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The Pedal Surfaces of (1,2)-Congruences with a One-Parameter Set of Ellipses

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Abstract. In the 3-dimensional Euclidean space (the model of the 3-dimensional projective space) among the surfaces of 4^{th} order with a nodal line those through the absolute conic are extracted as a special class which contains the pedal surfaces of (1,2)-congruences. One class of these surfaces is classified with regard to the number and types of their real singular points. These surfaces have been visualized using the program Mathematica 3.0.

Key Words: (1,2)-congruences, pedal surfaces of congruences, quartic surfaces

1 Introduction

Among the surfaces in the Euclidean space those which contain the absolute conic are of special interest. The cyclides are examples of such surfaces as they are 4^{th} order surfaces (quartics) which contain the absolute conic as a nodal curve. But the quartics with a nodal line which contain the absolute conic as an ordinary curve have interesting properties, too. In this paper we combine methods of synthetic and analytical geometry and we make use of the highly developed graphics facilities of the system *Mathematica* 3.0 in order to enumerate one class of symmetric quartics through the absolute conic with a nodal line. For some of these surfaces also the shapes are displayed.

2 Quartics with a nodal line

According to [8, p. 315] the quartics with a nodal line d belong to the class of n^{th} order surfaces with a multiple line of multiplicity $n-2$. According to [6, p. 1575], they form one of the four basis types in Kummer's classification of quartics which pass through conics [5]. These surfaces contain sixteen simple lines in eight planes of the pencil $[d]$ [10, p. 250], eight nodal points which do not lie on the line d [7, p. 218], four pinch-points on the nodal line d and, besides ∞^1 conics in the planes of the pencil $[d]$, other 128 conics which lie in 64 planes [6, p. 1633].

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Salmon [7] considered quartics with a nodal line as a special class of quartics with nodal curves. According to [7, p. 217], the general equation of these surfaces in the homogeneous Cartesian coordinates $(x:y:z:w)$ may be written as

$$u_4 + z\,u_3 + w\,u_3 + z^2\,t_2 + z\,v\,u_2 + w^2\,v_2 = 0 \tag{1}$$

where u_i, v_i, t_i are homogeneous polynomials in x and y of order i, and the z-axis is a nodal line. Merely in view of the variety of u_2, v_2, t_2 Salmon indicated some of the cases which would need to be considered in the complete classification of quartics with a nodal line [7, p. 217].

3 Quartics through the absolute conic and with a nodal line

We do not intend to give here a complete list of different kinds of these surfaces, but we merely indicate the first step towards their complete classification.

Lemma 1 *In homogeneous Cartesian coordinates $(x:y:z:w)$ the general equations of the intersection between the plane at infinity and the quartics through the absolute conic and with the z-axis as nodal line are as follows:*

$$\begin{aligned} (x^2+y^2+z^2)(a^2x^2-b^2y^2) &= 0, \qquad w=0 \\ (x^2+y^2+z^2)(ax-by)^2 &= 0, \qquad w=0 \\ (x^2+y^2+z^2)(a^2x^2+b^2y^2) &= 0, \qquad w=0 \end{aligned} \tag{2}$$

Proof: Since the point at infinity of the z-axis is a nodal point of the surface, the plane at infinity cuts the surface along the absolute conic and two lines through the point $(0:0:1:0)$. Therefore, the formulas (2) are direct consequences of the following facts:

- $(x^2+y^2+z^2) = w = 0$ are the equations of the absolute conic, and
- $a^2x^2 - b^2y^2 = 0$ or $(ax-by)^2 = 0$ or $a^2x^2+b^2y^2 = 0$, each combined with $w=0$, are the equations of two real, of one two-fold or of a pair of imaginary lines through the point $(0:0:1:0)$. □

In analogy to the affine types of conics we can call a quartic passing through the absolute conic and with a nodal line *hyperbolic*, *parabolic* or *elliptic*, if it has *two*, *one* or *no* real lines at infinity, respectively.

Theorem 1 *Equation (1) defines a quartic through the absolute conic and with the z-axis as a nodal if and only if*

$$u_4 = (x^2+y^2)t_2,\ u_3 = 0 \text{ and } t_2 = a^2x^2 - b^2y^2 \text{ or } t_2 = (ax-by)^2 \text{ or } t_2 = a^2x^2+b^2y^2.$$

Proof: For $w=0$ equ. (1) takes the form $u_4 + zv_3 + z^2t_2 = 0$. New we apply Lemma 1. □

4 Pedal surfaces of (1,2)-congruences

The locus of the feet of perpendiculars drawn from any fixed finite point P, called the *pole*, to the rays of an (n,m)-congruence is called the *pedal surface* of this congruence.

4.1 (1,2)-congruences

According to [9, p. 37] each 1st order and 2nd class congruence of lines $\mathcal{K}_2^1$ may be regarded as the system of lines meeting two directing curves, a conic c and a straight line d with only one common point O. In two planes (the plane of the conic c and the plane spanned by the line d and the tangent of the conic c at O) the rays of the congruence form two pencils of lines (O). All other lines of the bundle of lines $\{O\}$ are not regarded as the rays of the congruence.

A point is called a *singular point* of a congruence if ∞^1 rays are passing through it. Similarly, a plane is called a *a singular plane* of a congruence if it contains ∞^1 rays [4, p. 262]. The singular points of $\mathcal{K}_2^1(c,d,O)$ lie on the conic c or on the line d, and the singular planes of $\mathcal{K}_2^1(c,d,O)$ are the planes of the pencil $[d]$ and the plane of the conic c [1, p. 10].

According to the intersection of a congruence with the plane at infinity the congruences $\mathcal{K}_2^1(c,d,O)$ may be classified into six types:

- I: c is an ellipse, d is a finite line, O is a finite point. $\mathcal{K}_2^1(c,d,O)$ has a pair of imaginary rays through one real singular point at infinity.
- II: c is a parabola, d is a finite line, O is a finite point. $\mathcal{K}_2^1(c,d,O)$ has coinciding rays through two real singular points at infinity.
- III: c is a hyperbola, d is a finite line, O is a finite point. $\mathcal{K}_2^1(c,d,O)$ has two real rays through three real singular points at infinity.
- IV: c is a parabola, d^∞ is a line at infinity, O^∞ is a point at infinity. $\mathcal{K}_2^1(c,d^\infty,O^\infty)$ has the singular line d^∞ and the pencil of rays (O^∞) at infinity.
- V: c is a hyperbola, d^∞ is a line at infinity, O^∞ is a point at infinity. $\mathcal{K}_2^1(c,d^\infty,O^\infty)$ has the singular line d^∞ and the pencil of rays (R^∞) at infinity, where $R^\infty \in c$ and $R^\infty \neq O^\infty$.
- VI: c^∞ is a conic at infinity, d is a finite line, O^∞ is a point at infinity. $\mathcal{K}_2^1(c^\infty,d,O^\infty)$ has the singular conic c^∞ and the pencil of rays (O^∞) at infinity.

