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Closed Space Curves of Constant Curvature Consisting of Arcs of Circular Helices

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Abstract. A *closed κ_0-curve* is a closed regular curve of class C^r $(r \geq 2)$ in the Euclidean 3-space having constant curvature $\kappa_0 > 0$. We present various examples of nonplanar closed κ_0-curves of class C^2, which are composed of n arcs of circular helices. The construction of c starts from the spherical image (= tangent indicatrix) $c^\star$ of c, which then has to be a closed regular curve of class C^1 on the unit sphere S^2 consisting of n circular arcs and having the center $O^\star$ of S^2 as its center of gravity. The case $c^\star \subseteq S^2 \cap \Pi$ is studied in detail, assuming that Π is a cube, or, more generally, a regular polyhedron the edges of which are tangent to S^2. In order to describe and to visualize the curves $c^\star$ and c, and to derive c from $c^\star$, projection methods of Descriptive Geometry are used.

Keywords: closed (composite) space curve, constant curvature, circular helix, spherical image, tangent indicatrix, center of gravity, regular polyhedron.

MSC 1994: 53A04, 51M20, 51N05

1 Introduction

In the real 3-dimensional Euclidean space $\mathbb{E}^3$, curves c of class C^r $(r \geq 2)$ the first curvature κ of which is prescribed by a function $\kappa = \kappa(s)$ of class C^{r-2} depending on the arc length parameter s of c have been studied in different regards. There is e.g. a theorem of existence and uniqueness up to translations if additionally the spherical image $c^\star$ of c is given (see [4], 183–185; [9], 162–163), including considerations under which assumptions on $c^\star$ and $\kappa(s)$ the curve c is closed ([1], 78–79; [4], 183–185).

In [2], p. 182, E. Cesàro has considered the special case of space curves with *constant curvature* $\kappa(s) = \kappa_0 > 0$. These curves usually are called *twisted circles* of curvature κ_0 (in German: *windschiefe Kreise*), although they need not be closed, or shortly κ_0*-curves*. Special

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twisted circles having *constant torsion* $\tau(s) = \tau_0$ are the closed but planar *circles* ($\tau_0 = 0$) and the nonplanar and not closed *circular helices* ($\tau_0 \neq 0$).

In the differential-geometric literature it has been mentioned (see [1], 78–79) and proved (see [4], 183–185; compare [9], 162–163) that there exist nontrivial, i.e. *nonplanar*, closed twisted circles of curvature κ_0, but explicit examples of such nontrivial *closed* κ_0*-curves* seem to be missing.

In this note we present *examples of closed* κ_0*-curves* c *of class* C^2 which can be generated by *connecting arcs of circular helices of curvature* κ_0 (see Sections 5 and 6). Our construction, as developed in Sections 3.1 and 4, starts from the *tangent indicatrix* $c^\star$ of c which then must consist of *circular arcs on the unit sphere* S^2. In this context, we first need some differential geometric tools concerning closed κ_0-curves in general (see Section 2).

Note that, in the following sections, it is substantial, in particular with respect to the spherical image $c^\star$, to distinguish between a *"curve"* (a *regular curve*) as a bare point set and its parametrization — in the sense of an *oriented parametric curve* — which will be called a *"path"* (a *regular path*), respectively. Only by using the notion *path* it is e.g. possible to express that a circle or a circular arc is multiply traced by any moving point.

2 Differential geometrical tools

Proposition. *Construction of a closed* κ_0*-curve* c *of class* C^2 *from its tangent indicatrix* $c^\star$ *of class* C^1: In the Euclidean 3-space $\mathbb{E}^3$ let be given a regular closed curve $c^\star : \mathfrak{x}^\star(s^\star), s^\star \in I^\star := [0, L^\star]$ of class C^{r-1} $(r \geq 2)$ on the unit sphere S^2 with center $O^\star(0,0,0)$. Let $c^\star$, without loss of generality, be parametrized by its arc length parameter $s^\star$ $(0 \leq s^\star \leq L^\star; L^\star > 0)$ thus defining a path (oriented parametric curve) $c^\star$ of *path length* $L^\star$. Under these conditions which are equivalent to

$$(1)\qquad \begin{aligned} &\|\mathfrak{x}^\star(s^\star)\| = \|\frac{d\mathfrak{x}^\star}{ds^\star}(s^\star)\| = 1 \quad \forall s^\star \in I^\star, \\ &\mathfrak{x}^\star(0) = \mathfrak{x}^\star(L^\star), \quad \frac{d^k\mathfrak{x}^\star}{ds^{\star k}}(0) = \frac{d^k\mathfrak{x}^\star}{ds^{\star k}}(L^\star) \quad (k = 1, \ldots, r-1) \end{aligned}$$

the following propositions (a), (b), (c) hold[1]:

(a) For any $\kappa_0 > 0$, the curve $c \subset \mathbb{E}^3$ which is defined — according to (1) — by

$$(2)\qquad c: \quad \mathfrak{x}(s) = \frac{1}{\kappa_0}\int_0^{\kappa_0 s} \mathfrak{x}^\star(s^\star) ds^\star, \quad s \in I := [0, L], \quad L := \frac{L^\star}{\kappa_0}$$

is a regular space curve of class C^r with the arc length parameter s, the path length $L = L^\star/\kappa_0$ and the constant first curvature $\kappa(s) := \|\mathfrak{x}''(s)\| = \kappa_0$, i.e. c is a κ_0*-curve of class* C^r (compare [9], 162–163); corresponding arcs of $c^\star$ and c have proportional lengths $s^\star \in [0, L^\star]$ and $s = s^\star/\kappa_0 \in [0, L]$, respectively.

Note that, according to (2), any two closed κ_0-curves c_1 and c_2 of constant positive curvature κ_1, κ_2, respectively, corresponding to the *same spherical image* (tangent indicatrix) $c^\star$ are *similar* to each other: c_2 arises from c_1 via a *homothety of ratio* $\kappa_1/\kappa_2 > 0$.

[1] for the conditions of closure used here in $(1)_2$, compare [8], p. 21

(b) A κ_0-curve c of class C^r $(r \geq 2)$ defined by (2), (1) is *closed*, i.e., a closed κ_0-curve, if and only if the geometric center of gravity $G^\star$ of its tangent indicatrix $c^\star$ considered as a path coincides with the center $O^\star$ of S^2, that is if and only if (compare [4])

$$(3) \qquad L^\star \cdot \overrightarrow{O^\star G^\star} := \int_0^{L^\star} \mathfrak{x}^\star(s^\star) ds^\star = \mathfrak{o}.$$

(c) A closed κ_0-curve c of class C^r $(r \geq 2)$ defined by (2), (1), (3) is *nonplanar* if and only if its closed tangent indicatrix $c^\star \subset S^2$ as a point set is *not* a great circle of S^2.

Proof:

(a) From (2) we get

$$\frac{d\mathfrak{x}}{ds}(s) = \mathfrak{x}^\star(\kappa_0 s) \overset{(1)}{\Longrightarrow} \left(\frac{d\mathfrak{x}}{ds}\right)^2 \equiv 1 \ \forall s \in I, \quad \mathfrak{x}''(s) := \frac{d^2\mathfrak{x}}{ds^2}(s) = \kappa_0 \frac{d\mathfrak{x}^\star}{ds^\star}(\kappa_0 s).$$

Because of (1) this means that c is *regular* having s as its *arc length parameter* and the *constant* curvature $\kappa(s) = \|\mathfrak{x}''(s)\| = \kappa_0$.

(b) In consequence of (2), (1) we get:
A κ_0-curve $c : \mathfrak{x}(s), s \in I$ is *closed* $\iff$ $\mathfrak{o} = \mathfrak{x}(L) - \mathfrak{x}(0) = \frac{1}{\kappa_0} \int_0^{L^\star} \mathfrak{x}^\star(s^\star) ds^\star \iff$ (3).

