Journal for Geometry and Graphics
Volume 2 (1998), No. 2, 123–131

# Developing a Procedure to Transfer Geometrical Constraints from the Plane into Space

**Eva Knoll**

*308 Chemin du Tour, Ile Bigras,*
*Laval, Québec, Canada, H7Y 1H2*
*email: evak@speedware.com*

**Abstract.** In this paper at the interface between geometry and art design the relationship between planar and spatial geometry will be explored as a design element. The question will be answered whether it is possible, starting with a 2-dimensional system of design parameters, to construct a 3-dimensional object based on the spatial equivalents of the initial parameters. To illustrate this process, H. HINTERREITER's painting *"Opus 84"* will be geometrically analyzed and re-interpreted in space.

*Key Words:* Hans HINTERREITER, Constructivism, geometry, dimensions, design

## 1 Introduction

Topology teaches us that the two dimensional plane and three dimensional space have a comparable structure. In fact, this apparent parallel is deeply rooted in our consciousness and is applied in many domains, including various fields in the design industry, through the use of such tools as descriptive geometry and perspective drawing. From the particular point of view of the designer, however, this parallel in structure has often been simplified to plans, sections and elevations, i.e., 2-D slices through a 3-D object. It has therefore not been an integral part of the design process, but rather a tool of representation of the design process.

In the following paper, the relationship between plane and space will be explored as a design element. The question will be answered whether it is possible, starting with a 2-dimensional system of design parameters, to construct a 3-dimensional object based on the spacial equivalents of the initial parameters. To illustrate this process, the painting *"Opus 84"* of Hans HINTERREITER (1902-1992), a Swiss Concrete painter, will be re-interpreted in space.

*"Opus 84"* (Fig. 1) is a circular painting with a diameter of 82 centimeters. It was completed a first time in 1943, and in its final version in 1967. The work itself is determined entirely by three sets of parameters. First, the colors used throughout were chosen according

ISSN 1433-8157/\$ 2.50 

Figure 1: Hans HINTERREITER: *"Opus 84"* (1967)

to the color theory developed by Wilhelm OSTWALD. These colors were applied to areas determined by the combination of the other two sets of parameters (cf. [1], p. 46). The first of these leads to an orthogonal regular tiling of the plane, while the second deals with a transformation of the underlying orthogonal grid (Fig. 2). In this paper, only the last two sets of parameters will be considered, since only they are dependent on the number of dimensions.

## 2 The regular tiling system

In the diagram on the left of Fig. 2, a concave octagon has been grayed out, showing the basic closed shape that makes up the pattern. This shape, which resembles a fatted S, surrounds an intersection of axes. To facilitate the process, the shape will be sectioned into four sections, corresponding to the four quadrants defined by these axes (Fig. 3).

As can immediately be seen, these sections come in two types: a simple right-angled equilateral triangle (top right or bottom left quadrant), which will from now on be referred to as D; and a shape made of two triangular pieces (top left or bottom right quadrant), which will be referred to as Z, after the broken line that defines it. These sections (D and Z) always appear in the same combination throughout *"Opus 84"*, and the only symmetry group applied to the closed shape is simple translation by six units (see Fig. 3). We therefore are now faced