4.2 Quartic inversion

The quartic inversion $i_\Psi\colon P^3 \to P^3$, defined in [2], is a transformation of the projective space into itself where corresponding points A and $i_\Psi(A)$ are conjugate with respect to a regular quadric Ψ and lying on the rays of a congruence $\mathcal{K}_2^1(c,d,O)$.

This is a Cremona transformation with singular points on the curves d, c and e^6, where e^6 is the curve of contact between the quadric Ψ and a 6^{th} order ruled surface which is the intersection of $\mathcal{K}_2^1(c,d,O)$ and the complex of lines tangent to Ψ [2].

In [2, p. 191-194] it is proved that for every plane ϕ the image $i_\Psi(\phi)$ is a quartic with a nodal line d, which contains c and e^6.

There are two special cases where this quartic is reducible:

1. If ϕ is any plane of the pencil $[d]$, then $i_\Psi(\phi)$ splits into the cubic surface $i_\Psi(d)$ and the plane ϕ.
2. If ϕ is the plane of the conic c, then $i_\Psi(\phi)$ splits into the ruled cubic $i_\Psi(c)$ with a nodal line d and the plane ϕ.

For all other planes $i_\Psi(\phi)$ is an irreducible quartic cutting ϕ along two rays of $\mathcal{K}_2^1(c,d,O)$ and along the conic of intersection between ϕ and Ψ.

4.3 Pedal surfaces

Theorem 1.1 of [1, p. 10] says that the pedal surface of $\mathcal{K}_2^1(c,d,O)$ for the pole P is the image of the plane at infinity under the quartic inversion with respect to $\mathcal{K}_2^1(c,d,O)$ and any sphere centered at P.

It is clear that the pedal surface is a quartic with the nodal line d containing c and the absolute conic. It is also clear, according to the properties of the quartic inversion, that the pedal surface splits into the plane at infinity and a cubic surface if the congruence $\mathcal{K}_2^1(c,d,O)$ belongs to types IV, V or VI.

If we exclude the two special cases ([1, p. 15]), which will be mentioned in the subsection 5.5. of this paper, then the pedal surfaces for all other types of (1,2)-congruences are irreducible quartics with nodal line d which contain the absolute conic. If $\mathcal{K}_2^1(c,d,O)$ is of the type I, II or III then this quartic is elliptic, parabolic or hyperbolic in the sense of section 3.

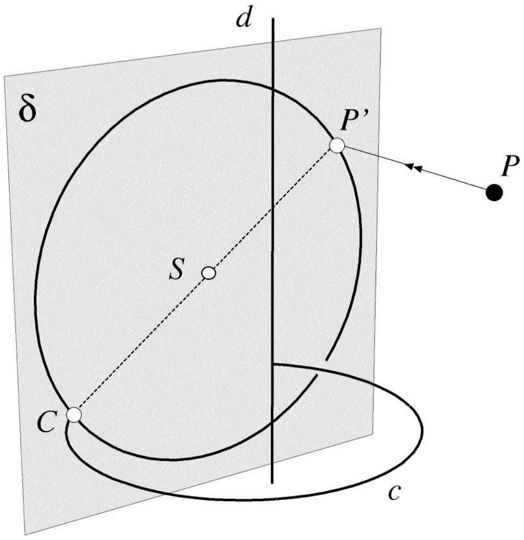

Figure 1: The illustration of Theorem 1.2 in [1]

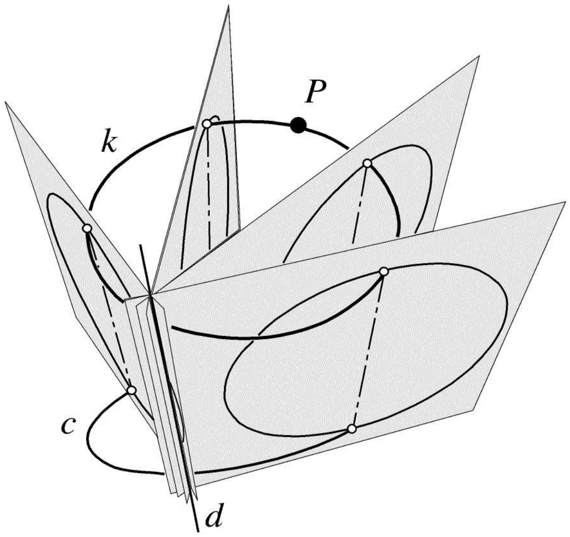

Figure 2: Generation of the pedal surfaces

The basis for the constructive treatment of pedal surfaces is Theorem 1.2 in [1, p. 10]. It says that each plane $\delta \in [d]$, which cuts the conic c in the point C, cuts the pedal surface along the double line d and the circle with the diameter $\overline{CP'}$, where P' is the foot of P in the plane δ (see Fig. 1).

Now we can comprehend the generation of the pedal surfaces as it is shown in Fig. 2: k is the circle in the plane through the pole P perpendicular to the line d. One of its diameters is enclosed by P and by the foot of P on the line d.

If the congruence belongs to the types IV or V, i.e. $\mathcal{K}_2^1(c,d^\infty,O^\infty)$, then the circle k splits into two lines. One is the line at infinity in the polar plane of O^∞ (the polarity is determined by any sphere with the center P), and the other is the perpendicular from P to the planes of the pencil $[d^\infty]$. One example of the pedal surface for a congruence of type IV is shown in Fig. 3.

If the congruence belongs to type VI, i.e. $\mathcal{K}_2^1(c^\infty,d,O^\infty)$, then the circles in the planes of the pencil $[d]$ split into two lines. In each plane one of these lines is the perpendicular from the point P' to the rays of the pencil (C^∞), and the other is the line at infinity. One example of the pedal surface for a congruence of type VI is displayed in Fig. 4.

