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On the Theorems of Central Axonometry

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Abstract. One of the important questions of central axonometry is to give a condition under which a central axonometric mapping is a central projection. The aim of this paper is to prove that the well-known STIEFEL's condition can be considered as a limiting case of a recent theorem proved by SZABÓ, STACHEL and VOGEL.

1 Introduction

Central axonometry, as a projective generalization of the classical axonometric mapping, plays an important role in descriptive geometry and computer graphics. A *central axonometric mapping* of the projective space onto the projective plane π can be defined as follows: Let a point O in the plane π be given as the image of the spatial origin. Three arbitrary lines x, y and z through O are in π the given images of the axes of a spatial cartesian coordinate system. Furthermore, let two distinct points on each axis, E_x, U_x, E_y, U_y and E_z, U_z be the images of the unit points and the vanishing points of the spatial axes, respectively (cf. Figure 2). Now the image of a spatial point $P = (p_x, p_y, p_z)$ is obtained by completing the projective scales on the axes. This means that the i-th coordinate p_i of P equals the cross ratio $p_i = (P_i E_i O U_i)$ on the axis, when P_i denotes the orthogonal projection of point P onto the i-th axis. The seven points $O, E_x, E_y, E_z, U_x, U_y, U_z$ form the so-called *reference system* of the central axonometric mapping.

One of the central questions in this field is the following: Under which conditions is a central axonometric mapping of an object a central projection of this object ? Or — in other words — in which way can the reference system of central projection be characterized ? The first (rather complicated) result has been given by KRUPPA [1], but this characterization is synthetic, so it cannot be applied immediately in computer graphics. An analytical condition for a particular case has been given by STIEFEL [2] (Theorem 1 of this paper), while finally SZABÓ et al. [3] proved a general theorem (Theorem 2 of this paper) with a simple analytical condition. In this paper we prove that STIEFEL's condition can be considered as a limiting case of this general theorem.

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2 The Theorems

With the notations introduced above, STIEFEL's condition can be expressed as follows:

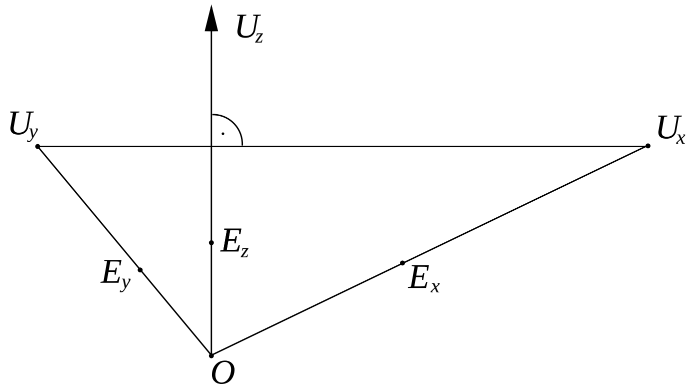

Figure 1: STIEFEL's particular case

Theorem 1 (STIEFEL): *If the vanishing point U_z of the central axonometric reference system $O, E_x, E_y, E_z, U_x, U_y, U_z$ is infinite and perpendicular to the line joining the vanishing points U_x, U_y of the x- and y-axis (see Figure 1), then this central axonometric mapping is a central projection if and only if the mutual distances obey*

$$(1) \qquad \left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2 = \left(\frac{U_xU_y}{OE_z}\right)^2 .$$

Theorem (SZABÓ, STACHEL, VOGEL): *When all points $O, E_x, E_y, E_z, U_x, U_y, U_z$ of a central axonometric reference system are finite, then the defined mapping is a central projection if and only if*

$$(2) \qquad \left(\frac{OE_x}{E_xU_x}\right)^2 : \left(\frac{OE_y}{E_yU_y}\right)^2 : \left(\frac{OE_z}{E_zU_z}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma ,$$

where $\alpha = \angle U_zU_xU_y, \beta = \angle U_xU_yU_z, \gamma = \angle U_yU_zU_x$ (see Figure 2).

3 The Limiting Process

Now we will prove that the general Theorem 2 implies Theorem 1 as its limiting case. First a lemma will be stated.

