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(n,2)-Axonometries and the Contour of Hyperspheres

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Abstract. The paper deals with special axonometric mappings of an n-dimensional euclidean space onto a plane π'. Such an $(n,2)$-axonometry is given by the image of a cartesian n-frame in π' and it is especially an isocline or orthographic axonometry, if the contour of a hypershere is a circle in π'.

The paper discusses conditions under which the image of the cartesian n-frame defines an orthographic axonometry. Also a recursive construction of the hypersphere-contour in case of an arbitrary given oblique axonometry is presented.

Keywords: multi-dimensional descriptive geometry, axonometric mappings.
MSC 1994: 51N05, 51N20

Introduction

In elementary Descriptive Geometry an axonometric mapping α or briefly an *axonometry* is the product of a *parallel projection* ψ and an *affinity*[1] κ from an auxiliary projection plane π (in space) onto the drawing plane π'. An axonometry α is well defined by the image of a cartesian coordinate system. If the auxiliary image plane π is orthogonal to the fibers of ψ, (i.e. ψ is an orthogonal projection), we call α an *ortho-axonometry*[2]. The well-known 3-dimensional *engineer's axonometry* (cf. [3], p. 63, [10], p. 73, DIN 5, ÖNORM A 6061) is an ortho-axonometry based on a particular axonometric reference system $\{x_1^\alpha, x_2^\alpha, x_3^\alpha\}$ in the drawing plane π' according to Fig. 1, which implies the distortion ratios $1:2:2$ of the x_1-, x_2- and x_3-axis. Thus the engineer's axonometry merges the advantages of distortion ratio-based (oblique) axonometries with that of ortho-axonometries (i.e. circular contours of spheres, 'direct' construction of the axes of the image-ellipses of circles).

[1]Due to the well-known theorem of Pohlke there exists always a factorirzation of α as a product of a parallel projection ψ and a *similarity* κ. From a more general point of view axonometry is a linear mapping acting on projective spaces and being the product of a (central) projection ψ of an n-space onto an auxiliary projective subspace of dimension m and a collineation κ of this subspace onto the m-dimensional image-space, cf. [1].

[2]We will also use the word 'normal' instead of 'orthographic' and briefly speak of 'ortho-axonometry' or 'ortho-projection' instead of 'orthographic axonometry' or 'orthographic projection'.

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The axonometric method is also a well suited tool to visualize objects of multidimensional spaces (cf. [8, 14]) and one will appreciate the advantages of an orthographic or at least *isocline*[3] projection, especially if one draws images of such objects by hand or by means of a CAD-system. However, as there are no commercial 'volume-based' CAD-systems for higher dimensions, one is restricted to 2D-CAD systems and to Descriptive Geometry. Thus there arises a need for constructive methods and of projections which are simple to handle.

In [29] WAN et al. gave a solution for an *ortho-axonometry* of the (euclidean) 4-space $\mathbb{E}^4$ onto the drawing plane π using an arbitrary axonometric 4-frame $\{x_1^n, x_2^n, x_3^n, x_4^n\}$ in the plane π and (rounded) distortion ratio $1 : 0.6 : 0.6 : 1$. They call such an axonometry an 'optimal' one: it can be factorized into an isometric ortho-projection $\psi_3^4 \colon \mathbb{E}^4 \to \mathbb{E}^3$ (mapping the unit segments on the x_1-, x_2-, x_3- and x_4-axis of $\mathbb{E}^4$ onto the 'height segments' of a regular tetrahedron of $\mathbb{E}^3$ with the orthocenter being the image of the origin $O \in \mathbb{E}^4$) and an ortho-projection $\psi_2^3 \colon \mathbb{E}^3 \to \pi$ providing finally a di-metric axonometry $\alpha \colon \mathbb{E}^4 \to \pi$ with the prechosen distortions.

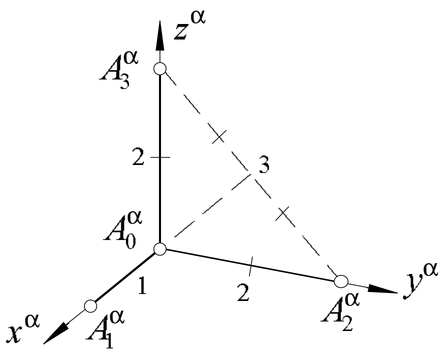

Figure 1: Engineer's axonometry

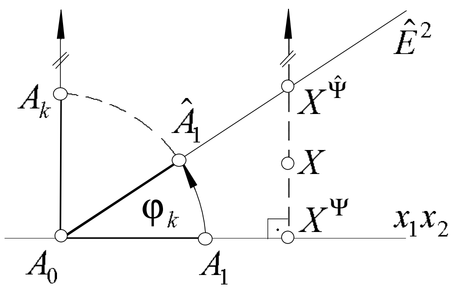

Figure 2:

The underlying paper extends classical results concerning ortho-(3,2)-axonometries α : $\mathbb{E}^3 \to \pi$ to ortho-(n,m)-axonometries, (cf. e.g. [24, 20, 21, 22, 27]), with special emphasis on (4,2)-axonometries. A consequence is a recursive construction of the contour-ellipse of a hypersphere in any $(n,2)$-axonometry.