(c) Any regular curve $c \subset \mathbb{E}^3 : \mathfrak{x}(s), s \in I$ of class C^1, having without loss of generality s as its arc length parameter, is *planar* if and only if there exists a vector $\mathfrak{a} \in \mathbb{R}^3 \backslash \{\mathfrak{o}\}$ such that $\mathfrak{a} \cdot \mathfrak{x}(s) = \mathfrak{a} \cdot \mathfrak{x}(s_0)$, $\forall s \in I$ $(s_0 \in I)$ $\iff$ $\mathfrak{a} \cdot \mathfrak{x}'(s) = 0 \ \forall s \in I$ $(*)$.
On the other hand, the tangent indicatrix $c^\star : \mathfrak{x}^\star(s) := \mathfrak{x}'(s)$, $s \in I$ (which is a curve of class C^0 on S^2) of such a regular curve c of class C^1 is contained in a great circle of S^2 if and only if $c^\star$ is contained in a plane having normal vector $\mathfrak{b}$ and containing $O^\star(0,0,0)$ $\iff$ there exists a vector $\mathfrak{b} \in \mathbb{R}^3 \backslash \{\mathfrak{o}\}$ such that $0 = \mathfrak{b} \cdot \mathfrak{x}^\star(s) = \mathfrak{b} \cdot \mathfrak{x}'(s) \ \forall s \in I$ $(**)$.
Since the conditions $(*), (**)$ coincide for $\mathfrak{a} = \mathfrak{b}$, one gets the local [global] result: A regular [closed regular] curve $c \subset \mathbb{E}^3$ of class C^1 is planar $\iff$ the point set of the tangent indicatrix $c^\star$ of c is contained in [equals] a great circle of S^2. q.e.d.

Remark: According to FENCHEL's Theorem (see [3], 238–245; [7], 139–145; [8], 31–32) and Eq. (2), the path length $L^\star$ of the tangent indicatrix $c^\star$ of a nonplanar closed κ_0-curve c and the length L of c satisfy the inequalities $L^\star > 2\pi$ and $L = L^\star/\kappa_0 > 2\pi/\kappa_0$, respectively.

3 Special tangent indicatrices of closed κ_0-curves, lying on regular polyhedra

3.1

Let Π be any *regular polyhedron* (regular tetrahedron, hexahedron (cube), octahedron, dodecahedron or icosahedron, respectively), of edge length $e > 0$, and denote by $O^\star$ the *point of symmetry* (= center of the circumsphere) of Π. It is well-known and obvious for reasons of symmetry that there is a unique sphere $S^2(O^\star, \rho)$ with center $O^\star$ and radius $\rho = \rho(e) > 0$ being tangent to all of the edges of Π touching them at their midpoints $A, B, C, \ldots$, compare [5], 84–91 for certain tetrahedra Π and [6], 436–438 for regular polyhedra Π. Obviously, this *edge-sphere* (German: *Kantenkugel*) of Π intersects each face of Π, which is a regular m-gon,

$m \in \{3,4,5\}$, in its incircle being tangent to the edges of this face at their midpoints. Let e be chosen in such a way that $\rho(e) = 1$, i.e. that the edge-sphere of Π is the unit sphere $S^2(O^\star, 1) =: S^2$ with center $O^\star$. In the case of a cube Π, which will be studied in detail later on, this means $e = \sqrt{2}$.

For reasons of symmetry, $O^\star$ is the center of gravity of the union $\mathcal{I}$ of the incircles of all faces of Π. Now, by Proposition (b), any closed regular path $c^\star \subseteq \mathcal{I}$ of class C^1, having $O^\star$ as its center of gravity, will be the spherical image (= the tangent indicatrix) of closed κ_0-curves c of class C^2, which are determined by $c^\star$ up to similarities, the ratio of similarity being $1/\kappa_0$. By this we have found a simple *geometric method* of constructing tangent indicatrices of closed κ_0-curves of class C^2 consisting of arcs of circular helices.

3.2

For first examples — which show that closed κ_0-curves consisting of helical arcs do exist — we consider in Fig. 1 three different regular polyhedra Π_i ($i = 1,2,3$), a *cube* Π_1, a regular *tetrahedron* Π_2, and a regular *octahedron* Π_3, each of them having an edge-sphere S^2 with radius 1. Each of the regular polyhedra Π_1, Π_2, Π_3 has regular m-gons as faces, with $m = 4,3,3$, respectively. Each of these polyhedra Π_i carries a spherical image $c_i^\star \subseteq \Pi_i \cap S^2$ of closed κ_0-curves c_i ($i = 1,2,3$): The paths $c_i^\star$ chosen in Fig. 1 consist of $n = 6,4,6$ circular arcs of central angle $2\pi/m$, which is the smallest central angle possible, respectively; none of these paths $c_i^\star$ is contained in a great circle of S^2, and each of them has the center $O^\star$ of its supporting regular polyhedron Π_i as its center of gravity, for each of the curves $c_1^\star, c_3^\star$ is symmetric with respect to the center $O^\star$ of Π_1, Π_3, respectively, and the curve $c_2^\star$ is mapped onto itself by a rotary reflection ρ which is composed of the reflection with respect to the plane $ABCD$ and a $(\pi/2)$- or $(-\pi/2)$-rotation about the normal of this plane passing through the center $O^\star$ of Π_2, which is the only fixed point of ρ. So each of the paths $c_i^\star$ ($i = 1,2,3$) in Fig. 1 is the *spherical image of nonplanar closed κ_0-curves* c_i (Examples 1, 2, 3). The shape of these curves c_i can be derived using the Proposition, Eq. (2) and Section 4, which may be left to the reader. For additional examples of nonplanar closed κ_0-curves, see the Examples 4–13 (Figs. 2–7), which are discussed in detail, and Example 14 in footnote 3.

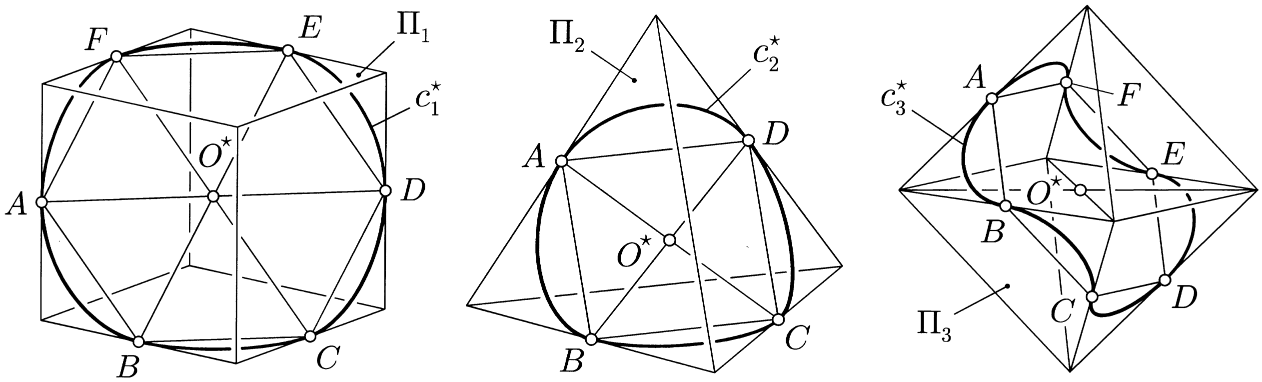

Figure 1: Examples of regular polyhedra carrying a tangent indicatrix of closed κ_0-curves.
$c_1^\star$: $+[1/4]ABCDEFA$, $c_2^\star$: $+[1/3]ABCDA$, $c_3^\star$: $+[1/3]ABCDEFA$

3.3

More generally, the way of composing the tangent indicatrix $c^\star$ of closed κ_0-curves c by n *congruent circular arcs of length* $k \cdot 2\pi/m$ ($k \in \mathbb{N}$) on the incircles of the faces (= regular m-gons, $m \in \{3,4,5\}$) of a regular polyhedron Π, can be described up to congruence and symmetry by a *structural formula* of the type (for $k = 1$, compare Fig. 1)

$$c^\star: \ \pm[k/m]A_1A_2\dots A_nA_1 \tag{4}$$

representing an algorithm of constructing $c^\star$ as follows. Looking at the outside of the polyhedron Π, each of the incircles of the faces of Π can be oriented either in the positive/counterclockwise or negative/clockwise sense.