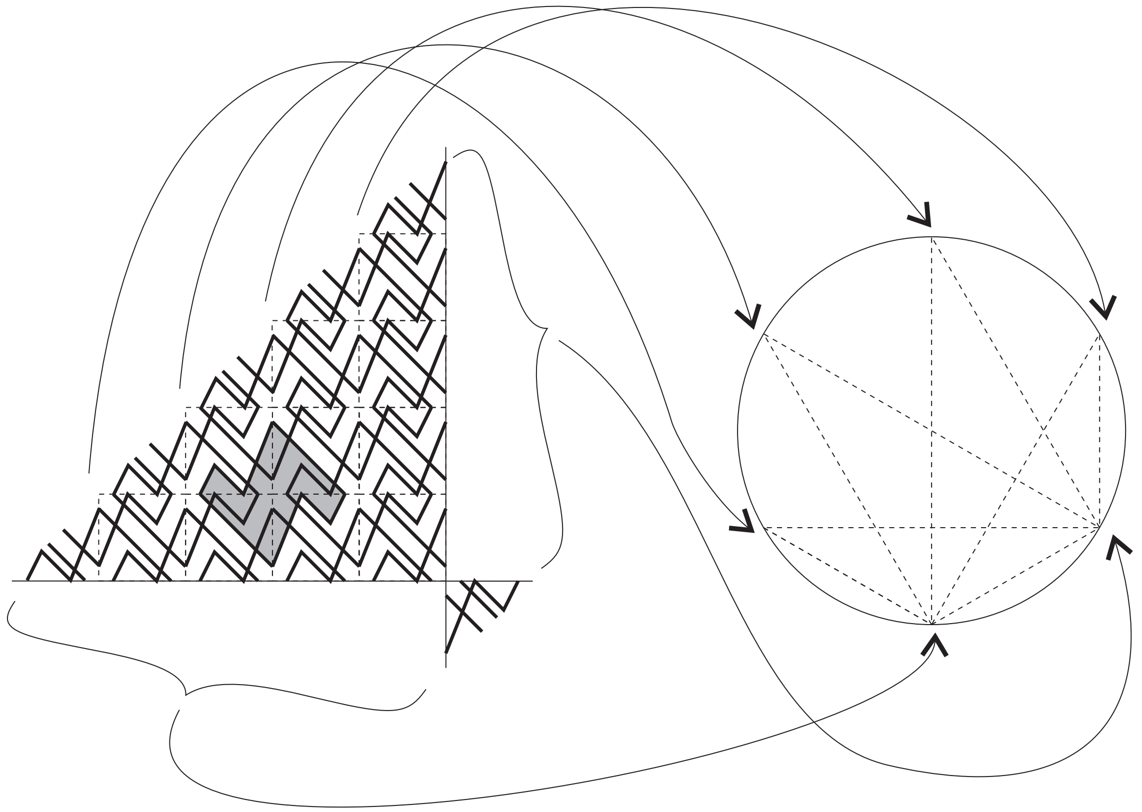

Figure 2: Combining the two systems

with a *polygon* that is *'centered on each intersection of axes', composed of a combination of predefined sections* (ZDZD) *and moved by translation only.*

Moving on into space, we are confronted with eight quadrants defined by three limiting planes and surrounding their intersection (equivalent to the four quadrants of the 2-D situation). What needs to be defined is the structure of the shapes that can be considered equivalent to the D and Z of the 2-D situation. In 2-D, each section was defined as a *polygon*, two *sides* of which are *collinear* with the *axes* and touching the *intersection of these axes*. The other *side*, or *sides* are all contained in the given *quadrant*. Similarly, in the 3-D situation, the sections will be defined as *polyhedra*, *three faces* of which are *coplanar* with the *limiting planes* of the quadrant and touching the *intersection of these limiting planes*.

What remains to be defined is the structure of the faces contained in each quadrant. To achieve this, we must begin by defining the shape of the faces that are coplanar with the limiting planes. These can be borrowed directly from the 2-D situation, D and Z. If we keep with the rules of the painting, on each limiting plane the same configuration should be found as in the 2-D situation (ZDZD). This means that three fatted S's will intersect at right angles to each other to determine a new polyhedron. But this is not sufficient, since several combinations of the three S shapes are possible. It is simpler at this point to begin with the definition of the individual polyhedra. Obviously, the simplest of these would be defined by three D's on the three limiting planes. This would give us a right angled tetrahedron pushed down into the corner of the quadrant (DDD in Fig. 5). As soon as we begin making use of the Z section, however, it must be noted that where it is used in a given quadrant, in the neighboring one it is reversed! This new section comes about through an inherent property of space, namely the possible simultaneous existence of a 2-D object and its mirror image