Analytical treatment of ortho-axonometries

Let $\alpha \colon \mathbb{E}^n \to \mathbb{E}^{m\prime}$ be an ortho-(n,m)-axonometry from a euclidean n-space $\mathbb{E}^n$ onto a euclidean m-space $\mathbb{E}^{m\prime}$, i.e. the product of a *normal* (n,m)-projection $\psi_m^n \colon \mathbb{E}^n \to \mathbb{E}^m$, ($\mathbb{E}^m \subset \mathbb{E}^n$), and a *similarity* $\sigma \colon \mathbb{E}^m \to \mathbb{E}^{m\prime}$.

Any (n,m)-axonometry α of a euclidean n-space $\mathbb{E}^n$ onto a euclidean m-space $\mathbb{E}^{m\prime}$ is uniquely defined by the image $\mathfrak{B}^\alpha \subset \mathbb{E}^{m\prime}$ of a cartesian frame $\mathfrak{B} := \{A_0; A_1, \ldots, A_n\} \subset \mathbb{E}^n$, whereby e.g. $\{A_0^\alpha; A_1^\alpha, \ldots, A_m^\alpha\}$ span the image space $\mathbb{E}^{m\prime}$, (cf. [2]). Following [20] we choose an additional frame $\tilde{\mathfrak{B}} := \{A_0; \tilde{A}_1, \ldots, \tilde{A}_n\}$ such that ψ_m^n becomes the ortho-projection onto the coordinate subspace $\mathbb{E}^m = A_0 \vee \tilde{A}_1 \vee \cdots \vee \tilde{A}_m$. The congruence transformation replacing

[3] Cf. [20]

the original basis $\mathfrak{B}$ by $\tilde{\mathfrak{B}}$ defines a certain orthogonal matrix

$$(1) \qquad \mathfrak{M} := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (\bar{\mathsf{a}}_1, \bar{\mathsf{a}}_2, \ldots, \bar{\mathsf{a}}_n) = \begin{pmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \\ \vdots \\ \mathsf{a}_n \end{pmatrix}.$$

We notice that the columns $\bar{\mathsf{a}}_i$ as well as the rows a_i fulfill the conditions

$$(2) \qquad \bar{\mathsf{a}}_i \cdot \bar{\mathsf{a}}_k = \delta_{ik}, \quad \mathsf{a}_i \cdot \mathsf{a}_k = \delta_{ik}, \quad i,k \in \{1,\ldots,n\}$$

Hence the matrix $\mathfrak{N} =: (\bar{\mathsf{a}}_1^\alpha, \bar{\mathsf{a}}_2^\alpha, \cdots, \bar{\mathsf{a}}_n^\alpha)$ of the (affine linear) mapping α consists of the first m rows a_j of $\mathfrak{M}$. With

$$(3) \qquad \lambda_j^2 := a_{1j}^2 + a_{2j}^2 + \cdots + a_{mj}^2, \; j = 1,\ldots,n,$$

follows

$$(4) \qquad \sum_{j=1}^{n} \lambda_j^2 = \sum_{k=1}^{m} \|\mathsf{a}_k\|^2 = m.$$

Especially for an ortho-axonometry which is *congruent*[4] to an ortho-projection the values λ_j (3) are the distortions along the j^{th} coordinate axis $A_0 \vee A_j$ of $\mathfrak{B}$. With respect to given unit segments in $\mathbb{E}^n$ and $\mathbb{E}^{m\prime}$, (4) is a necessary condition for α being an ortho-axonometry. According to [20] (4) is also sufficient (cf. also [28]); so we state

Lemma 1: *Let α be an (n,m)-axonometry of a euclidean n-space $\mathbb{E}^n$ onto a euclidean m-space $\mathbb{E}^{m\prime}$ defined by the image $\mathfrak{B}^\alpha \in \mathbb{E}^{m\prime}$ of a cartesian frame $\mathfrak{B} \in \mathbb{E}^n$. Then α is an* ortho-axonometry, *if and only if the distortions λ_j $(j = 1,\ldots,n)$ along the coordinate axes fulfill the condition*

$$\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 = m.$$

Let the image-space $\mathbb{E}^{m\prime} =: \pi'$ be two-dimensional, so it admits the interpretation as Gaussian plane: Put $A_j^\alpha \mathrel{\hat=} z_j := a_{1j} + i\, a_{2j} \in \mathbb{C}$, $(j = 1,\ldots,n)$, and $A_0^\alpha \mathrel{\hat=} 0 \in \mathbb{C}$. Then, from (2) follows the 'Gauss-*condition*'

$$(5) \qquad \sum_{j=1}^{n} z_j^2 = \bar{\mathsf{a}}_1^2 + 2i\, \bar{\mathsf{a}}_1 \cdot \bar{\mathsf{a}}_2 - \bar{\mathsf{a}}_2^2 = \bar{\mathsf{a}}_1^2 - \bar{\mathsf{a}}_2^2 = 0,$$

which is independent from an arbitrary dilatation-factor of $\sigma\colon \mathbb{E}^2 \to \pi$, $(\mathbb{E}^2 \subset \mathbb{E}^n)$. For $n = 3$ this result is well-known (cf. [3]), for $n \geq 4$ it is mentioned in [21].