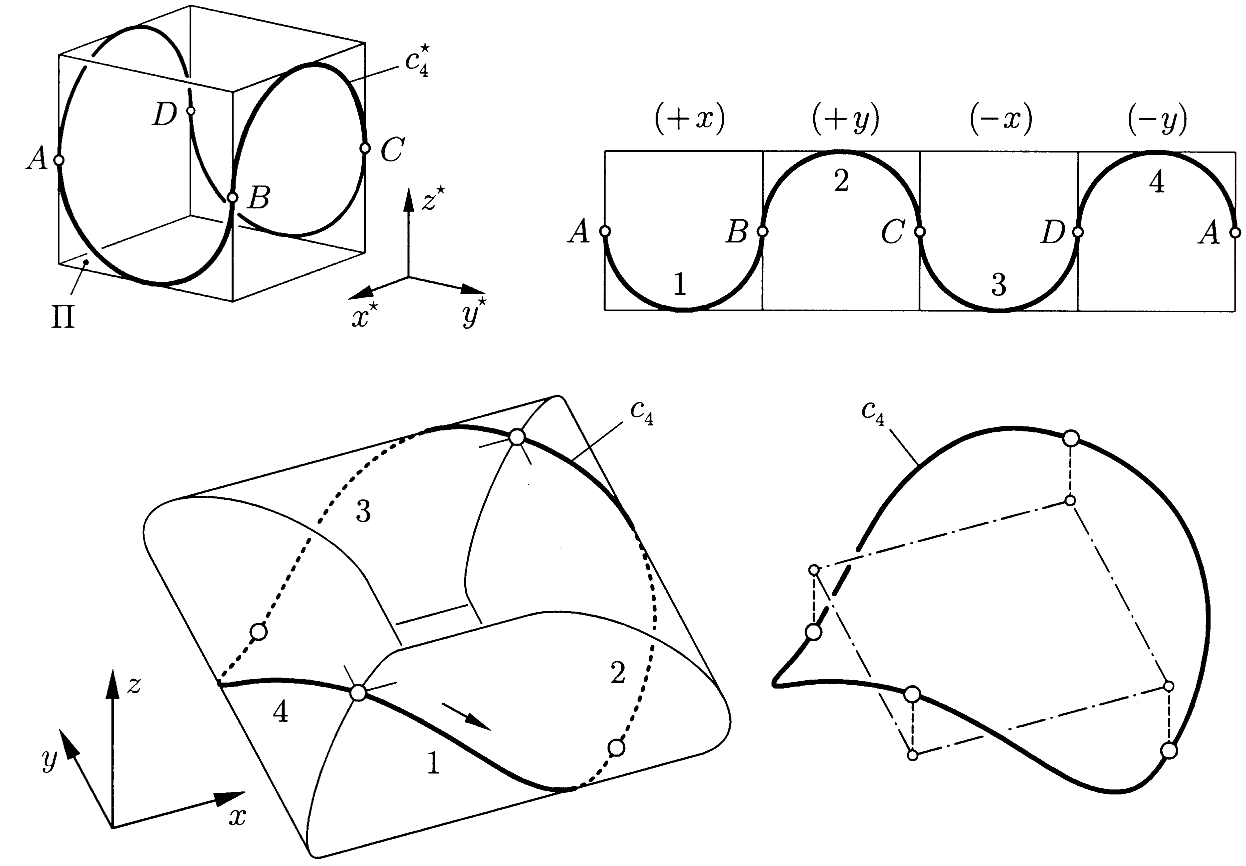

Figure 2: (Example 4) Closed κ_0-curve c_4 of class C^2, consisting of (the minimum number of) four congruent helical arcs. Structural formula of $c_4^\star$: $+[1/2]ABCDA$

Eq. (4) tells us to construct a *regular* path $c^\star \subset S^2$ of class C^1 by connecting, one after each other, each of the n pairs $\{A_1, A_2\}, \{A_2, A_3\}, \dots, \{A_n, A_1\}$ of edge-midpoints (= bisecting points of the edges) of Π by a circular arc which is uniquely determined by (i) lying in the unique face of Π which contains both edge-midpoints which are to be connected, (ii) having the central angle $2\pi \cdot k/m$, and (iii) having correct orientation. The *correct orientation* of the circular arcs is to be determined as follows: the first ("starting") arc $\widehat{A_1A_2}$ of $c^\star$ is oriented in the sense corresponding to the sign $+$ or $-$, respectively, at the beginning of Eq. (4). Proceeding on $c^\star$, any circular arc $\widehat{A_iA_{i+1}}$ ($i = 2, \dots, n \bmod n$) must be oriented in the same or the opposite way as its preceding arc $\widehat{A_{i-1}A_i}$, according to if one remains in the same face of Π, which then contains the three edge-midpoints A_{i-1}, A_i, A_{i+1}, or if a new face of Π has to be entered ($\Leftrightarrow$ the face of Π being uniquely determined by A_i, A_{i+1} is different from the face containing A_{i-1} and A_i).

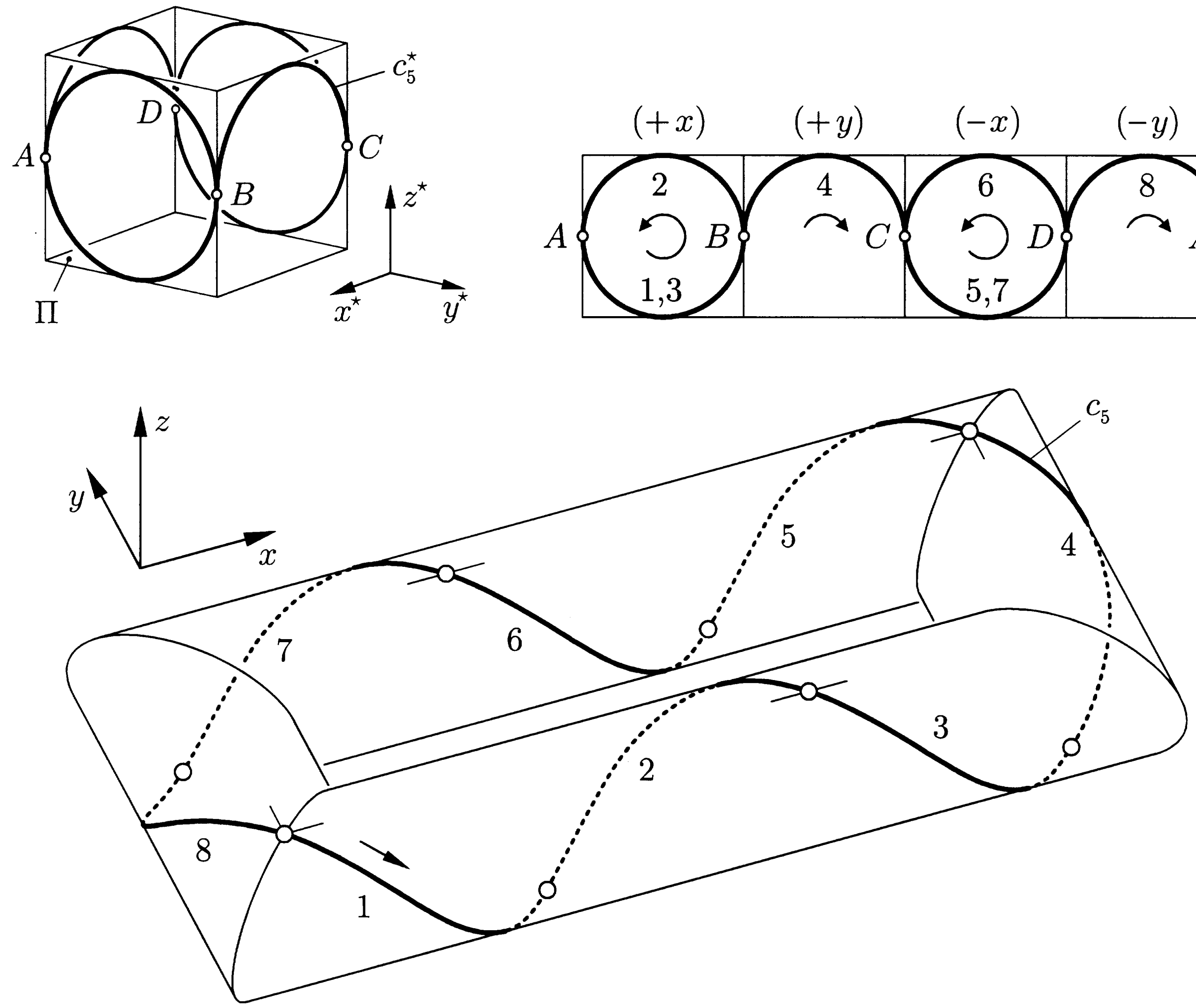

Figure 3: (Example 5) Closed κ_0-curve c_5 of class C^2, generated from Example 4 (Fig. 2) by insertion of a pair of congruent helical arcs 2, 3 and 6, 7 of the angle of rotation 2π. This can also be achieved by extending each of the helical arcs 1, 3 of c_4 to its threefold length.