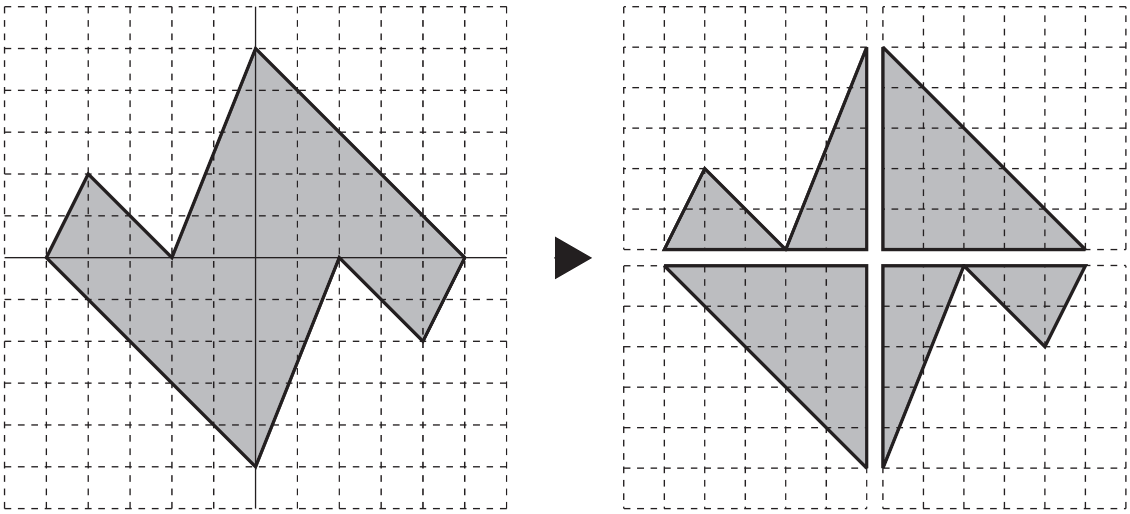

Figure 3: The basic S shape and its components

without the aid of that symmetry transformation. This section will from now on be referred to as S. We now have three different 2-D sections to be used in defining our 3-D sections.

Combinatorially, and after having eliminated equivalent configurations, we are left with seven different sections in 3-D: DDD, DDZ, DZZ, DZS, DSZ, ZZZ and ZZS. The first of these sections has already been discussed. For DDZ, however, a problem presents itself already: if the polyhedron is triangulated using the given edges and vertices, a new edge which is collinear to one of the limiting planes is introduced (Fig. 4). The analog does not happen in Hinterreiter's *"Opus 84"*.

This is therefore unfeasible and must be eliminated. To solve this, an additional point has been introduced inside the quadrant in order to 'lift' the surface away from the limiting planes. The choice of location for this additional point was made using similar coordinates to the lifting points of the intermediate points in the Z (or S) variation: 2 and 4 units from the origin of the quadrant (see Fig. 3). The same rationale is used in the development of the remaining sections. Since the addition of vertices becomes necessary only with the introduction of the Z or S variations, they are always located in proximity of the 'humps' that distinguish these variations from D.

Once the location of all the additional points has been established, however, a new dilemma arises. In the DSZ and ZZS cases, two possible triangulations can result from each configuration of vertices (Fig. 5, DSZ, DSZ*, ZZS and ZZS*). The difference between these two solutions is that in one case (DSZ and ZZS), the additional vertex is joined with all the other possible vertices whereas in the other case (DSZ* and ZZS*), the resultant polyhedron is actually made of two parts that touch only by a point located on one of the intersecting lines of the limiting planes.

## 3 Assembling the shapes

Now that all the pieces have been defined, it remains to be resolved in what combinations these can be assembled. As was discussed previously, on each of the three limiting planes, the 'fatted S' appears in its entirety. This means that for each variation shown in Fig. 5, only a limited set of neighbors are valid. Furthermore, by virtue of the internal symmetry of the