While the Gauss-condition characterizes a *parallel-projection* ψ_2^n for being an *ortho-projection* this is not true for parallel-*axonometries*, if $n > 3$, as it is proved in the following:

Let a $(n,2)$-axonometry α be product of an *ortho-projection* $\psi_2^n\colon \mathbb{E}^n \to \mathbb{E}^2 = (x_1 \vee x_2)$ (with matrix $\mathfrak{N}$ according to (3)) and a *similarity* $\sigma\colon \mathbb{E}^2 \to \pi'$ (with dilatation-factor d and an orthogonal matrix S). Then α is described by the matrix $\mathfrak{A} = S \cdot \mathfrak{N}$. On the other hand, the *oblique projection* $\widehat{\psi}_2^n$ of $\mathbb{E}^n$ parallel to the coordinate space $A_0 \vee \widetilde{A}_3 \vee \cdots \vee \widetilde{A}_n$ onto the plane

[4]This will make sense, if we embed $\mathbb{E}^{m\prime}$ into $\mathbb{E}^n$ such that we use the 'same' unit segments in $\mathbb{E}^n$ and in the image-space $\mathbb{E}^{m\prime}$. From a more general point of view, if we refrain from embedding $\mathbb{E}^{m\prime}$ into $\mathbb{E}^{m\prime}$, we may still use the concept 'congruent' with respect to chosen unit segments in $\mathbb{E}^n$ and in $\mathbb{E}^{m\prime}$. Cf. [20].

$\widetilde{\mathbb{E}}^2 := A_0 \vee B_1 \vee B_2$ ($B_1 := (1,0,\mathsf{b}_1)^\top$, $B_2 := (0,1,\mathsf{b}_2)^\top$, $\mathsf{b}_i \in \mathbb{R}^{n-2}$) (cf. Fig. 2), is described by the transformation matrix

$$(6) \qquad \widetilde{\mathfrak{M}} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \mathsf{b}_1 & \mathsf{b}_2 & \mathsf{o} & \cdots & \mathsf{o} \end{pmatrix}, \qquad \mathsf{b}_i, \mathsf{o} \in \mathbb{R}^{n-2},$$

and differs from the orthogonal projection $\psi_2^n \colon \mathbb{E}^n \to \mathbb{E}^2$ by an affine mapping, namely the perspectivity $\sigma := \widehat{\psi}_2^n | \mathbb{E}^2$. This affine mapping σ obviously turns out to be a *similarity* if and only if

$$(7) \qquad \mathsf{b}_1 \cdot \mathsf{b}_2 = 0 \ \wedge \ \mathsf{b}_1^2 = \mathsf{b}_2^2.$$

Thus follows that conditions (7) can become true only if $\dim \mathbb{E}^n \geq 4$!

Let conditions (7) be fulfilled. Then, σ is described by

$$(8) \qquad S := \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \quad d := \sqrt{1 + \mathsf{b}_1^2},$$

and, in the subspace $\mathbb{E}^4 := \mathbb{E}^2 \vee \widehat{\mathbb{E}}^2 \subset \mathbb{E}^n$, the planes $\mathbb{E}^2$ and $\widehat{\mathbb{E}}^2$ are "*isocline*" with the (two equal) main angles $\varphi_1 = \varphi_2$ with $\cos\varphi_i = 1/\sqrt{1 + \mathsf{b}_1^2}$. Hence an ortho-axonometry $\alpha \colon \mathbb{E}^n \to \pi'$ can always be factorized into an *isocline oblique projection*[5] $\widehat{\psi}_2^n$ and a (suitable) similarity σ.