Structural formula of $c_5^\star$: $+[1/2]ABABCDCDA = +[3/2]AB[1/2]BC[3/2]CD[1/2]DA$

This *rule of orientation* influences in the case $k/m \neq 1/2$ the selection of the next possible edge-midpoint A_{i+1}, too. This results from the demand that there are *no cusps* on the path $c^\star := \widehat{A_1A_2} \cup \widehat{A_2A_3} \cup \ldots \cup \widehat{A_nA_1}$, i.e., that $c^\star$ is a *regular path* (a regular closed curve of class C^1). Note that this rule of orientation must be satisfied for the transition from the last arc $\widehat{A_nA_1}$ to the starting arc $\widehat{A_1A_2}$, too!

After having selected such a regular closed path $c^\star \subseteq \mathcal{I}$ on Π, one has to check if the center of gravity $G^\star$ of the path $c^\star$ coincides with $O^\star$. In most of the cases considered in this note, this property results more or less from symmetry. For instance, all of the paths $c^\star$ of the Figs. 1(a),(c), 5, 6, 7 and of footnote 3 (Example 14) are symmetric with respect to $O^\star$, and all of the paths $c^\star$ of the Figs. 2, 3, 4 are symmetric at least to the line which is parallel to the $z^\star$-axis and contains $O^\star$. If $G^\star = O^\star$ — and only then — $c^\star$ can be interpreted as the *tangent indicatrix of closed κ_0-curves* which then are described by (2).

3.4

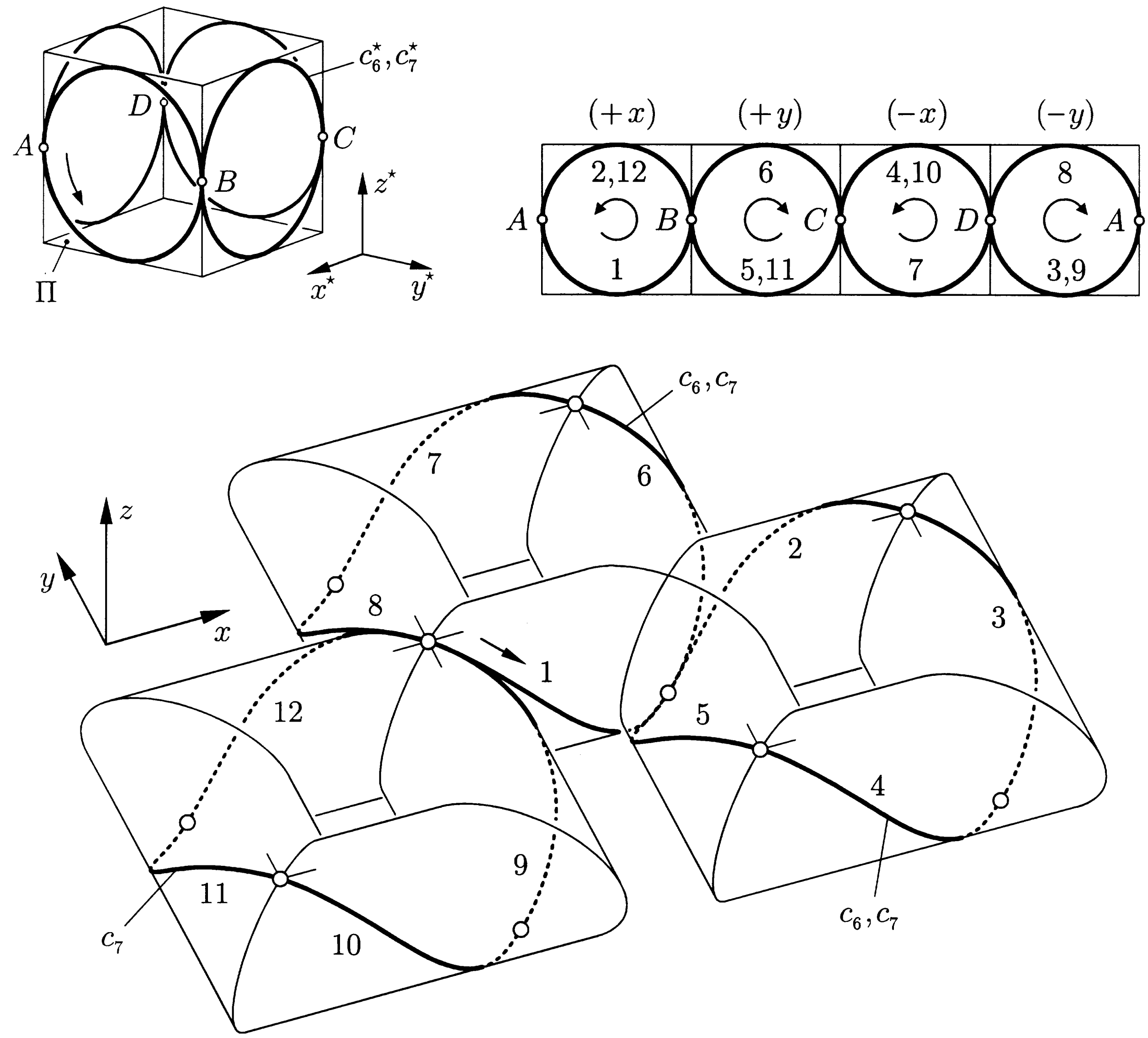

Figure 4: (Examples 6, 7) Closed κ_0-curves c_6 (arcs $1, \dots, 8$) and c_7 (arcs $1, \dots, 12$) of class C^2, generated by joining two or three copies, respectively, of Example 4 (Fig. 2);

$$c_6^\star: \ +[1/2]ABADCBCDA = +[1/2]BADCBCDAB,$$
$$c_7^\star: \ +[1/2]ABADCBCDADCBA = +[3/2]BA[1/2]ADCBCDADCB$$

In the most general case under consideration, we admit that $c^\star$, being contained in the intersection of the faces of a regular polyhedron Π and its edge-sphere S^2 of radius 1, is composed from $n \geq 4$ circular arcs of *different lengths* $2\pi \cdot k_i/m$ ($k_i \in \mathbb{N}$, $i = 1, 2, \dots, n$; $n \geq 4$), see Figs. 3, 4, 6. The structural formula (4) now is to be generalized to expressions of the type

$$(5) \qquad c^\star: \ \pm[k_1/m]A_1A_2 \dots A_p[k_2/m]A_pA_{p+1} \dots A_nA_1$$

containing two or more changes of the lengths of the circular arcs within $c^\star$, by inserting one or more *commands of length* $[k_1/m]$, $[k_2/m]$, and so on. Each of these commands determines the lengths of the following circular arcs until the next command of length appears.

4 Closed κ_0-curves consisting of arcs of circular helices

In order to determine the shape and analytical representation of *nonplanar closed* κ_0-curves c of class C^2 consisting of $n \geq 4$ *arcs of circular helices* we now have to start from their tangent indicatrices $c^\star$. These special curves/paths $c^\star$ are characterized by (i) being closed regular curves of class C^1 on the unit sphere S^2 with center $O^\star(0,0,0)$, which consist of n *circular arcs* $h_i^\star$ $(i = 1, \dots, n)$ of S^2 and (ii) having $O^\star$ as their *center of gravity*.