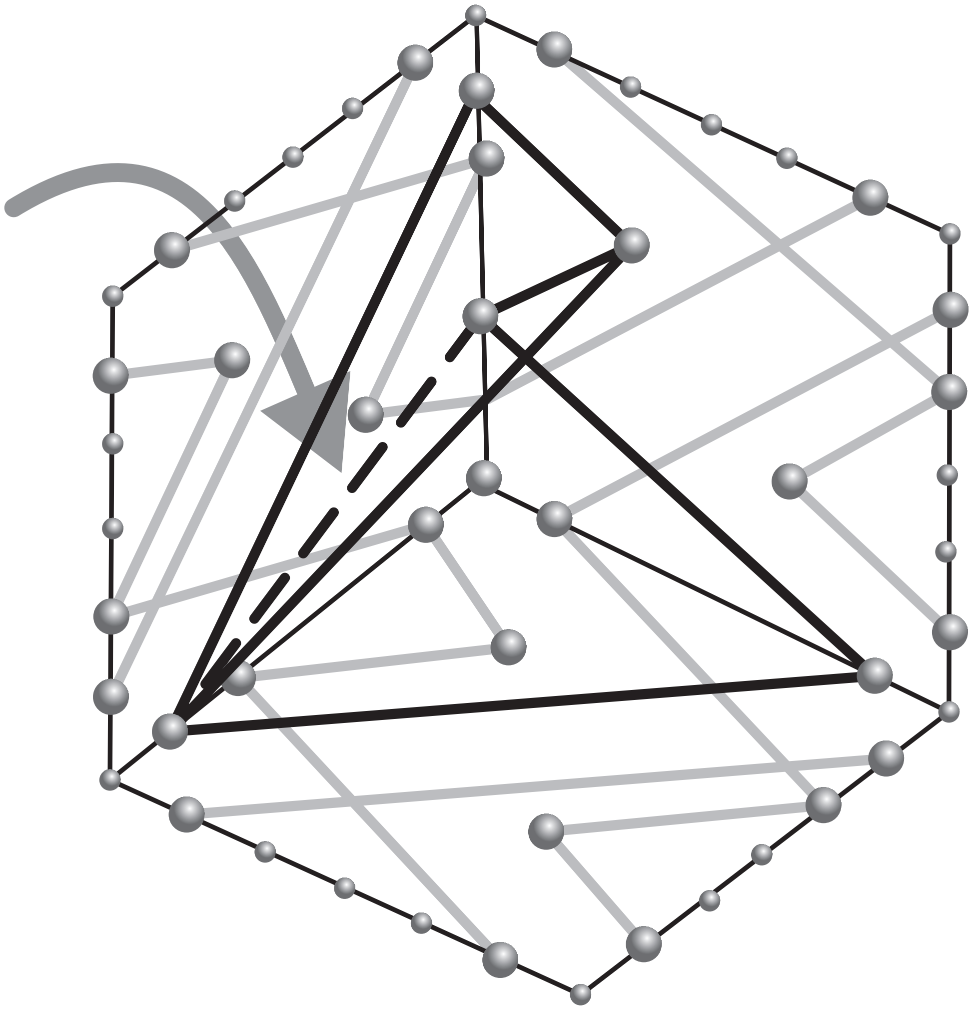

Figure 4: The DDZ configuration

'fatted S', it can be deduced that for each quadrant, the mirror image of the chosen variation will be positioned in the opposite quadrant (e.g., Top-Left-Front and Bottom-Right-Back). This means that there are really only four quadrants of eight that need to be determined. The other four will be their mirrors.

Combinatorially (again), and after having eliminated equivalent configurations, we are left with four possible combinations. The first one is based on the DDD variation, around which three DZZ variations are arranged in such a way that their D face is adjacent to the DDD quadrant. It follows that the other four quadrants contain one DDD and three DSS (the mirror image of DZZ) variations, in such a way that they all oppose their mirror image. This means that if the first DDD is in the top left front quadrant, the other will be in the bottom right back quadrant. This creates an object with a central symmetry as well as an axial rotational one of degree 3 (120°) around the axis of symmetry of the DDD quadrant.

The next solution begins again with the DDD variation, but this time one DZZ, one DZS and one DSZ variations surround it in such a way that their common faces match. The other four quadrants, again, contain the mirror images of the first four respectively. This particular solution is illustrated in Fig. 6, where the top four views show the top quadrants, each view showing one quadrant. The bottom four views each show the quadrant immediately under