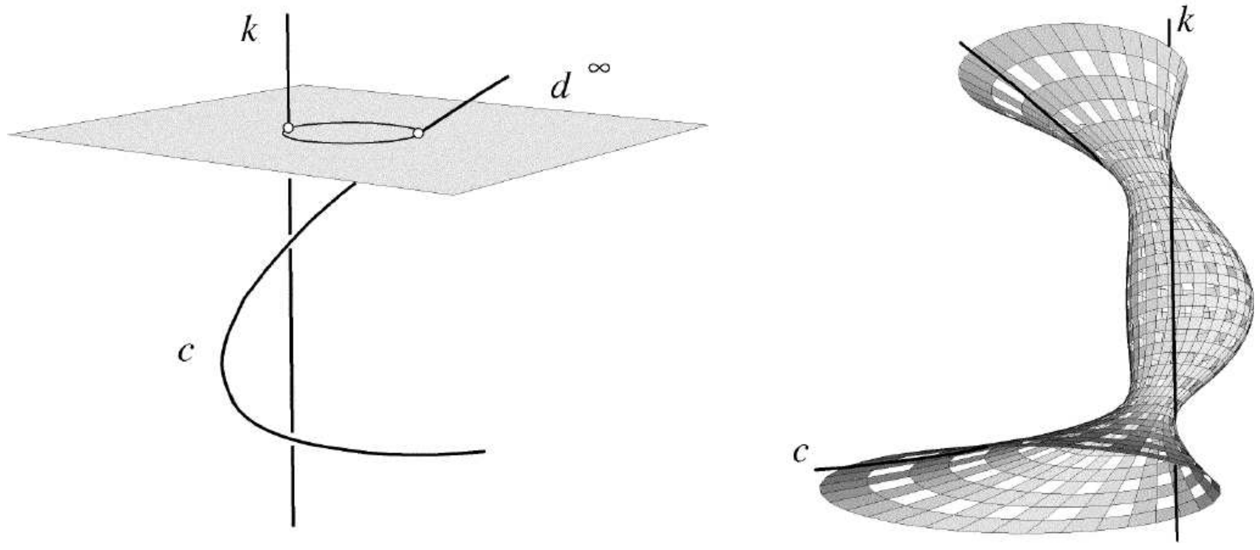

Figure 3: The line d^{∞} is determined by one directing plane parallel to the axis of the directing parabola c. The pedal surface is the cubic through the curves d^{∞} and k and the parabola c

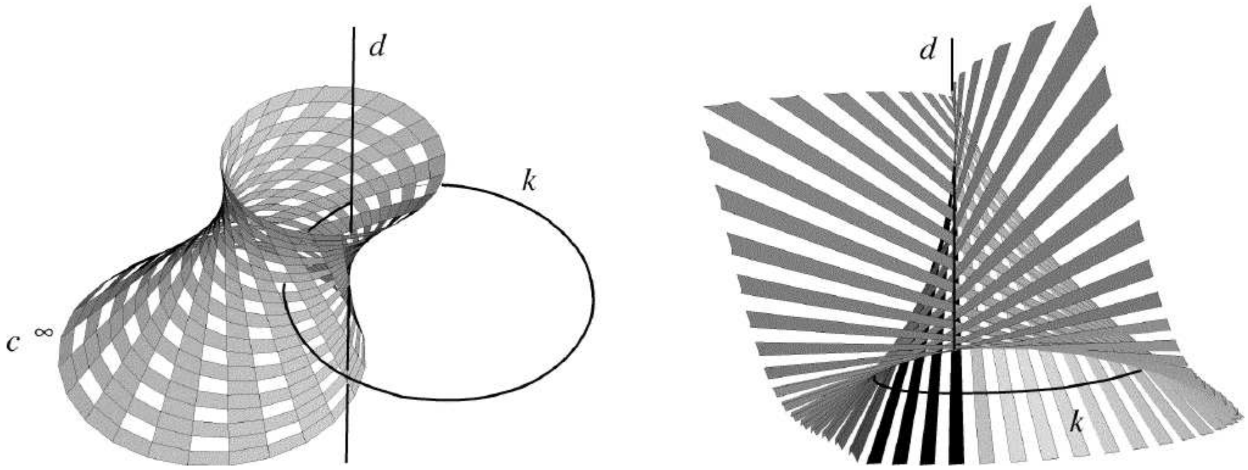

Figure 4: The conic c^{∞} is determined by a one-sheet hyperboloid which contains the line d as a generator. The pedal surface is the conoid of 3^{rd} degree with the nodal line d and it contains the circle k and the conic c^{∞}.

5 Classification and construction of the pedal surfaces of (1,2)-congruences with a one-parameter set of ellipses

For the pedal surfaces which will be the topic of this section the directing conic of $\mathcal{K}_2^1$ is an ellipse e, different from a circle. e lies in a plane perpendicular to the directing line d which intersects e at a given point O (see Fig. 5). These surfaces have the special property that they are symmetric with respect to a plane which is not contained in the pencil $[d]$.

In the Cartesian coordinate system (O, x, y, z) the directrices e and d and point P can be given in the following way:

$$\begin{array}{lll} e & \ldots & b^2(x-x_E)^2+(y-b\sqrt{1-x_E^2})^2=b^2,\ z=0, \qquad (0<b\le 1, \quad 0\le x_E\le 1) \\ d & \ldots & x=0,\ y=0 \\ P & \ldots & (p,q,r). \end{array}$$

Here b is the minor semi-axis and x_E the abscissa of the center E of the ellipse e (see Fig. 5). Without loss of generality we assumed that the major semi-axis a of the ellipse e is 1.

Since the pedal surface of $\mathcal{K}_2^1(e,d,O)$ for the pole P is uniquely determined by the five numbers b, x_E, p, q, r, we will denote it by $F_E[b,x_E,p,q,r]$.