Lemma 2: *The* Gauss*-condition (5) characterizes $(n,2)$-axonometries of a euclidean n-space $\mathbb{E}^n$, $n \geq 4$, onto the drawing plane π' for being* isocline[6] *axonometries.*

Isocline- and ortho-(4,2)-axonometries

In the following we restrict the discussion to dimensions $n = 4$ and $m = 2$. Then, in the Gaussian plane π', (5) is represented by a *quadrangle* $\mathcal{Q}$ with bars parallel to the vectors $\overrightarrow{0z_i^2}$ (cf. Fig. 3). Assuming a given unit segment in π' then, because of (4) and $|z_i^2| = \lambda_i^2$, an *ortho*-axonometry is characterized by a quadrangle $\mathcal{Q}$ with a circumference of length 2. Note that four segments with lengths $|z_i^2|$ can form a quadrangle only if a set of *quadrangle-conditions* holds; i.e.

$$(9) \qquad |z_i^2| \leq \sum_{j \in I \setminus i} |z_j^2|; \quad (I = \{1, \ldots, 4\}).$$

As we may commute the numbers z_i^2, the quadrangle $\mathcal{Q}$ is determined only up to the sequence of its sides. As a quadrangle with given side lengths still is flexing, we may interpret $\mathcal{Q}$ as a *four-bar mechanism* (cf. Fig. 3). Let one axis (e.g. the axis x_4^α with unit point A_4^α) coincide with the imaginary axis in π, then the 'base pivots' 0 and z_4^2 of the four-bar linkage $\mathcal{Q}$ are the origin and an arbitrary point on the negative real axis. To make sure that α becomes an *ortho*-axonometry, $\mathcal{Q}$ must have bars with total length 2, according to Lemma 1.

Thus follows

[5] We call an oblique projection $\widehat{\psi}_2^n$ 'isocline', if its image plane is isocline (cf. [20]) to a plane $\mathbb{E}^2$ (totally) orthogonal to the fibers of $\widehat{\psi}_2^n$ and therefore isocline to the fibers themselves.

[6] Cf. [20]; of course 'isocline' includes 'orthogonal' as a subcase.

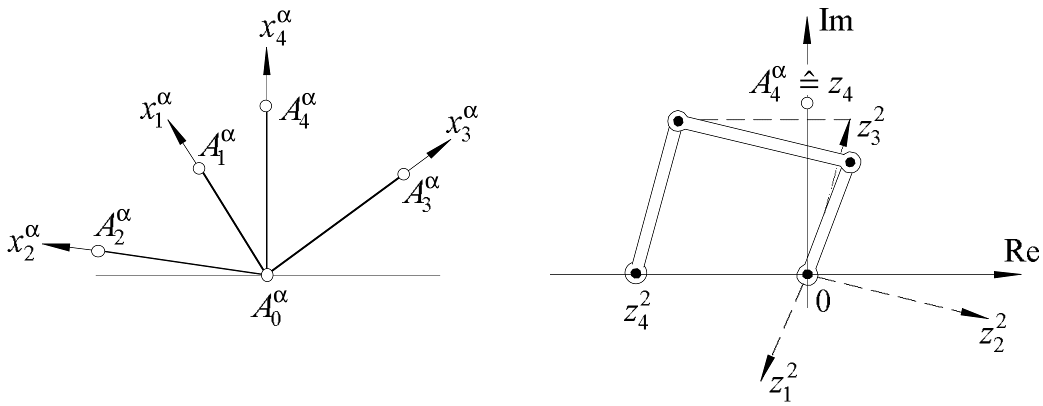

Figure 3: Axonometric image of a 4-frame and the corresponding four-bar linkage

Lemma 3: *To any (labelled) quadrangle in π with circumference 2, interpreted as zero-sum of four complex numbers z_i^2, there exists a unique ortho-axonometric 4-frame with the origin $A_0^\alpha \triangleq 0$ and unit points $A_i^\alpha \triangleq z_i$.*

Lemma 3 provides an easy exact construction[7] of an ortho-axonometric 4-frame by just determining square-roots of complex numbers, a process that easely can be implemented in computer software. E.g. to prescribed distortion ratios $\lambda_1 : \cdots : \lambda_4$ and one angle between two axes (e.g. $\sphericalangle\,(x_1^\alpha, x_4^\alpha)$) the corresponding coupler linkage is (over the field $\mathbb{C}$ of complex numbers) ambiguously determined (cf. Fig. 4 with $\lambda_1 : \cdots : \lambda_4 = 1 : 2 : 2 : 2$ and $<$) $(x_1^\alpha, x_4^\alpha) = 135°$. The corresponding ortho-axonometric 4-frame is *di-metric* and generalizes the engineer's axonometry Fig. 1).

The axonometry α belonging to such a 4-frame is (in general) *similar* to an ortho-projection of $\mathbb{E}^4$ onto π. (The factor of similarity is equal to the radius of the contour circle of the unit sphere $\Omega_4 \subset \mathbb{E}^4$; in the latter we present a construction of this contour.)

With regard to the set of four-bar linkages we conclude that *apart from similarities there exists a four-parametric set of essentially different ortho-(4,2)-axonometries.*

Special examples of ortho-(4,2)-axonometries

Let us demand that the restriction of α to one coordinate plane (e.g. the x_1x_2-plane) is a similarity. Then we obtain equally distorted and orthogonal axes x_1^α, x_2^α, hence

$$z_1^2 + z_2^2 = 0.$$

Then, from (5) follows $z_3^2 + z_4^2 = 0$. In other words, the restriction of α to the x_3x_4-plane also is a similarity and α turns out to be *di-metric.* These special axonometries α are subcases of di-metric axonometries with two pairs of equal distortions; the corresponding four-bar linkages have two pairs of bars of equal length too. For the special subcases the four-bar linkages form a twice covered two-bar linkage, cf. Fig. 5. (The coupler motion ramifies in case of such di-metric linkages.) Furthermore, from Fig. 5 follows immediately that *any two isosceles right*

[7]contrary to WAN's construction of the 'optimal (4,2)-axonometry'.