1. *First step*: We determine the helical arc $h \subset c$ having a given circular arc $h^\star \subset c^\star$ with the central angle $\omega > 0$ and radius $r^\star =: \sin\alpha$, $\alpha \in]0, \pi/2]$ as its tangent indicatrix. This is since $0 < r^\star \leq 1$. Up to a proper or improper motion, a parametrization of $h^\star$ depending on its angle parameter t can be given with respect to the Cartesian $x^\star y^\star z^\star$-coordinate system by

$$h^\star : \quad \mathfrak{x}_h^\star(t) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos t \\ \sin\alpha\sin t \end{pmatrix}, \quad t \in [0, \omega]; \tag{6}$$

thus $h^\star$ is contained in the plane $x^\star = \cos\alpha$. The arc length parameter $s^\star$ of the curve $h^\star \subset c^\star$, defined by (6), is (without loss of generality) $s^\star(t) = t\sin\alpha$; then, in respect of Proposition (a), the arc length parameter s of the κ_0-curve $h \subset c$, corresponding to the parametrization (6) of $h^\star$, is given by $s(t) = (\sin\alpha/\kappa_0)t$. Using (6) and $s^\star(t)$, $s(t)$ the formula (2) yields the parametrization $\mathfrak{x}_h(t)$ of the curve h of constant curvature κ_0 depending on the parameter t:

$$h : \quad \mathfrak{x}_h(t) = \frac{\sin\alpha}{\kappa_0} \begin{pmatrix} t\cos\alpha \\ \sin\alpha\sin t \\ -\sin\alpha\cos t \end{pmatrix}, \quad t \in [0, \omega]. \tag{7}$$

Obviously h is an arc of a *circular helix*, including the special case of a *circular arc* if $\alpha = \pi/2$, having the radius

$$r = r(\alpha, \kappa_0) = \frac{\sin^2\alpha}{\kappa_0}, \tag{8}$$

which equals the radius of the circular cylinder containing the helix h, and having the pitch $(\sin 2\alpha/\kappa_0)\pi$ (thus having the *angle of slope* $\pi/2 - \alpha$) and the angle of rotation ω; the axis of h is parallel to the x-axis.

2. *Second step*: Each of the circular arcs $h_i^\star \subset c^\star$ having the radius $r_i^\star =: \sin\alpha_i$ $(0 < \alpha_i \leq \pi/2)$ and the central angle $\omega_i > 0$ is congruent by some motion μ_i, via $h_i^\star = \mu_i h^\star$, to the circular arc $h^\star$ in (6) if we choose $\alpha = \alpha_i$ and $\omega = \omega_i$ $(i = 1, \dots, n)$. Then the helical arc $h_i \subset c$ corresponding to $h_i^\star$ is uniquely determined via $h_i = \tau_i \circ \mu_i h$, taking h according to (7) and setting $\alpha = \alpha_i$, $\omega = \omega_i$; furthermore, τ_i is a translation which is to be chosen uniquely in such a way that the endpoint with parameter $t = \omega_{i-1}$ of the helical arc h_{i-1} coincides with the starting point with parameter $t = 0$ of the helical arc h_i $(i = 1, \dots, n)$. The "helical arc" h_0 formally appearing here is without loss of generality to be set as the single point O^*, which is both starting point *and* endpoint of h_0.

The figures in this note have been produced using an algorithm based on the formulas and techniques from above.

5 Explicit examples of nonplanar closed κ_0-curves, the tangent indicatrix of which lies on a cube

5.1 General considerations:

In this section, we consider closed κ_0-curves c the tangent indicatrix $c^\star$ of which is contained as a point set in the union $\mathcal{I}$ of the incircles of the faces of a *cube* Π of edge length $e = \sqrt{2}$, thus having an edge-sphere $S^2 \supset c^\star$ of radius 1.

Using the method described in Section 3.1, various examples of spherical images $c_i^\star \subseteq \Pi \cap S^2$ ($i = 1$ and $i = 4, \dots, 13$) of nonplanar closed κ_0-curves c_i are presented, see the parts (a) of Figs. 1–6 for $i = 1; 4, \dots, 12$ and, for $i = 13$, also Fig. 7(a) in Section 6, where Π is replaced by a regular prism[2]. The center $O^\star$ of S^2 and of the cube or prism Π, coincides with the center of gravity $G_i^\star$ of each curve $c_i^\star$, and none of these curves is contained in a great circle of S^2 ($i = 1; 4, \dots, 13$). Thus the κ_0-curve c_i corresponding to $c_i^\star$ by Proposition (a), Eq. (2) is *closed and nonplanar*[3]; it is of class C^2 since $c_i^\star$ is of class C^1. Furthermore c_i consists of helical arcs which correspond to the circular arcs of $c_i^\star$ and can be explicitly found according to Section 4. In this context, the way how the *path* $c_i^\star$ is traced is essential: it is formally determined by the structural formula of $c_i^\star$ which is additionally visualized in Figs. 2–5 by a *planar diagram* of the path $c_i^\star$, which is gained by developing those faces of the cube Π which contain arcs of $c_i^\star$, and by numbering in the same way the circular arcs of $c_i^\star$ and their corresponding helical arcs of c_i (compare Figs. 2–5, parts (b), (c)).

In Fig. 6 the shape of the curve c_{12}, and how this curve is traced, is visualized — in a different way — by indicating the correspondence between the endpoints (linking points) $A, \dots, K$ of the circular arcs of $c_{12}^\star$ and the pairs $\{A_1, A_2\}, \dots, \{K_1, K_2\}$ of corresponding image points of c_{12}. The fact that the condition $G_i^\star = O^\star$ is really satisfied for all of the curves/paths $c_i^\star$ ($i = 1, \dots, 14$) of this note can be deduced either from symmetry (for instance, $O^\star$ is the point of symmetry of $c_i^\star$ for $i = 1, 3, 8\text{-}14$) or directly from the planar diagram (see the curves $c_i^\star$, $i = 4\text{-}8$, in the Figs. 2–5).

For practical reasons the two cases $c_i^\star \neq \mathcal{I}$ and $c_i^\star = \mathcal{I}$ are now to be distinguished in the following Sections 5.2, 5.3 which deal with the Examples 1, 4–12, referring to a cube Π.

5.2 Case $c^\star \neq \mathcal{I}$

The simplest of the Examples 1, 4–8 (Figs. 1(a), 2–5) of this case are the nonplanar closed κ_0-curves of the Examples 1 (Fig. 1(a)), 4 (Fig. 2) and 8 (Fig. 5) which consist of *congruent* helical arcs, the congruent circular arcs of the corresponding tangent indicatrix having the central angle $\pi/2$, π and $3\pi/2$, respectively. Further examples of closed κ_0-curves can be derived from the above curves by *insertion* of a pair of helical arcs being congruent (by translation, see Example 5 in Fig. 3, or by reflection with respect to the axis of the first helical arc and translation, see Example 12 in Section 5.3) and by *joining* two or more closed κ_0-curves (Examples 6, 7 in Fig. 4; see Examples 9–11 in Section 5.3, too). Note: the insertion

[2]For further examples $c_i^\star$ ($i = 2, 3, 14$) of spherical images of nonplanar closed κ_0-curves c_i see Section 3.2, Fig. 1(b),(c) for $i = 2, 3$, and footnote 3 for $i = 14$.

[3] and, moreover, no arc of c_i ($i = 1, \dots, 13$) lies in a plane. On the other hand, there also are nonplanar closed κ_0-curves c of class C^2 being composed of planar and nonplanar arcs, e.g. ≥ 2 circular arcs and ≥ 2 helical arcs. The spherical image $c^\star =: c_{14}^\star$ of a closed κ_0-curve $c =: c_{14}$ of this kind (Example 14) may be composed, for example, as a path $c_{14}^\star := s_1 \cup g_1 \cup s_2 \cup g_2$ consisting of a pair of semicircles g_1, g_2 of a great circle $g \subset S^2$ and two small circles s_1, s_2 on S^2 each of which is tangent to g at one of the two common endpoints of g_1, g_2 in such a way that s_2 is symmetric to s_1 with respect to the center $O^\star$ of S^2.