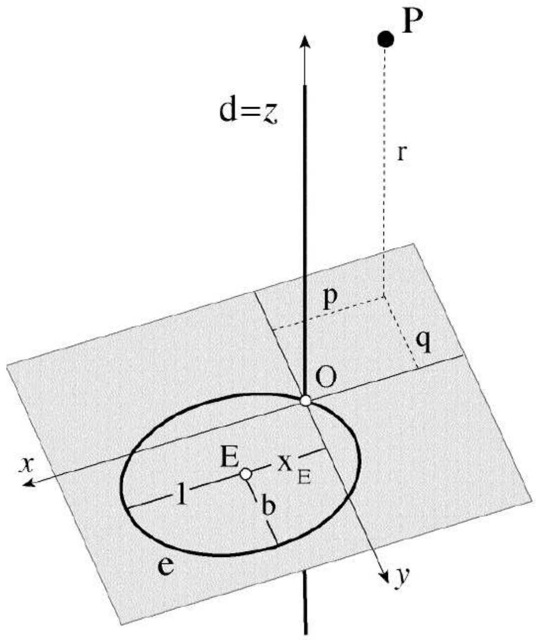

Figure 5

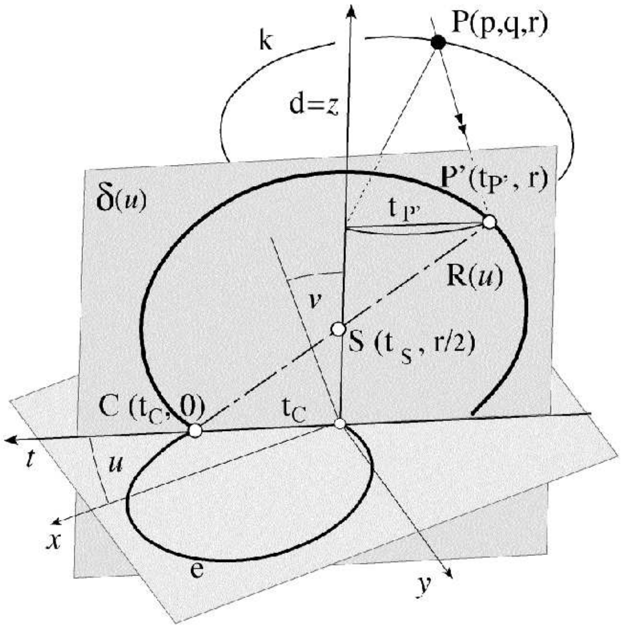

Figure 6

5.1 The equations of $F_E[b, x_E, p, q, r]$

Between the pencil of planes $[d]$ and the half-closed interval $[-\pi/2,\pi/2)$ there is a one-to-one mapping $u\leftrightarrow\delta(u)$ (Fig. 6). According to Theorem 1.2 of [1, p. 11], each plane $\delta(u)$ cuts $F_E[b,x_E,p,q,r]$ along the circle $c(u)$ with diameter $\overline{CP'}$. In the plane $\delta(u)$ we introduce the Cartesian coordinate system (O,t,z): The t-axis is the intersection of $\delta(u)$ and the xy-plane and its positive orientation points into the semiplane $x\ge 0$ (Fig. 6).

It is clear that in the *cylindrical coordinate system* (O,t,u,z) the equation of the surface $F_E[b,x_E,p,q,r]$ can be written as

$$(3) \qquad (t-t_S(u))^2+\left(z-\frac{r}{2}\right)^2=R^2(u), \qquad u\in[-\frac{\pi}{2},\frac{\pi}{2})$$

where $t_S(u)$ is the t-coordinate of the center and $R(u)$ is the radius of the circle $c(u)$ in the plane $\delta(u)$. $t_S(u)$ and $R(u)$ depend on the t- and z-coordinates of C and P' according to

$$(4) \qquad t_S(u)=\frac{1}{2}(t_C(u)+t_{P'}(u)),$$

$$(5) \qquad R(u)=\frac{1}{2}\sqrt{(t_C(u)-t_{P'}(u))^2+r^2}\,.$$

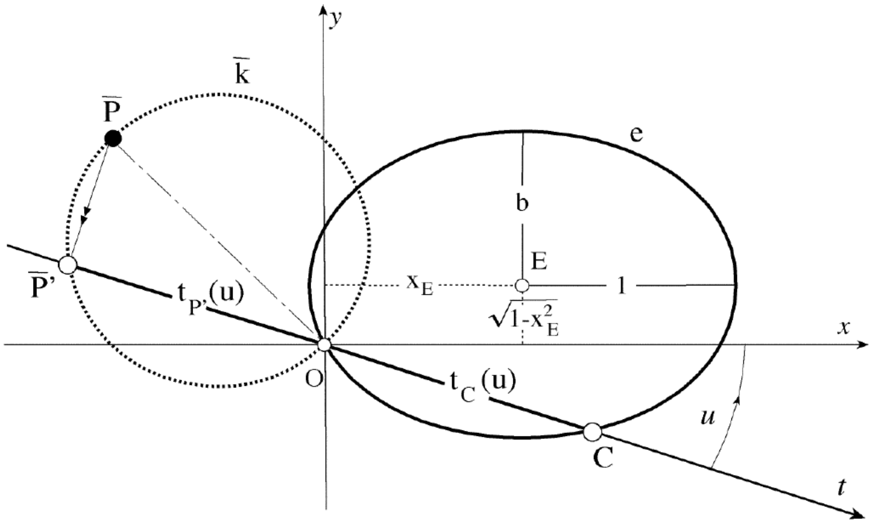

Figure 7: The circle $\overline{k}$ is the normal projection of the circle k into the xy-plane

$t_C(u)$ is a function of b and x_E and it is given by

$$t_C(u) = \frac{2b(b\,x_E \cos u + \sqrt{1-x_E^2}\sin u)}{b^2\cos^2 u + \sin^2 u}. \tag{6}$$

$t_{P'}(u)$ depending on p and q (see Fig. 7) obeys

$$t_{P'}(u) = p\cos u + q\sin u\,. \tag{7}$$

After the substitution of $t = \sqrt{x^2+y^2}$ and $u = \arctan\frac{y}{x}$ into equation (3) we can write the equation of $F_E[b, x_E, p, q, r]$ in the *homogeneous Cartesian coordinates* $(x:y:z:w)$ in the following Salmon form:

$$\begin{aligned}(8)\quad &(x^2+y^2)(b^2x^2+y^2) - [(px+qy)(b^2x^2+y^2) + 2b(b\,x_E x + \sqrt{1-x_E^2}\,y)(x^2+y^2)]w + \\ &+ z^2(b^2x^2+y^2) - r(b^2x^2+y^2)zw + 2b(b\,x_E x + \sqrt{1-x_E^2}\,y)(px+qy)w^2 = 0\end{aligned}$$

Since

$$t(v) = R(u)\sin v + t_S(u), \quad z(v) = R(u)\cos v + \frac{r}{2}, \quad v \in [0, 2\pi) \tag{9}$$

are parametric equations of the circle $c(u) \subset \delta(u)$ in the coordinate system (O, t, z) and $x = t\cos u$, $y = t\sin u$, we obtain

$$\begin{aligned}(10)\qquad x(u,v) &= \cos u\ (R(u)\sin v + t_S(u))\\ y(u,v) &= \sin u\ (R(u)\sin v + t_S(u))\\ z(u,v) &= R(u)\cos v + \frac{r}{2}, \qquad u \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad v \in [0, 2\pi)\end{aligned}$$

as the *parametric equations* of the surface $F_E[b, x_E, p, q, r]$ in the coordinate system (O, x, y, z). It is clear that the *v-curves* are the circles $c(u)$ in the planes $\delta(u)$ of the pencil $[d]$.

5.2 Singular points of $F_E[b, x_E, p, q, r]$ on the nodal line

A point on the nodal line d is a *binode*, a *pinch-point* or an *isolated double point* of $F_E[b, x_E, p, q, r]$, if there exist two real, coinciding or imaginary tangent planes, respectively.

It was mentioned earlier that a quartic surface with a nodal line has four pinch-points ([6], [7]), but these need not be real and distinct points.

Theorem 2 *The pinch-points of the surface $F_E[b, x_E, p, q, r]$, $r \neq 0$, are pairwise coinciding if and only if $p = q = 0$.*

Proof: According to [7, p. 218] the pinch-points are pairwise coinciding if and only if the polynomials t_2, u_2, v_2 in equation (1) have a common factor. For the surface $F_E[b, x_E, p, q, r]$ equ. (1) takes the form (8). Since the polynomial $b^2x^2 + y^2$ is irreducible, the polynomials t_2, u_2, v_2 in (8) may have a common factor only if v_2 is equal to zero.