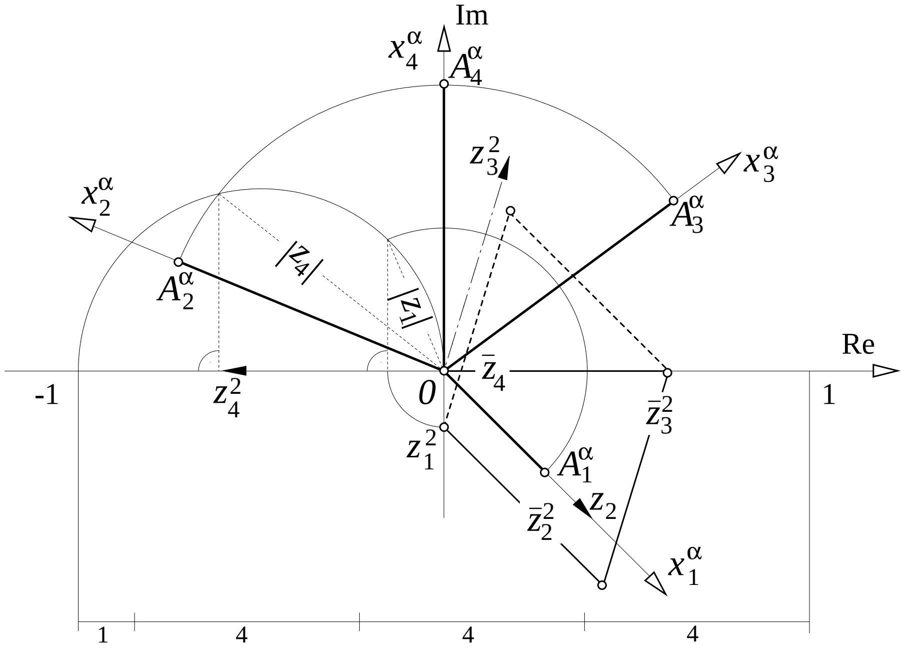

Figure 4: A 4D-version of engineer's axonometry

angle hooks (with common vertex A_0^α) in the euclidean plane π form the axonometric 4-frame of an ortho-(4,2)-axonometry.

As the restriction of α to any coordinate-3-space of $\mathfrak{B}$ is an ordinary 'frontal (3,2)-axonometry', i.e. similar to an oblique projection onto a coordinate plane, constructive treatment of objects of $\mathbb{E}^4$ becomes extremely well-arranged. But the most important advantage seems to be that, generalizing the usual top- and front-projection, one can handle constructions according to e.g. Kruppa [12] in the 'system of two images' based on the restrictions of α to the x_1x_2- and the x_3x_4-plane (cf. also [6, 25, 26, 30]).

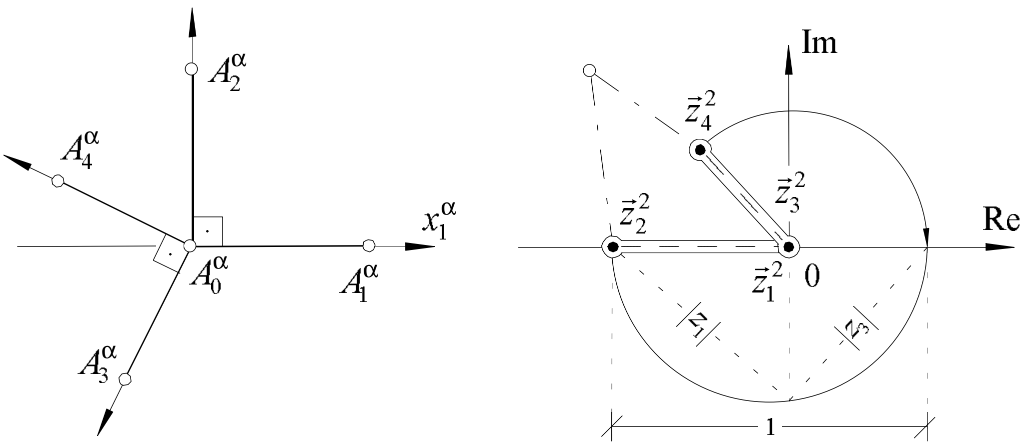

Figure 5: A 'frontal' (4,2)-axonometry

The contour of a hypersphere

Let $\mathfrak{B}^\alpha \subset \pi$ be the (plane) image of a cartesian n-frame $\mathfrak{B} \subset \mathbb{E}^n$ under an arbitrary $(n, 2)$-axonometry $\alpha\colon \mathbb{E}^n \to \pi$. Note that in case of $m = 2$ there always exist planes $\mathbb{E}^2 \subset \mathbb{E}^n$ such that a generalized version of theorem of POHLKE holds, i.e. α is the product of a parallel projection $\psi\colon \mathbb{E}^n \to \mathbb{E}^2$ and a *similarity* $\sigma\colon \mathbb{E}^2 \to \pi$ (cf. [23, 20]). In the following we describe an iterative construction of the contour-ellipse u_n^α of the unit-hypersphere Ω_n based on the well-known construction of the axonometric image of an ordinary sphere Ω_3 (cf. [15], p. 244):