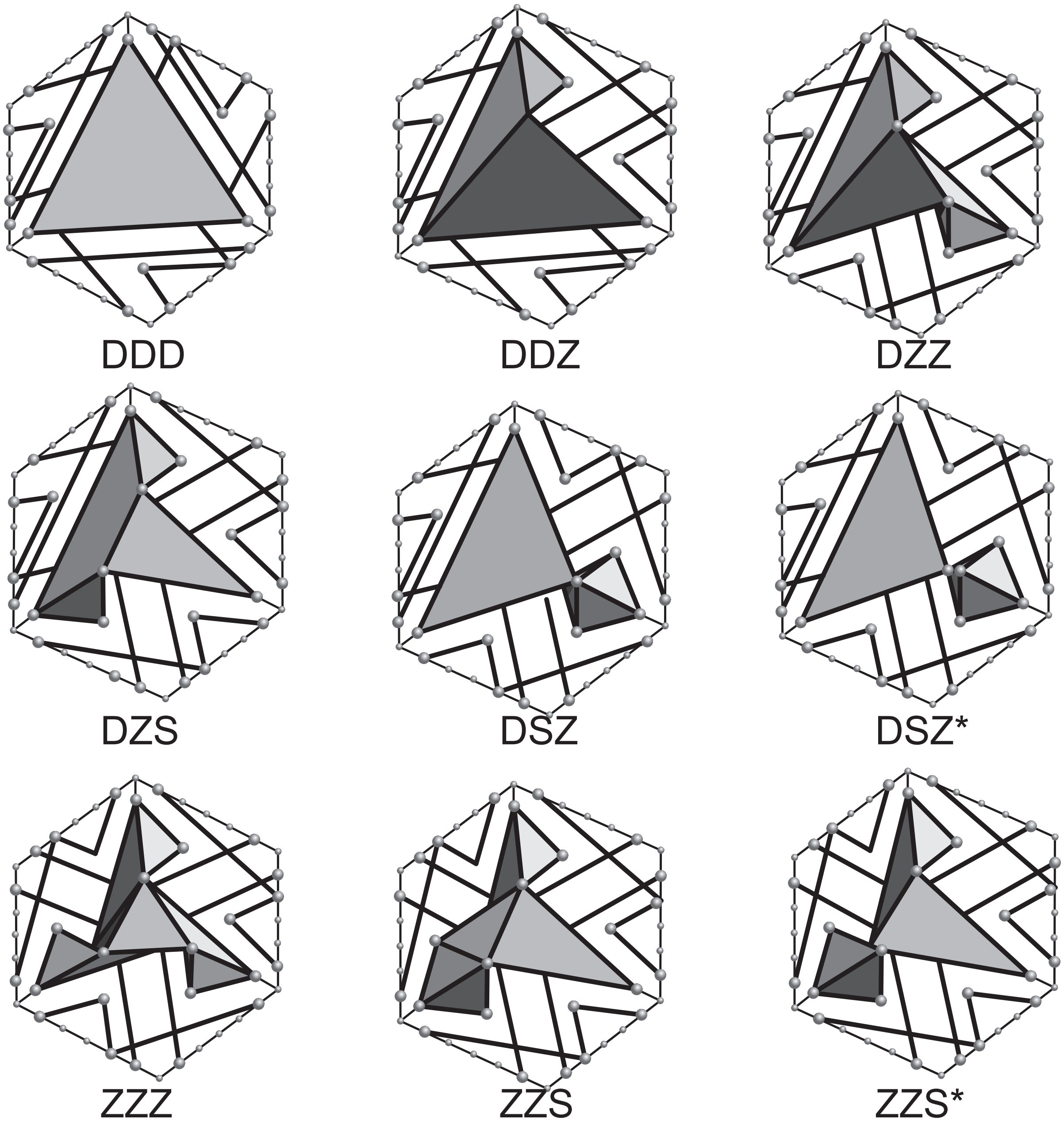


Figure 5: Nine 3-D variations

the view above it. The third solution is based on the ZZZ variation surrounded by three DDS's (the mirror images of DDZ). This version, again, possesses a central symmetry (as do all), and an axial rotational symmetry of degree 3. The last solution is composed of the ZZS variation surrounded by two DDS and one DDZ variation. The other four quadrants contain one SSZ, two DDZ, and one DDS variation.

Finally, in each case, the resultant polyhedron is translated along the three axes at the same interval of six units, as the 'fatted S' was in the 2-D situation.