Because of the assumed values for the numbers b and x_E the polynomial v_2 in (8) vanishes only in the case $p = q = 0$.

From the construction of the pedal surfaces it is clear that all circles $c(u)$ on such a surface are passing through the points $P(0, 0, r)$ and $O(0, 0, 0)$ (see Fig. 12). □

Corollary 1 *The four pinch-points of the surface $F_E[b, x_E, p, q, r]$ coincide if and only if $p = q = r = 0$.*

Proof: For $r = 0$ it is a direct consequence of Theorem 3 (see Fig. 13). □

Now, let us consider the other cases, i.e., $p \neq 0 \vee q \neq 0$:

Lemma 2 *The circles of the system $c(u)$, $u \in [-\frac{\pi}{2}, \frac{\pi}{2})$, intersect the nodal line d in pairs (B_1, B_2) of a symmetric involution.*

Proof: For each binode B_1 on the line d there are two circles of the system $c(u)$, $u \in [-\frac{\pi}{2}, \frac{\pi}{2})$, passing through ([2, p. 192]). Since the surface $F_E[b, x_E, p, q, r]$ is symmetric with respect to the plane $z = \frac{r}{2}$ (z-coordinate of the centers of the circles $c(u)$) these two circles intersect once more in the binode B_2 which is symmetric to B_1 with respect to the plane $z = \frac{r}{2}$. □

Lemma 3 *The parameters u_1, u_2 of circles $c(u_1)$, $c(u_2)$ sharing the binodes (B_1, B_2) on d obey the equations:*

(a) if $p = 0$, $x_E = 0$, $q \neq 0$ or $q = 0$, $x_E = 1$, $p \neq 0$ then

$$\tan u_1 = -\tan u_2\,, \tag{11}$$

(b) in other cases

$$\tan u_1 \cdot \tan u_2 - A(\tan u_1 + \tan u_2) - b^2 = 0, \tag{12}$$

where $A = b(bq\sqrt{1 - x_E^2} - px_E)/(bqx_E + p\sqrt{1 - x_E^2})$.

Proof: For $t = 0$ equation (3) gives the z-coordinates of the binodes (B_1, B_2):

$$z_{1,2} = \pm\sqrt{R^2(u) - t^2(u)} + \frac{r}{2}\,. \tag{13}$$

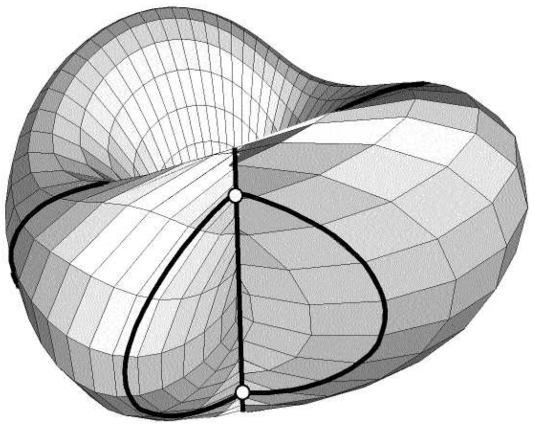

Figure 8: $F_E[0.4, 0.5, -1, 1, -3/4]$, $u_1 = -0.0785$, $u_2 = 0.4357$

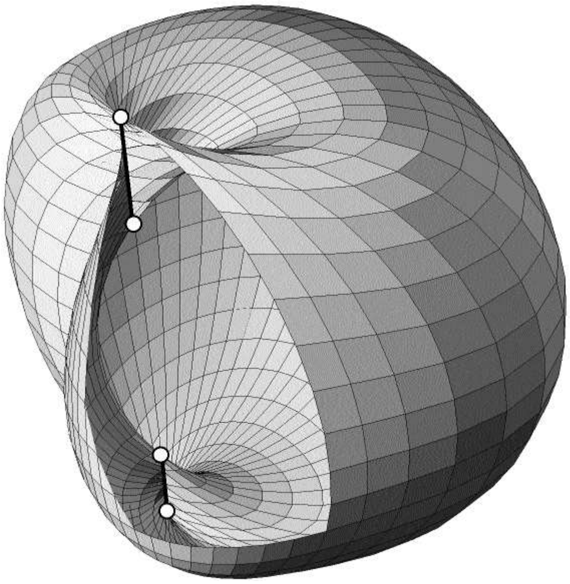

Figure 9: $F_E[0.4, 0.5, 1, 1, 3.5]$

According to the relations given in (4), (5), (6) and (7) the equation $z^2(u_1) = z^2(u_2)$ may be written as

$$(p + q\tan u_1)\frac{2b(b\,x_E + \sqrt{1-x_E^2}\tan u_1)}{b^2 + \tan^2 u_1} = (p + q\tan u_2)\frac{2b(b\,x_E + \sqrt{1-x_E^2}\tan u_2)}{b^2 + \tan^2 u_2}.$$

If the trivial case $u_1 = u_2$ and the case $p = 0 \wedge q = 0$ are excluded, then the above equation takes the form (11) or (12). □

This property is illustrated in Fig. 8.

Lemma 4 *The pair of tangent planes of the surface $F_E[b, x_E, p, q, r]$ at the corresponding points $B_1, B_2 \in d$ are $\delta(u_1), \delta(u_2)$, where u_1, u_2 are determined by the circles of the system $c(u)$, $u \in [-\pi/2, \pi/2)$, meeting at the points B_1, B_2.*

Proof: At any point B on the nodal line d each of the two tangent planes is spanned by d and by the tangent of any curve which lies on the surface and passes through B. □

Theorem 3 *The surface $F_E[b, x_E, p, q, r]$*

(a) has four *real pinch-points if $p = 0, x_E = 0, q < 0$ or $q = 0, x_E = 1, p < 0$, and it has* two *real pinch-points if $p = 0, x_E = 0, q > 0$ or $q = 0, x_E = 1, p > 0$.*

(b) In all other cases it has four, three or two real pinch-points if and only if

$$r^2 \gtreqqless \frac{4(bq + \sqrt{1-x_E^2}\sqrt{p^2+b^2q^2})^2}{\sqrt{b^2q^2+p^2} + bq\sqrt{1-x_E^2} - px_E}.$$

Proof: The pinch-points of a surface are points on its double curve at which two tangent planes coincide. Therefore, according to Lemma 4, the pair (P_1, P_2) will be the pair of pinch-points if the planes $\delta(u_1)$, $\delta(u_2)$ coincide. Now, according to Lemma 4 and Lemma 3, by substituting $u_1 = u_2$ into the equations (11) and (12) we get the angles u_{I}, u_{II}

$$u_{\mathrm{I}} = 0, \ u_{\mathrm{II}} = -\frac{\pi}{2} \qquad \text{for (a)} \tag{14}$$

$$u_{\mathrm{I,II}} = \arctan(A \pm \sqrt{A^2 + b^2}) \qquad \text{for (b)} \tag{15}$$

which determine the circles $c(u_{\mathrm{I}})$, $c(u_{\mathrm{II}})$ meeting the line d in the pinch-points. For the values u_{I}, u_{II} the equations (13) give the z-coordinates of the pinch-points.