Lemma: *When the vertex C of a triangle ABC tends to infinity along the fixed altitude of C, then according to the notation in Figure 3*

$$\lim_{C\to\infty} m_c \sin\gamma = c .$$

Proof of the Lemma: From the area of the triangle ABC in Figure 3 we obtain

$$ab\sin\gamma = m_c c \quad \text{or} \quad m_c \sin\gamma = c\frac{m_c^2}{ab} .$$

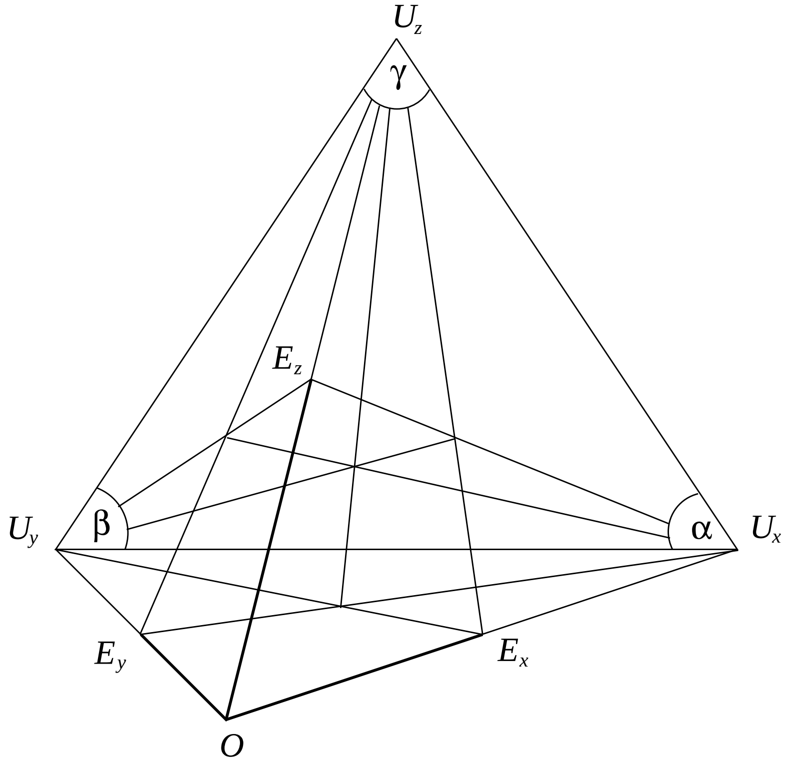

Figure 2: Central axonometric reference system $O, E_x, E_y, E_z, U_x, U_y, U_z$

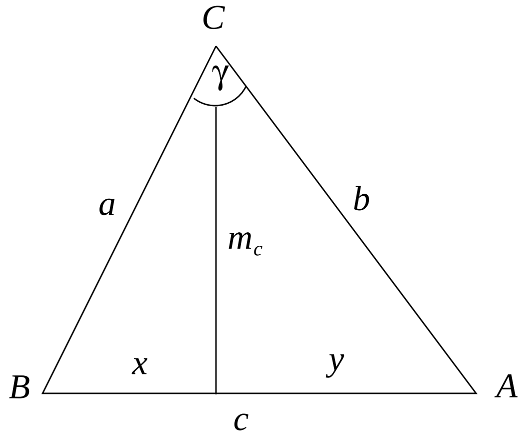

Figure 3:

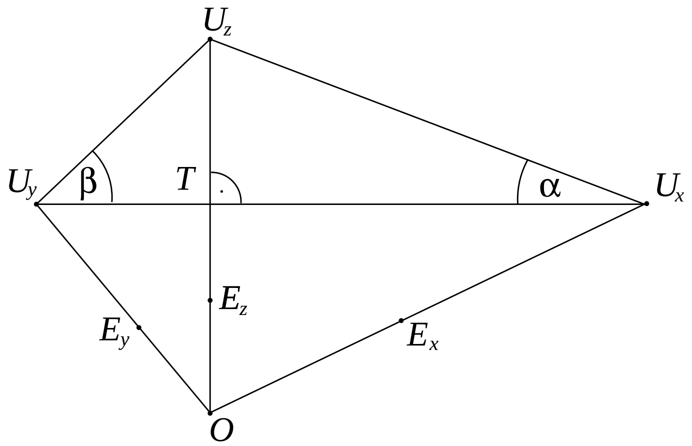

Figure 4:

Using this equation and the fact that $C \to \infty$ implies $m_c \to \infty$ we obtain

$$\lim_{C\to\infty} m_c \sin\gamma = \lim_{C\to\infty} c\,\frac{m_c^2}{ab} = c \lim_{C\to\infty} \frac{{m_c}^2}{ab} = c \lim_{C\to\infty} \frac{1}{\frac{\sqrt{m_c^2+x^2}}{m_c}\,\frac{\sqrt{m_c^2+y^2}}{m_c}} = c\cdot 1 = c$$

where x and y according to Figure 3 denote the lengths of the sections of AB separated by the altitude of C. □

Now we can prove the main result:

Proof (Theorem 2 ⇒ Theorem 1): From formula (2) we deduce

$$\cot\alpha = \left(\frac{E_xU_x}{OE_x}\right)^2\left(\frac{OE_z}{E_zU_z}\right)^2\cot\gamma \quad \text{and} \quad \cot\beta = \left(\frac{E_yU_y}{OE_y}\right)^2\left(\frac{OE_z}{E_zU_z}\right)^2\cot\gamma$$

which imply

$$\cot\alpha + \cot\beta = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right]\left(\frac{OE_z}{E_zU_z}\right)^2\cot\gamma\,. \tag{3}$$

On the other hand, let us consider a special case of the coordinate system, which is partially equal to STIEFEL's case: Let the z-axis be perpendicular to the line U_xU_y, but let the point U_z be finite (see Figure 4). Then the point T of intersection between the z-axis and the line U_xU_y obeys

$$\cot\alpha = \frac{U_xT}{U_zT} \quad \text{and} \quad \cot\beta = \frac{U_yT}{U_zT}\,,$$

therefore

$$\cot\alpha + \cot\beta = \frac{U_xU_y}{U_zT}\,.$$

The right side of this equation must be equal to the right side of (3):

$$\frac{U_xU_y}{U_zT} = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right]\left(\frac{OE_z}{E_zU_z}\right)^2\cot\gamma.$$

Multiply both sides of the equation with $\dfrac{U_xU_y\cdot U_zT}{OE_z{}^2}$:

$$\left(\frac{U_xU_y}{OE_z}\right)^2 = \left[\left(\frac{E_xU_x}{OE_x}\right)^2 + \left(\frac{E_yU_y}{OE_y}\right)^2\right]\frac{U_xU_y\cdot U_zT\cdot\cos\gamma}{E_zU_z{}^2\cdot\sin\gamma}\,.$$

Note that this latter equation is the same as (1) apart from the last factor of the right side. Hence it is sufficient to prove that if the point U_z tends to infinity along the fixed z-axis, then this factor tends to 1, that is

$$\lim_{U_z\to\infty}\frac{U_xU_y\cdot U_zT\cdot\cos\gamma}{E_zU_z{}^2\cdot\sin\gamma} = 1\,.$$

From Figure 4 one can deduce that $E_zU_z = U_zT \pm TE_z$. So this formula can be written as

$$\frac{U_zT}{E_zU_z}\cdot\frac{U_xU_y\cdot\cos\gamma}{U_zT\cdot\sin\gamma \pm TE_z\cdot\sin\gamma}\,.$$

Now in order to compute this limit we decompose the term into the following parts:

1. $\displaystyle\lim_{U\to\infty}\frac{U_zT}{E_zU_z} = 1$, since $E_zU_z = U_zT \pm TE_z$ and TE_z is constant.
2. $\displaystyle\lim_{U\to\infty} U_xU_y\cdot\cos\gamma = U_xU_y$, since $\displaystyle\lim_{U\to\infty}\gamma = 0$.
3. $\displaystyle\lim_{U\to\infty} TE_z\cdot\sin\gamma = 0$.
4. $\displaystyle\lim_{U\to\infty} U_zT\cdot\sin\gamma = U_xU_y$ due to the Lemma.

Finally 1–4 imply

$$\lim_{U\to\infty}\frac{U_xU_y\cdot U_zT\cdot\cos\gamma}{E_zU_z{}^2\cdot\sin\gamma} = \frac{U_xU_y}{U_xU_y} = 1$$

which completes the proof. □

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