The great circle $k_3 \subset \Omega_3$ in the projecting plane through the axis A_0A_3 is mapped onto a segment $[U_1^\alpha, U_2^\alpha]$ with center A_0^α, cf. Fig. 6. Choose $B \in k_3$ such that $\sphericalangle\,(A_3A_0B) = \pi/2$. Then, if γ measures the angle between the projecting rays s and the segment $[U_1^\alpha, U_2^\alpha]$, and with $\beta := \sphericalangle\,(s, A_0B)$ follows

$$d(A_0^\alpha, A_3^\alpha) = \frac{\cos(\beta)}{\cos(\gamma)}, \quad d(A_0^\alpha, B^\alpha) = \frac{\sin(\beta)}{\cos(\gamma)}, \tag{10}$$

and a contour point $U_1 \in k_3$ is mapped onto U_1^α with

$$d^2(A_0^\alpha, U_1^\alpha) = d^2(A_0^\alpha, B^\alpha) + d^2(A_0^\alpha, A_3^\alpha). \tag{11}$$

Assume that the great circle Ω_2 through the unit points A_1, A_2 has an elliptic α-image u_2^α. If V^α denotes the image of one of the two contour points of Ω_2, (the tangent to u_2^α is parallel to $A_0^\alpha A_3^\alpha$), then the contour ellipse u^α of Ω_3 is determined by the pair of conjugate half-diameters $([A_0^\alpha, U_1^\alpha], [A_0^\alpha, V_1^\alpha])$.

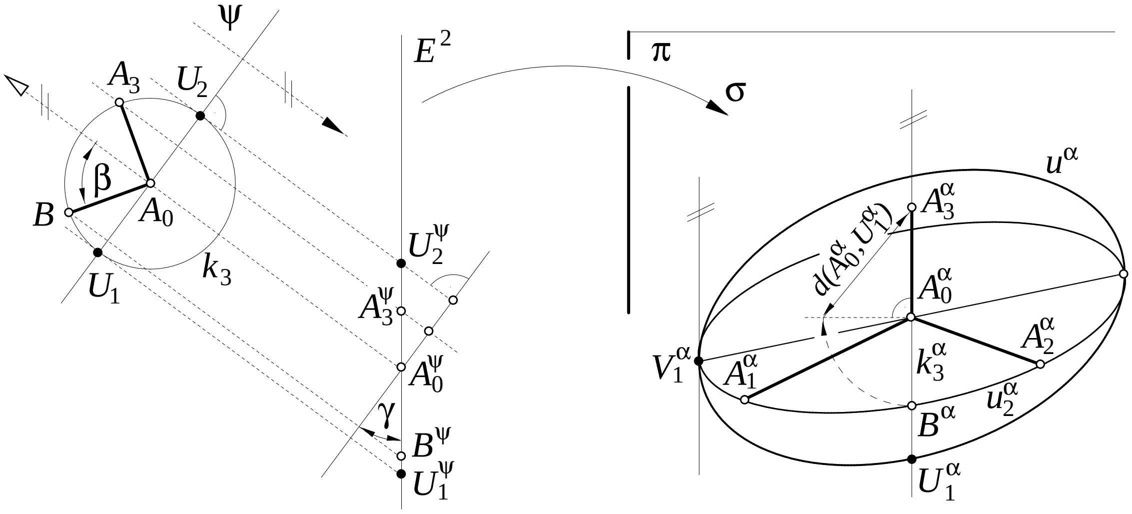

Figure 6: Axonometric contour of the unit-sphere

Remark 1): The length of the minor semiaxis of u^α equals the factor of the similarity $\sigma\colon \mathbb{E}^2 \to \pi$.

Remark 2): For any pair of conjugate half diameters $[A_0^\alpha, A_1^\alpha] =: z_1$ and $[A_0^\alpha, A_2^\alpha] =: z_2$ of e.g. the ellipse u_2^α in the GAUSSIAN plane π' with origin A_0^α yields (cf. [24])

$$f_{1,2} := \sqrt{z_1^2 + z_2^2} = const.\,,$$

whereby the complex numbers f_1 and f_2 describe the focal points F_1, F_2 of u_2^α.