(a) For $p = 0, x_E = 0, q \neq 0$: $z_{1,2} = \frac{r}{2} \pm \frac{r}{2}$, $z_{3,4} = \frac{r}{2} \pm \sqrt{-2bq}$.
For $q = 0, x_E = 1, p \neq 0$: $z_{1,2} = \frac{r}{2} \pm \sqrt{-2p}$, $z_{3,4} = \frac{r}{2} \pm \frac{r}{2}$.

(b) For all other cases

$$z_{1,2} = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{(bq-\xi)^2}{\sqrt{p^2+b^2q^2}-\eta}}; \ z_{3,4} = \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{(bq+\xi)^2}{\sqrt{p^2+b^2q^2}+\eta}}$$

with $\xi = \sqrt{1-x_E^2}\sqrt{p^2+b^2q^2}$ and $\eta = bq\sqrt{1-x_E^2} - px_E$.
The denominator $\sqrt{p^2+b^2q^2} - \eta$ in the first of the above equations is positive. Namely, it is clear that the function $f(b, x_E, p, q) = \sqrt{p^2+b^2q^2} - bq\sqrt{1-x_E^2} + px_E$ is equal to zero only in the cases $(p = q = 0)$ or $(p = 0, x_E = 0)$ or $(q = 0, x_E = 1)$ which in case (b) are excluded.
For other values, it follows from $\partial f/\partial b = \partial f/\partial x_E = \partial f/\partial p = \partial f/\partial q = 0$ that $x_E = -p/\sqrt{p^2+b^2q^2}$ and that the extremum of the function f is $(b^2q^2 - |bq|bq)/\sqrt{p^2+b^2q^2} \geq 0$.

Now, we can conclude that the points P_1, P_2 are always real and that the points P_3, P_4 are real and distinct, real and coinciding or a pair of imaginary points if and only if $r^2 \gtreqless (4(bq+\xi^2)/(\sqrt{b^2q^2+p^2}+\eta)$ (see Fig. 9, Fig. 10, Fig. 11). □

5.3 Singular points of $F_E[b, x_E, p, q, r]$ not lying on the nodal line

Before we start to analyze the singular points of $F_E[b, x_E, p, q, r]$ which do not lie on the nodal line we will cite some of Salmon's considerations:

A quartic with a nodal line may have also double points. Two of the eight planes which meet the surface in straight lines will coincide with the plane joining the nodal line with one of the nodal points. Each such plane intersects the surface twice in the nodal line and in two lines meeting at the nodal point. But any such plane may meet the surface besides the nodal line in a two-fold line containing two nodal points; the surface may thus have eight nodal points ([7, p. 218]).

Accepting these general considerations we can formulate the following

Theorem 4 *An elliptic quartic passing through the absolute conic and with a nodal line may have* three *real nodal points which do not lie on the nodal line.*

Proof: Two of the eight planes which meet the surface in straight lines are the pair of imaginary planes determined by the nodal line and the pair of imaginary lines at infinity. Since only six of the eight planes which meet the surface in straight lines may coincide in pairs, for such a surface there exist at most three planes through the nodal line and the nodal points. Every such plane must meet the surface in the nodal line twice, and in the pair of isotropic lines (split circle) intersecting at the real nodal point. □

Corollary 2 *Only in the case $r = 0$ the pedal surfaces $F_E[b, x_E, p, q, r]$ can have nodal points, which are not located on the nodal line. At most three real nodal points of the surface are points of intersection between the ellipse e and the circle k and different from O.*

Proof: Each plane through the nodal line cuts the surface in the circle with radius

$$R(u) = \frac{1}{2}\sqrt{(t_C(u) - t_{P'}(u))^2 + r^2}.$$

Since the circle splits only if $R(u) = 0$, the surface has nodal points only if $r^2 = 0$ and $t_C(u) = t_{P'}(u)$. □

According to Cor. 2 the surface $F_E[b, x_E, p, q, 0]$ has at most three real nodal points not lying on the nodal line. In these cases the circle k and the ellipse e, on which the end points of the diameters of the circles $c(u)$ are lying, belong to the xy-plane. The circle k and the ellipse e are intersecting at point O and at three other points. Since these points may be all real and distinct or coinciding or they may be pairwise coinciding or imaginary, the surfaces $F_E[b, x_E, p, q, 0]$ have different numbers and different types of real nodes.

5.4 The classification of the surfaces $F_E[b, x_E, p, q, r]$

According to the subsections 5.2. and 5.3. we can classify the surfaces $F_E[b, x_E, p, q, r]$ with regard to the number and type of their real singular points in the following way:

Type 1 (Fig. 9) 4 pinch-points, binodes and isolated double points on the line d

Type 2 (Fig. 10) 3 pinch-points, binodes and isolated double points on the line d

Type 3 (Fig. 11) 2 pinch-points, binodes and isolated double points on the line d

