lines the envelope (31) is a *Steiner-hypocycloid* ([10], p. 142f., [9], p. 143, [5], p. 637), which also is rational of fourth order and third class and which is tangent to the sides of the triangle ABC.
2. If the axes $f(a:b:c)$ (satisfying conditions (a) to (d) in 2.1) lie in the pencil of lines with the pencil center $(Y_0:Y_1:Y_2)$, then the corresponding circumconics $k(a:b:c)$ lie in the *pencil of conics*

   $$b[(Y_0-Y_1)\xi_0\xi_1+Y_0\xi_1\xi_2-Y_1\xi_2\xi_0]+c[-Y_2\xi_0\xi_1+Y_0\xi_1\xi_2+(Y_0-Y_2)\xi_2\xi_0]=0. \tag{32}$$

   The pencil of conics (32) has as basic points the vertices of the triangle A, B, C and the point $(\eta_0:\eta_1:\eta_2)$ with

   $$\begin{aligned}\eta_0 &= (Y_0-Y_1+Y_2)(Y_0+Y_1-Y_2),\\ \eta_1 &= (Y_1-Y_2+Y_0)(Y_1+Y_2-Y_0),\\ \eta_2 &= (Y_2-Y_0+Y_1)(Y_2+Y_0-Y_1).\end{aligned} \tag{33}$$

   The mapping $(Y_0:Y_1:Y_2)\mapsto(\eta_0:\eta_1:\eta_2)$ is a quadratic Cremona transformation.
3. The three tangents of the envelope (31), (counted algebraically), containing a fixed point $\overline{X}=(x_0:x_1:x_2)$, are determined by the parameters $(\xi_1:\xi_0)$ that satisfy the equation following from (30):

   $$\begin{aligned}&bu^2(x_2-x_0)\xi_1^3+(Wx_0+Mx_1+Sx_2)\xi_1^2\xi_0+\\ &+(Lx_0+Nx_1+Vx_2)\xi_1\xi_0^2+av^2(x_2-x_1)\xi_0^3 \quad =0\end{aligned} \tag{34}$$

   with

   $$\begin{aligned}&W:=ua(v-b), \quad M:=ub^2, \quad S:=u[ab+u(b+u)],\\ &L:=va^2, \quad N:=vb(u-a), \quad V:=v[ab+u(a+v)].\end{aligned} \tag{35}$$

   If (34), for the point $\overline{X}$, has three coincident zeros $(\xi_1:\xi_0)$, then the three envelope tangents passing through $\overline{X}$ coincide and $\overline{X}$ is one of the three singular points (cusps) of the rational envelope of third class (31). From (34) follows:
   For $\overline{X}\in ZB$ $(x_2-x_2=0)$, $(\xi_0:\xi_1)=(0:1)$ is a solution of (34). From this follows with (31): *ZB is always the envelope tangent through $\overline{X}$ at the envelope point*

   $$(x_0(1:0):x_1(1:0):x_2(1:0))=\left(b^2:-v(u+w):b^2\right).$$

For $\overline{X} \in ZA$ ($x_1 - x_2 = 0$), $(\xi_0 : \xi_1) = (1 : 0)$ is the solution of (34). From this follows with (31): *ZA is always the envelope tangent through $\overline{X}$ at the envelope point*

$$(x_0(0:1) : x_1(0:1) : x_2(0:1)) = \left(-u(w+v) : a^2 : a^2\right).$$

For $\overline{X} \in ZC$ ($x_0 - x_1 = 0$), $(\xi_0 : \xi_1) = (-u : v)$, is the solution of (34). From this follows with (31): *ZC is always the envelope tangent through $\overline{X}$ at the envelope point*

$$(x_0(-u:v) : x_1(-u:v) : x_2(-u:v)) = \left(c^2 : c^2 : -w(v+u)\right).$$

4. For $a = b = c$ holds: *The cusps of the envelope* (31) *lie in the points*:

$$\begin{array}{lll} (-8:1:1) & \text{on the line } AZ \text{ for} & (\xi_0:\xi_1) = (1:0), \\ (1:-8:1) & \text{on the line } BZ \text{ for} & (\xi_0:\xi_1) = (0:1), \\ (1:1:-8) & \text{on the line } CZ \text{ for} & (\xi_0:\xi_1) = (-1:1); \end{array}$$

the axis f is incident with the unit line e and consequently the points Z'_A, Z'_B, Z'_C lie on f. The envelope (31) of the projective Wallace lines osculates for $a = b = c$ the sides of the triangle at the points $Z_A \in BC$, $Z_B \in CA$, $Z_C \in AB$.

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Received October 6, 1997; final form November 14, 1997