Thus, according to Fig. 6, the conjugate half diameters $[A_0^\alpha, V_1^\alpha] =: v$, $[A_0^\alpha, B^\alpha] =: w$ of u_2^α fulfill

$$z_1^2 + z_2^2 = v^2 + w^2 \ .$$

As the contour ellipse u^α of the unit sphere Ω_3 possesses the conjugate half diameters $[A_0^\alpha, V_1^\alpha]$ and $[A_0^\alpha, U_1^\alpha] =: u$ with $u^2 = w^2 + z_3^2$ because of (11) and collinearity of $\{u, w, z_3\}$, the focal points E_1, E_2 of u^α are described by the complex numbers

$$e_{1,2} := \sqrt{u^2 + v^2} = \sqrt{z_1^2 + z_2^2 + z_3^2} = \sqrt{f^2 + z_3^2}. \tag{12}$$

From the Gauss-condition (5), the construction in Fig. 6, and using the concept of *anti-focal points*[8] $\widetilde{F}_1, \widetilde{F}_2$ of u_2^α we can conclude (cf. [3]),

Lemma 4: *The contour u^α of the unit sphere Ω_3 under a general $(3,2)$-axonometry α is an ellipse in the image-plane π' and the focal points E_1, E_2 of u^α are represented by the complex numbers*

$$e_{1,2} = \sqrt{z_1^2 + z_2^2 + z_3^2}.$$

(The complex numbers z_i and 0 represent the unit points A_i^α and the origin A_0^α of the axonometric three-frame in π'.)

The contour u^α of Ω_3 is a circle, i.e. α is an ortho-(3,2)-axonometry, if and only if A_3^α happens to coincide with an *antifocus* of the image u_2^α of the 'equator' Ω_2 of Ω_3.

A construction of the length a of the main axis of u^α, in case u^α is an ellipse with the focal points E_1, E_2, can be based on the ortho-projection of the axonometric frame $\{A_0^\alpha, \dots, A_3^\alpha\}$ onto the axis $A_0^\alpha E_1$ of u^α:

Let r_j be the distance from A_0^α to the image point $A_j^{\alpha\prime}$ of A_j^α and let ϕ_j be the angle between the axis $A_0^\alpha E_1$ and $A_0^\alpha A_j^\alpha$. According to (3) yields

$$\overline{A_0^\alpha A_j^\alpha} = |z_j| = \lambda_j \text{ and } r_j = \lambda_j \cos\phi_j. \tag{13}$$

Then the half semiaxis of u^α has the length[9]

$$a = \sqrt{r_1^2 + r_2^2 + r_3^2}. \tag{14}$$

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[8]Applying a quarter-turn to the focal points F_1, F_2 of an ellipse u_2 about its center we obtain the so called *anti-focal points* of u_2. They are the focal points of the 'antipolarity' with respect to the conic u_2. (The antipolarity to a conic u_2 possessing a center is the product of the polarity with respect to u_2 and the central reflection in the conic's center (cf. [2]).)

[9]The transpose of the matrix $\mathfrak{R}^T$ (2) maps any unit vector y of the image plane π' onto a vector x of $\mathbb{E}^n$ which is orthogonal to the fibers of α. It turns out that x^2 equals the momentum a_y^2 of the point set $\{\dots, A_j^\alpha, \dots\}$ with respect to the line $\mathsf{y}\mathbb{R}$, cf. [2], p. 22. The vector $a_\mathsf{y}^{-1}\mathsf{y}$ ends in a point of the 'ellipsoid of inertia', while $a_\mathsf{y}^{-1}\mathsf{x}$ describes a point of the contour subsphere of the unit hypersphere Ω_n in $\mathbb{E}^n$. The α-image of the last vector is a point of the contour u^α of Ω_n. In this manner Havlicek (unpublished exercise material to a lecture on 'Geometry with Maple', Vienna 1997) constructs u^α of Ω_n with respect to any given $(n,2)$-axonometry α. Especially if y is a unit vector in $A_0^\alpha E_1$, the momentum $a_\mathsf{y}^2 =: a^2$ measures the square of the length of the main semiaxis of u^α. Thereby yields, analogously to (13) and (14)

$$a^2 = r_1^2 + \cdots + r_n^2.$$

Let now $\alpha = \psi \cdot \sigma$ be a $(n,2)$-axonometry. The $(n,2)$-projection ψ has $(n-2)$-dimensional (totally parallel) fibers and can be generated successively by a sequel of $(k,2)$-projections, $(k = 3, \ldots, n)$. Thus the contour of the unit-hypersphere Ω_n can be determined as follows (cf. Fig. 7):