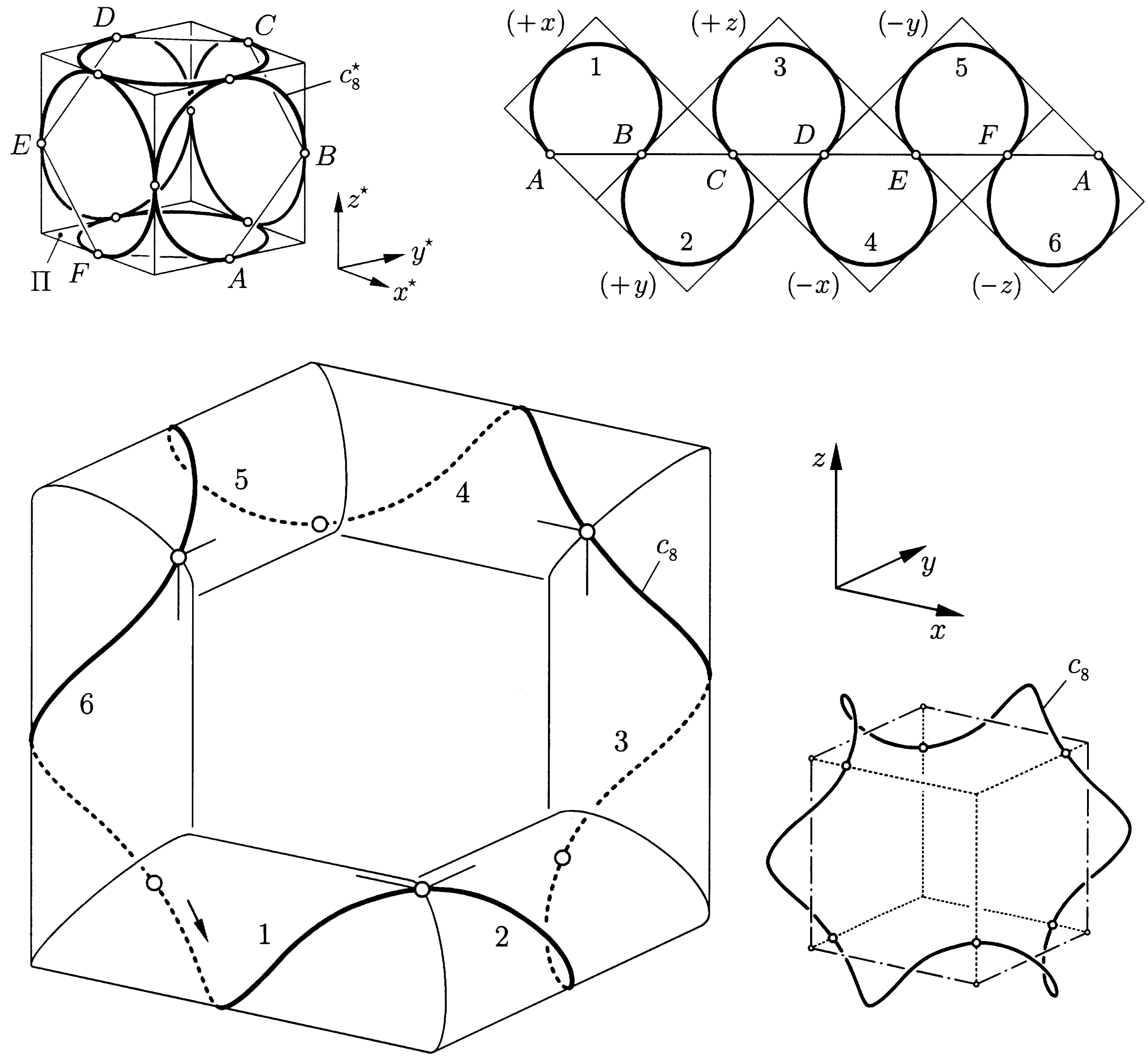

Figure 5: (Example 8) Closed κ_0-curve c_8 of class C^2, consisting of six congruent helical arcs having three pairwise orthogonal axial directions; $c_8^\star$: $-[3/4]ABCDEFA$

of a pair of helical arcs, being congruent as described above and having the angle of rotation $l \cdot 2\pi$ ($l \in \mathbb{N}$), into a closed κ_0-curve c is *equivalent* to the insertion a pair of complete incircles — each of them being covered l times — of opposite faces of the cube Π into the tangent indicatrix $c^\star$ of c; this action does not change the center of gravity $G^\star = O^\star$ of $c^\star$.

5.3 Case $c^\star = \mathcal{I}$:

Finally, we show that there are closed κ_0-curves having a spherical image $c^\star$ lying in a cube Π of edge length $e = \sqrt{2}$ and having the property $c^\star = \mathcal{I}$ in the following sense: Except for the 12 edge-midpoints of Π, each of which then will be covered twice, the path $c^\star$ covers each point of $\mathcal{I}$, which equals the union of the six incircles of the faces of the cube Π, once and only once. Referring to Fig. 6(a), one can easily verify that each of the four curves $c_i^\star \subseteq \mathcal{I}$

($i = 9, 10, 11, 12$), the path on $\Pi \cap S^2$ of which is described by the structural formula:

$$
\begin{aligned}
c_9^\star: \quad & -[3/4]ABCDEFA[1/4]AFEDCBA = -[1/4]BA[3/4]ABCDEFA[1/4]AFEDCB \\
& = -[1/2]\underline{BKB}[3/4]BCDEF[1/2]\underline{FJF}[1/4]FEDCB, \\
c_{10}^\star: \quad & +[1/2]AHDGA[1/4]AJ[1/2]\underline{JCJ}[1/4]JG[1/2]GDH[1/4]HK[1/2]\underline{KEK}[1/4]KA, \\
c_{11}^\star: \quad & +[1/2]AHD\underline{GAG}[1/4]GE[1/2]\underline{EKE}[1/4]EDC[1/2]\underline{CJC}[1/4]CH[1/2]HA, \\
c_{12}^\star: \quad & +[1/4]AB[1/2]\underline{BIB}[1/4]BH[1/2]HD\underline{GAG}DH[1/4]HK[1/2]\underline{KEK}[1/4]KA,
\end{aligned}
$$

respectively (complete circles are underlined), satisfies the property $c_i^\star = \mathcal{I}$. Therefore the center $O^\star$ of the cube Π coincides with the center of gravity of the path $c_i^\star$ ($i = 9, 10, 11, 12$), too. Consequently, each of these pathes $c_i^\star$ is the tangent indicatrix of closed κ_0-curves c_i. These *four examples* 9, 10, 11, 12 *of closed κ_0-curves* the tangent indicatrix of which is $c_9^\star, c_{10}^\star, c_{11}^\star, c_{12}^\star$, respectively, are quite different one from each other. Each of the three curves $c_9^\star, c_{10}^\star, c_{11}^\star$ can be separated into a pair $c_i^\star =: a_i^\star \cup b_i^\star$ ($i = 9, 10, 11$) of independent tangent indicatrices of closed κ_0-curves being characterized by the structural formulas (referring to Fig. 6(a)):

$$
\begin{aligned}
& a_9^\star: \; -[3/4]ABCDEFA, \quad b_9^\star: \; +[1/4]AFEDCBA; \\
& a_{10}^\star: \; +[1/2]AHDGA, \\
& b_{10}^\star: \; -[1/4]AJ[1/2]\underline{JCJ}[1/4]JG[1/2]GDH[1/4]HK[1/2]\underline{KEK}[1/4]KA; \\
& a_{11}^\star = a_{10}^\star, \quad b_{11}^\star: \; -[1/2]AG[1/4]GE[1/2]\underline{EKE}[1/4]EDC[1/2]\underline{CJC}[1/4]CH[1/2]HA.
\end{aligned}
$$

Complete circles are underlined. One can easily verify that $O^\star$ is the center of gravity of each of the six curves $a_i^\star, b_i^\star$ ($i = 9, 10, 11$), too. Note that curves like $a_9^\star, b_9^\star$ and $a_{10}^\star = a_{11}^\star$ have already occurred (except for $a_9^\star = c_8^\star$, with different notation) in the Examples 8, 1, 4 (see Figs. 5, 1(a), 2), respectively.