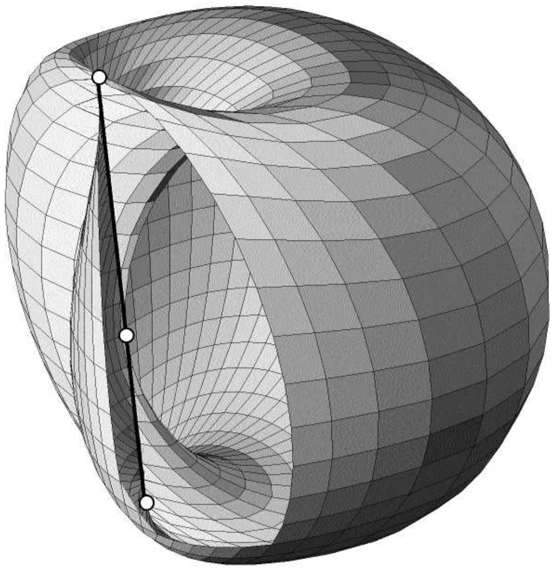

Figure 10: $F_E[0.4, 0.5, 1, 1, 2.774]$

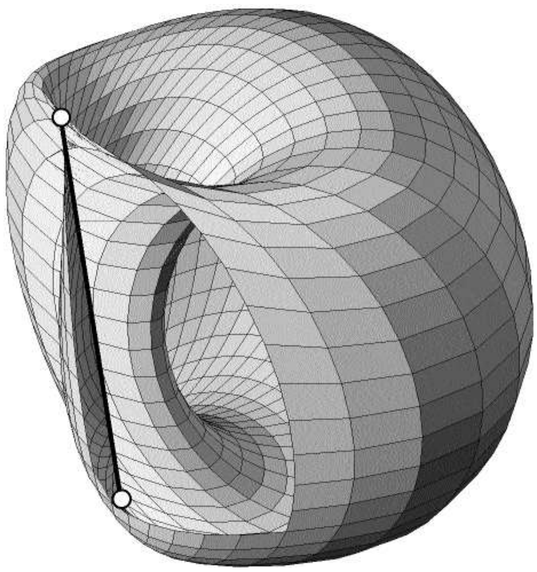

Figure 11: $F_E[0.4, 0.5, 1, 1, 2]$

Type 4 (Fig. 12) 2 pinch-points and isolated double points on the line d

Type 5 (Fig. 13) 1 pinch-point and isolated double points on the line d

Type 6 (Fig. 14) 2 pinch-points, binodes and isolated double points on the line d, 3 conical points

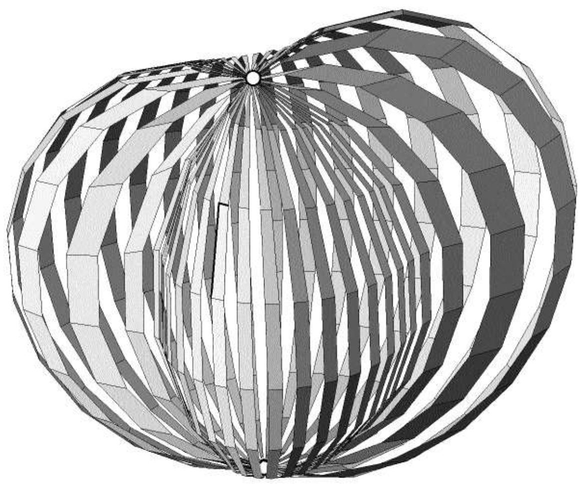

Figure 12: $F_E[0.4, 0.5, 0, 0, 2]$

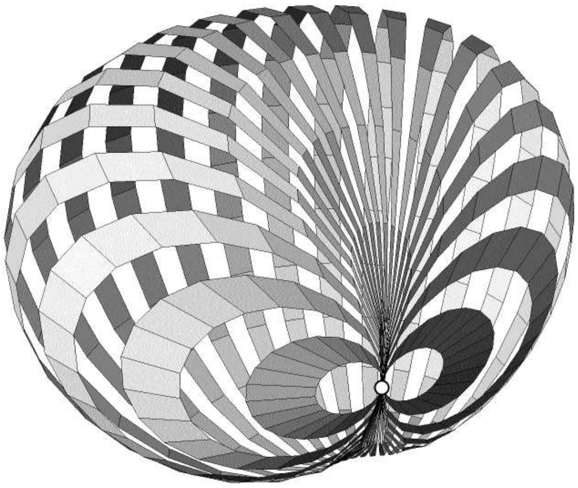

Figure 13: $F_E[0.8, 0, 0, 0, 0]$

Type 7 (Fig. 15) 2 pinch-points, binodes and isolated double points on the line d 1 conical point, 1 point of double contact

Type 8 (Fig. 16) 2 pinch-points, binodes and isolated double points on the line d, 1 point of triple contact

Type 9 (Fig. 17) 2 pinch-points, binodes and isolated double points on the line d, 1 conical point

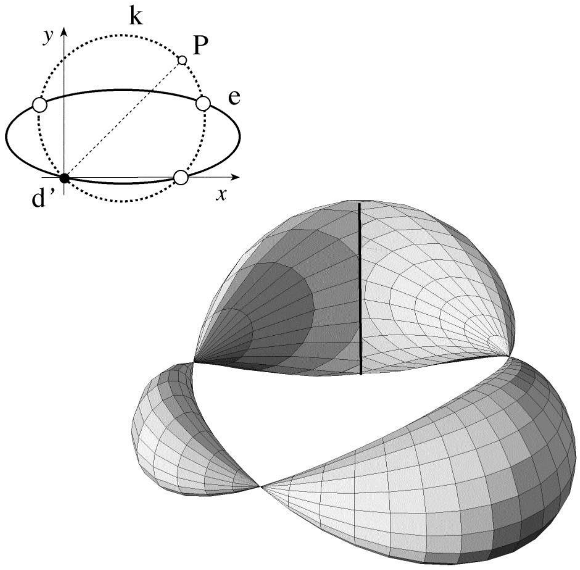

Figure 14: $F_E[0.4, 0.5, 1, 1, 0]$

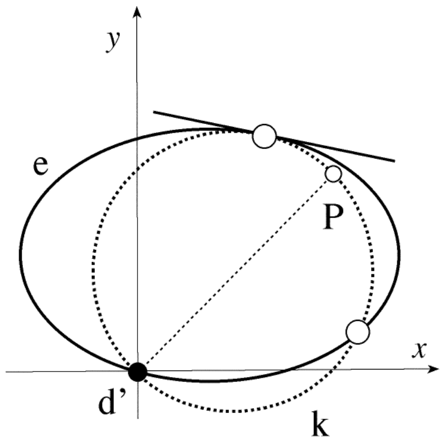

Figure 15: $F_E[2/3, 0.38, 1, 1.78, 0]$

Type 10 (Fig. 18) 1 point of double contact and isolated double points on the line d, 2 conical points

Type 11 (Fig. 19) 1 point of double contact and isolated double points on the line d, 1 point of double contact

Type 12 (Fig. 20) 2 pinch-points, binodes and isolated double points on the line d, 1 point of double contact on the line d