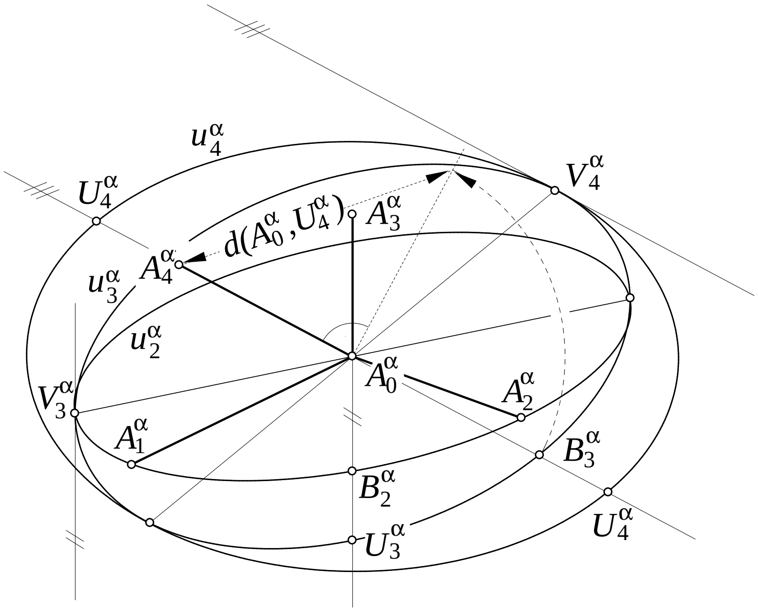

Figure 7: Axonometric contour of a 4D-hypersphere

(a) construct the contour-ellipse u_3^α of the unit-2-sphere Ω_3 according to Fig. 6;

(b) construct one contour point V_3^α of u_3^α with respect to the contour-ellipse u_4^α of the unit-3-sphere Ω_4, (the tangent in V_3^α to u_3^α is parallel to $A_0^\alpha A_4^\alpha$), and construct one intersection point B_3^α of u_3^α with $A_0^\alpha A_4^\alpha$;

(c) the ellipse determined by the pair of conjugate half-diameters $([A_0^\alpha, V_3^\alpha], [A_0^\alpha, B_3^\alpha])$ is the contour-ellipse u_4^α of Ω_4, because (6) and (7) (with index 4 instead of 3) still hold;

(d) repeat (b) and (c) by increasing the indices step by step until you end up with the contour-ellipse u_n^α.

Remark 3): The construction of u_n^α allows any permutation of the set of indices $\{1, \ldots, n\}$. So, if the axonometric frame $\mathfrak{B}^\alpha$ has one pair of orthogonal axes of equal length, one will of course start step (a) with this pair.

Remark 4): Any 2D-CAD-software which is able to handle ellipses given by conjugate diameters can follow the step by step construction of the axonometric contour of a hypersphere.

The recursive construction described above leads to an obvious generalization of (12) and of Lemma 4. We summarize these results in the following

Theorem: *Let $\alpha : \mathbb{E}^n \to \pi'$ be a $(n,2)$-axonometry defined by the image $\mathfrak{B}^\alpha \subset \pi'$ of a cartesian n-frame $\mathfrak{B} = \{A_0, A_1, \ldots, A_n\} \subset \mathbb{E}^n$ and let $A_0^\alpha, A_1^\alpha, \ldots, A_n^\alpha$ be described by the complex numbers $0, z_1, \ldots, z_n$ in the* Gaussian *plane π'. Then the contour u_n^α of the unit sphere Ω_n is an ellipse u_n^α with focal points E_1, E_2 such that their describing complex numbers e_1, e_2 fulfill*

$$e_{1,2} = \sqrt{z_1^2 + \cdots + z_n^2}.$$

The contour u_n^α *of* Ω_n *is a circle, i.e.* α *is* similar *to an* isocline-$(n,2)$-projection ψ, *if and only if* A_n^α *happens to coincide with an* antifocus *of the image* u_{n-1}^α *of the 'equator-subsphere'* Ω_{n-1} *of* Ω_n.

The length a of the main semiaxis of u^α, in case u^α is an ellipse with the focal points E_1, E_2, is the squareroot of the momentum of $\{A_1^\alpha, \ldots, A_n^\alpha\}$ with respect to the axis $A_0^\alpha E_1$. According to (13) and footnote 9 follows

$$a = \sqrt{r_1^2 + \ldots, r_n^2} \text{ with } r_j = \lambda_j \cos \phi_j,$$

where λ_j is the length of the distorted unit segment $[A_0^\alpha, A_j^\alpha]$, ϕ_j its angle to the axis E_1E_2, and r_j the orthoprojection of $[A_0^\alpha, A_j^\alpha]$ onto E_1E_2.

This Theorem provides a simple method to determine an isocline-$(n,2)$-axonometry α by prescribing the images $\{A_0^\alpha; A_1^\alpha, \ldots, A_{n-1}^\alpha\} \subset \pi'$ of $\mathfrak{B} \subset \mathbb{E}^n$ and defining A_n^α as one of the anti-focal points of u_{n-1}^α.

For $n = 4$ Stachel [21] gave another very simple criterion for an ortho-(4,2)-axonometry using two great *circles* of Ω_4 in totally orthogonal coordinate-planes.

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