The spherical images $b_{10}^\star$ or $b_{11}^\star$ from above obviously arise from a spherical curve $c^\star$ of type $-[1/2]AGDHA$ by inserting a pair of (complete) circles lying in opposite faces ($z^\star = \pm e/2 = \pm 1/\sqrt{2}$) of the cube Π, namely the circles being described by $+[1/2]JCJ$, $-[1/2]KEK$ or $-[1/2]EKE$, $+[1/2]CJC$, respectively. Compare here Example 4 in Fig. 2; note that the path $-[1/2]AGDHA$ arises from $a_{10}^\star$ by reflection with respect to the plane being parallel to the $z^\star$-axis and passing through A, D)

Hence follows that each of the closed κ_0-curves b_{10} or b_{11}, the tangent indicatrix of which is $b_{10}^\star$ or $b_{11}^\star$, respectively, arises from a curve c of Example 4, having the same constant curvature κ_0, by inserting — in different ways — a pair of congruent arcs of circular helices of curvature κ_0 and angle of rotation 2π, the axes of which are parallel to the z-axis and thus *orthogonal* to the axes of the helical arcs of c, which are parallel to the x- or y-axis, respectively. Consequently, the curves b_{10} and b_{11} are (other than the curves a_{10} and a_{11} corresponding to $a_{10}^\star$ and $a_{11}^\star$, respectively) *new examples* of closed κ_0-curves, being different — though generated by "insertion", too — from those of Example 5 in Fig. 3. The explicit drawing of a closed κ_0-curve having $c_9^\star, c_{10}^\star$ or $c_{11}^\star$ as its tangent indicatrix may be left to the reader.

The curve $c_{12}^\star$ and the corresponding closed κ_0-curves c_{12} do not separate: according to the structural formula of $c_{12}^\star$ and/or Fig. 6(b), the curve c_{12} can obviously be generated from a closed κ_0-curve c_6 as Example 6 (Fig. 4) being characterized, with respect to Fig. 6(a),

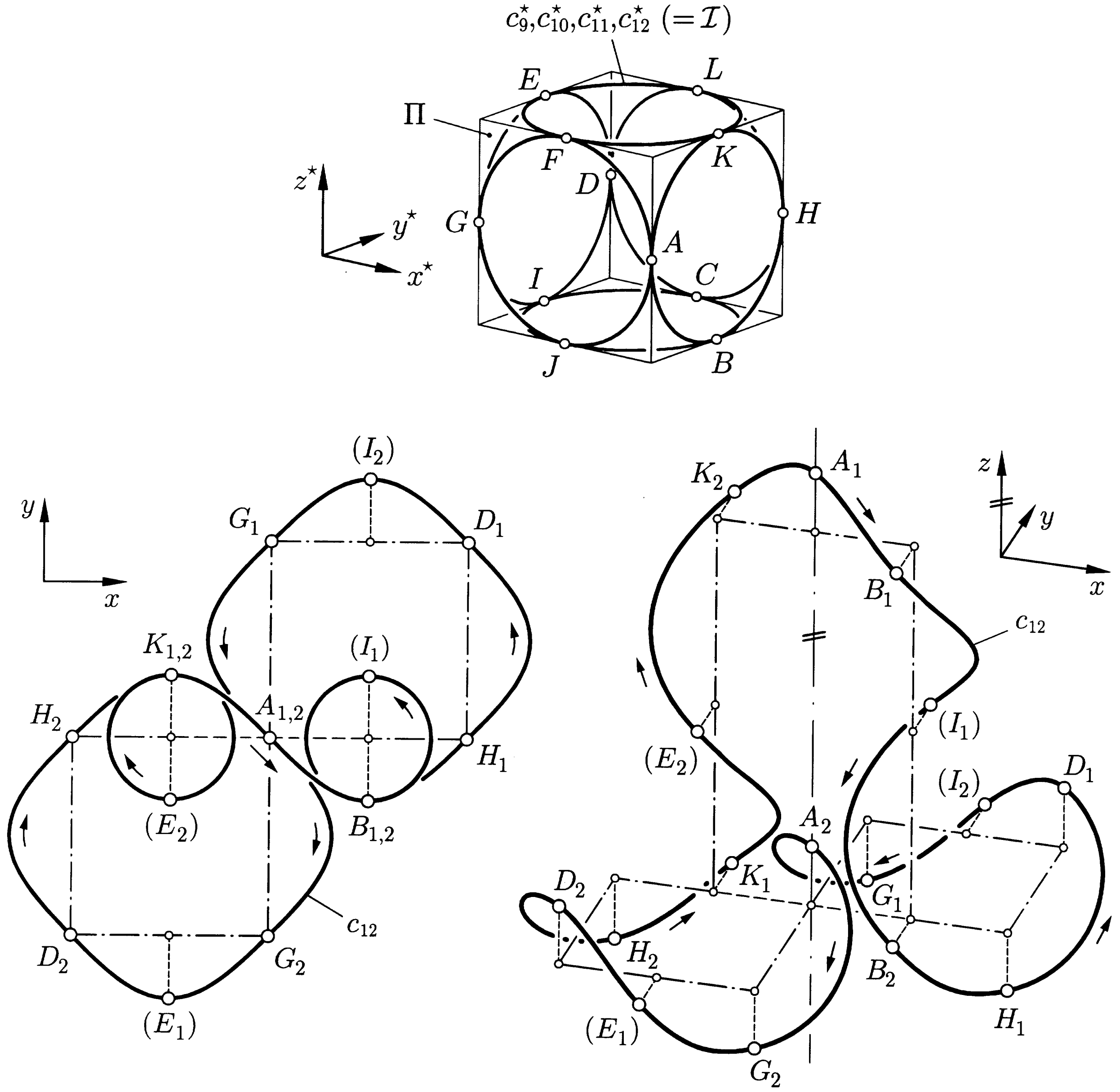

Figure 6: (Examples 9, 10, 11, 12) Closed "separating" κ_0-curves c_9, c_{10}, c_{11} (not shown in this figure) and the "non-separating" closed κ_0-curve c_{12}, each of them of class C^2, having the common tangent indicatrix $c_i^\star = \mathcal{I}$ $(i = 9, \ldots, 12)$: except for the 12 edge midpoints of the cube Π — each of which is covered twice — $c_i^\star$ covers each point of the six incircles of the faces of Π once and only once. The curve c_{12} is symmetric with respect to the line A_1A_2

by the rewritten structural formula[4] $c_6^\star$: $+[1/2]AHDGAGDHA$ by inserting two congruent helical arcs $\widehat{B_1(I_1)B_2}$, $\widehat{K_1(E_2)K_2}$, corresponding to the pair of complete circles $-[1/2]BIB$, $-[1/2]KEK$ of c^*, at the midpoint — corresponding to B or K — of the helical arc of c_6 which corresponds to $+[1/2]AH = +[1/4]ABH$ or $+[1/2]HA = +[1/4]HKA$, respectively. The shape and the course of a closed κ_0-curve c_{12} corresponding to $c_{12}^\star$ can be recognized, for instance, by drawing a pair of orthographic views of c_{12}, compare the top view of c_{12} in

[4] The points A, B, C, D of $c_6^\star$ in Fig. 4(a),(b) now are to be replaced by H, A, G, D, respectively.