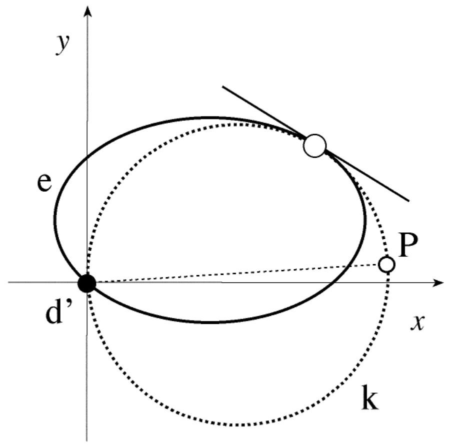

Figure 16: $F_E[2/3, 0.8, 1.95, 0.1, 0]$

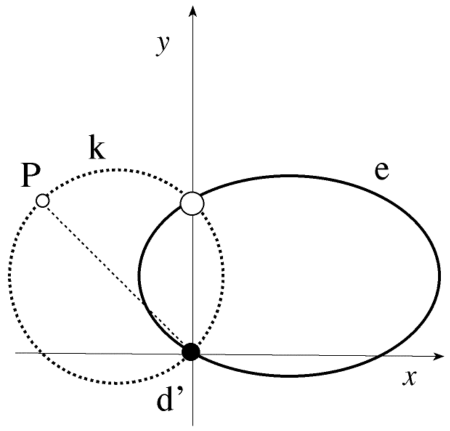

Figure 17: $F_E[2/3, \sqrt{3/7}, -1, 1, 0]$

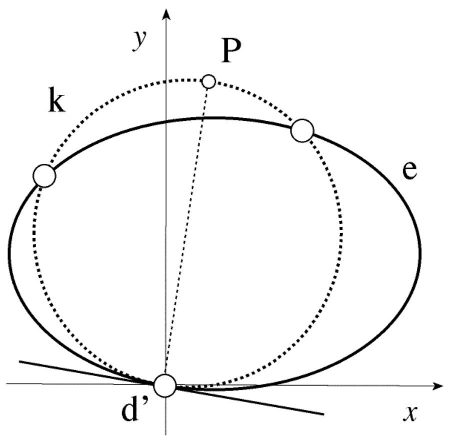

Figure 18:
$F_E[2/3, 1/4, 1/4, 0.375.\sqrt{15}, 0]$

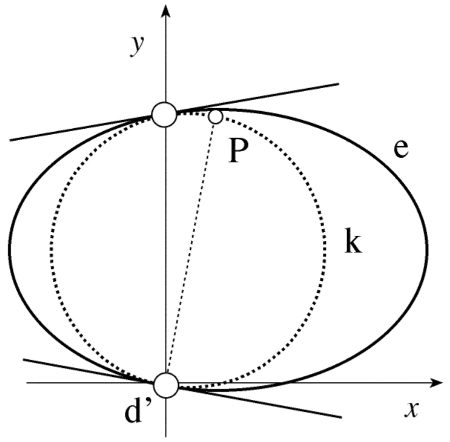

Figure 19:
$F_E[2/3, 1/4, 2/9, \sqrt{5/3}, 0]$

Type 13 (Fig. 21) 1 point of double contact and isolated double points on the line d

Type 14 (Fig. 22) 1 point of triple contact and isolated double points on the line d, 1 conical point

Type 15 (Fig. 23) 1 point of fourfold contact and isolated double points on the line d

5.5 Special cases of $F_E[b, x_E, p, q, r]$

The special class of the surfaces $F_E[b, x_E, p, q, r]$ is obtained for $b = 1$, $x_E = 1$. In this case the ellipse e is a circle. Since the circles e and k lie in parallel planes, the absolute points of the xy-plane are double points of the surface, and any plane parallel to the xy-plane cuts it into a bicircular quartic. These surfaces are elaborated in [1] and [3].

The surface $F_E[b, x_E, p, q, r]$ degenerates only in two cases:

1. $F_E[1, 1, 0, 0, r]$ splits into the sphere and the pair of isotropic planes through the line d, and

2. $F_E[1, 1, 2, 0, 0]$ degenerates into the circle e and the line d. Namely, each plane through d cuts the surface in d and in the pair of isotropic lines through a point on the circle e.

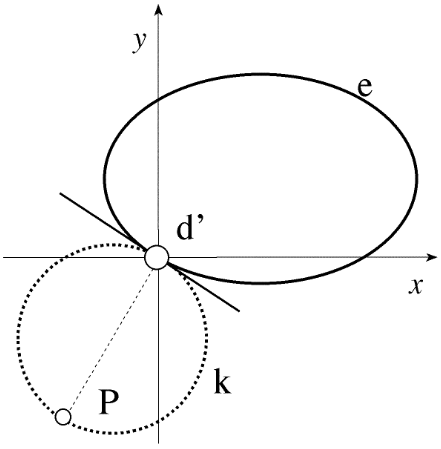

Figure 20:
$F_E[2/3, \sqrt{3/7}, -0.6, -0.6\sqrt{3}, 0]$

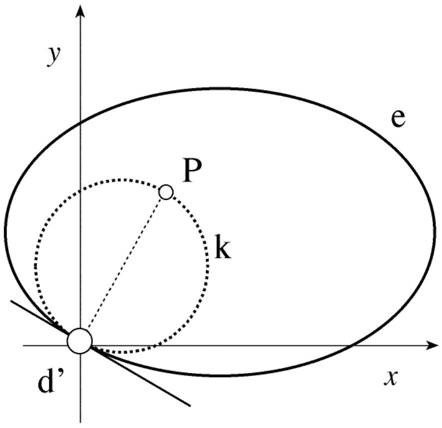

Figure 21:
$F_E[2/3, \sqrt{3/7}, 0.4, 0.4\sqrt{3}, 0]$

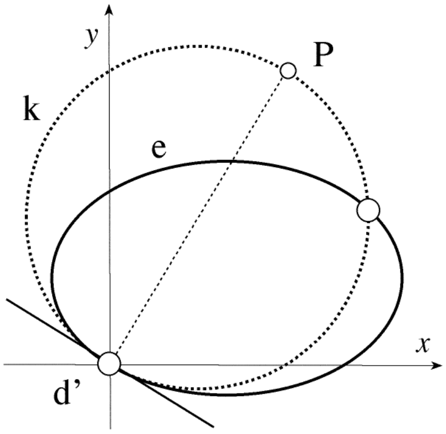

Figure 22:
$F_E[2/3, 2/3, 1, 0.75\sqrt{5}, 0]$

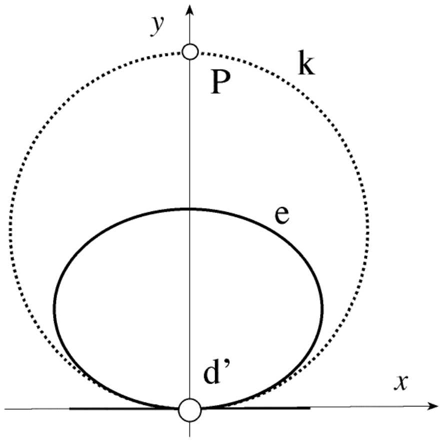

Figure 23: $F_E[0.75, 0, 0, 8/3, 0]$

6 Conclusion

According to the facts that the surfaces $F_E[b, x_E, p, q, r]$ (classified in the subsection 5.4) have a higher number of singular points than the general pedal surfaces of the type I, and that most of the theorems are valid or analogous theorems can be easily derived for the pedal surfaces of types II and III, the same procedure of classification could be applied also for the pedal surfaces of these (1,2)-congruences. It is still open whether all quartics passing through the absolute conic and containing a nodal line can be classified in this way.

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