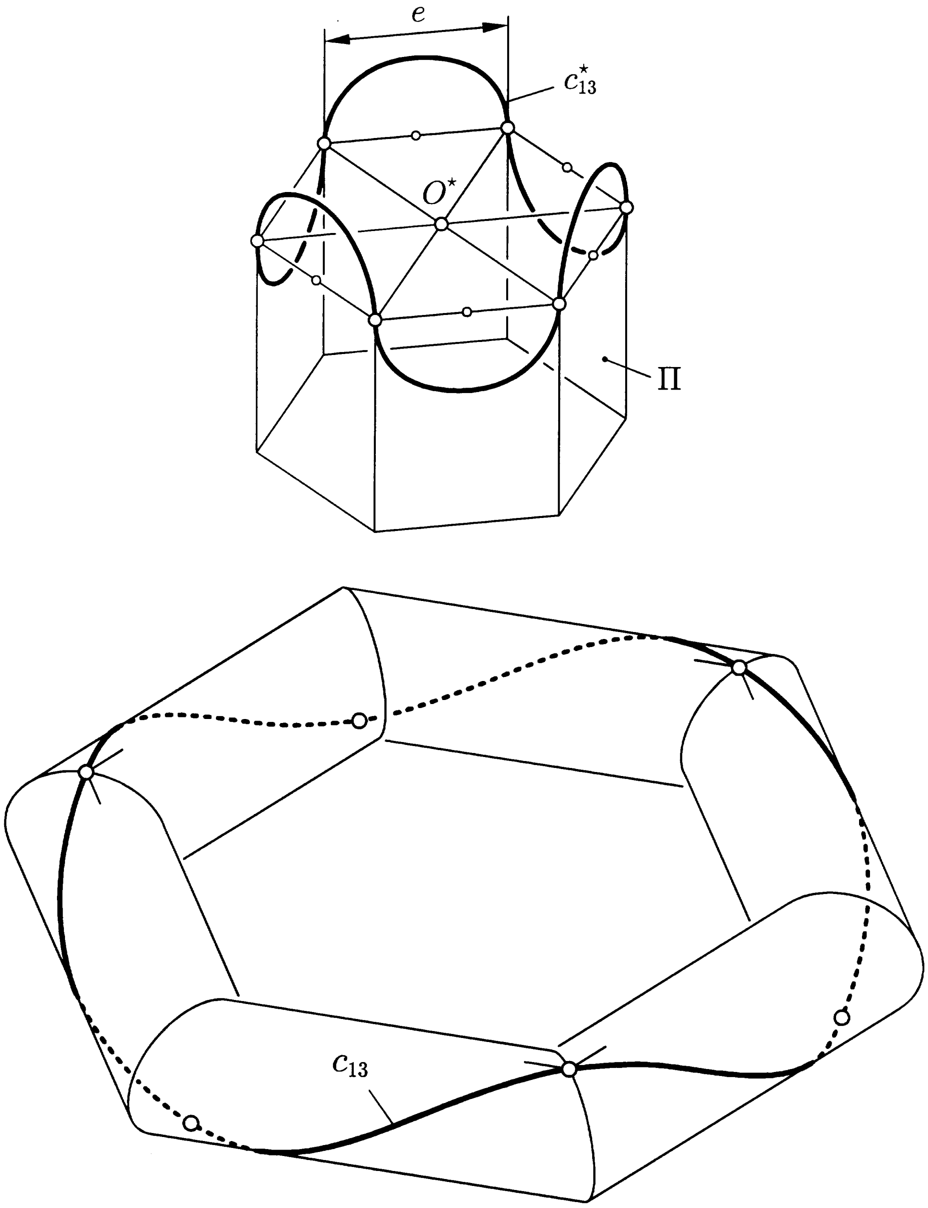

Figure 7: (Example 13) Closed κ_0-curve c_{13} of class C^2, consisting of six congruent helical arcs having three complanar axial directions

Fig. 6(b). Note that the helical arcs of c_{12} in our case ($\alpha = \pi/4$; orthogonal projection with respect to a coordinate plane) project into arcs of either *circles* of (with respect to (8)) radius $1/(2\kappa_0)$ or of *sinusoidal curves* having π/κ_0 as *length of period* and the *amplitude* $1/(2\kappa_0)$.

Fig. 6 shows in (b) the normal projection of c_{12} into the plane $z = 0$ (top view) and in (c) an axonometric representation of c_{12}. The closed κ_0-curve c_{12} is composed of 12 arcs (see below) of a pair of helices being inversely congruent (anti-congruent). Note that all of the *circular arcs* of $c^\star_{12}$ being contained in a *pair of opposite faces* $x^\star = \pm e$ / $y^\star = \pm e$ /$z^\star = \pm e$ of Π are *oriented in the same sense* $+/-/-$, respectively, thus corresponding to arcs of c_{12} which belong to *circular helices* being directly congruent, the axes of which are parallel to the x-/y-/z-axis, respectively. As a point of $c^\star_{12}$, each of the 8 edge-midpoints A, B, D, E, G, H, I, K of the cube Π appearing in the formula of $c^\star_{12}$ has a *pair of image* *points* $\{A_1, A_2\}, \ldots, \{K_1, K_2\}$ on c_{12}. Using 12 of these points, corresponding to the six points A, B, D, G, H, K of $c^\star_{12}$, the course of the closed κ_0-curve c_{12} is determined by the sequence of the points $A_1B_1B_2H_1D_1G_1A_2G_2D_2H_2K_1K_2A_1$: c_{12} is composed of 12 helical arcs $\widehat{A_1B_1}, \widehat{B_1B_2}, \ldots, \widehat{K_2A_1}$ being numerated by $1, 2, \ldots, 12$, each two consecutive arcs having *anti-congruent supporting circular helices* and thus *differing in the sign of the torsion.* With respect to the positive z-direction, the six helical arcs #1-3, 5, 7, 9 of c_{12} are *descending*, the other six arcs #4, 6, 8, 10-12 are *ascending.* To ease the understanding, the Figs. 6(b),(c) additionally show the four points I_1, I_2, E_1, E_2 of c_{12} which correspond to the points I, E of $c^\star_{12}$. More in detail, the course of c_{12} now is described by the sequence of the points $A_1B_1(I_1)B_2H_1D_1(I_2)G_1A_2G_2(E_1)D_2H_2K_1(E_2)K_2A_1$: the four additional points I_1, I_2, E_1, E_2 — which are put in brackets in the point sequence and in the Figs. 6(b),(c) since they are no endpoints of helical arcs of c_{12} — are the midpoints of the helical arcs # 2, 5, 8, 11 respectively. These are the arcs $\widehat{B_1B_2}, \widehat{D_1G_1}, \widehat{G_2D_2}, \widehat{K_1K_2}$.

6 Outlook: Further possibilities of constructing closed κ_0-curves

In Fig. 2 all of the circular arcs of the spherical curve $c^\star_4$ are contained in the $n := 4$ "vertical" faces of the cube Π, being parallel to the $z^\star$-axis. More generally, the cube Π can be replaced by a *right prism* Π with center $O^\star$, its lateral edges being parallel to the $z^\star$-axis and its bases, lying in the parallel planes $z^\star = \pm 1$, being regular n-gons ($n \geq 3$) of circumradius 1, which can be represented by the congruent regular n-gon $A_1A_2 \ldots A_n$ cut from Π by the plane $z^\star = 0$ (and generalizing the square = regular 4-gon $ABCD$ in Fig. 2(a)). Let $\mathcal{I}$ denote the intersection of the right prism Π and the sphere S^2, having the center $O^\star$ and the radius 1; so $\mathcal{I}$ consists of a chain of n congruent complete circles. Then any regular curve/path $c^\star \subseteq \mathcal{I}$ of class C^1 having $O^\star$ as its center of gravity can be chosen to be the tangent indicatrix of nonplanar κ_0-curves of class C^2 consisting of helical arcs, the axes being parallel to the plane $z = 0$. In the simplest case, $c^\star$ consists of n, if n is even, or $2n$ if n is odd, congruent semicircles: then the path $c^\star$ can be described by the structural formula $\pm[1/2]A_1A_2 \ldots A_nA_1$ (n even) or $\pm[1/2]A_1A_2 \ldots A_nA_1A_2 \ldots A_nA_1$ (n odd), respectively. For $n = 4$ and $n = 6$, see Example 4 (Fig. 2) and Example 13 (Fig. 7), respectively.

Nonplanar closed κ_0-curves of class C^r ($r \geq 3$), in particular of class C^ω, can also be determined, for instance by analytic representations using *elliptic* integrals, using Proposition and certain properties of symmetry of the tangent indicatrix $c^\star$. Explicit examples of *nonplanar closed κ_0-curves of class C^ω* will be presented in a forthcoming note.

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