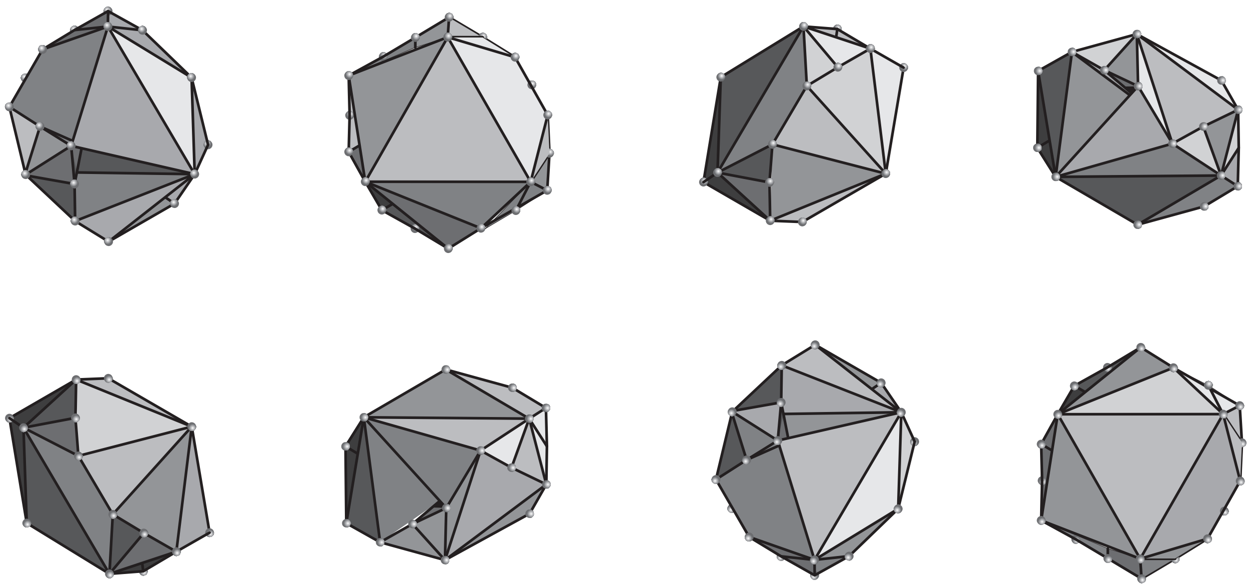

Figure 6: Solution using two DDD, two DZZ, two DZS and two DSZ

## 4 Deformation constraints

The previous section served to demonstrate that there are four valid non-equivalent solutions to the transfer into 3-D of the tiling pattern used in *"Opus 84"*. The deformation grid shown in Fig. 2 poses a different problem. It is not a specific illustration of a set of parameters. Rather, it is a set of parameters that can be applied to another; a modifier. What is to be defined here, therefore, is a 3-D deformation grid equivalent to the one on the left of Fig. 2. From that diagram, we can deduce that the deformed grid used in *"Opus 84"* by Hans Hinterreiter is as follows: *two sets of straight lines radiate from two points on the limiting circle. The points are at* 60° *of each other in relation to the center of the circle, and the straight lines are at* 30° *of each other in each set of five, starting with the line joining the two points.*

The equivalent situation in 3-D can be defined as follows: *three sets of planes radiate from three axes defined by three points on the surface of the limiting sphere. The axes are defined by three points at* 60° *to each other in relation to the center of the sphere, and the planes are at* 30° *to each other starting with the plane joining the three points.*

This new set of parameters is illustrated in Fig. 7. The view on the left is taken through the axis of symmetry of the configuration. The center view is a section taken through the one of the three planes of symmetry. The view on the right is taken perpendicularly to the plane touching the three foci.

## 5 Conclusion

With the deformed grid and the polyhedron defined, the only step remaining is the integration of the two components. This is achieved by placing the chosen polyhedron at every intersection of three planes in the above configuration in such a way that the instances are foreshortened as well as skewed proportionally to the grid. Because the symmetry of the polyhedra is different from that of the deformation grid, the solutions to this combination of two systems are more numerous than they were at the preceding steps. The representation of even a single one of

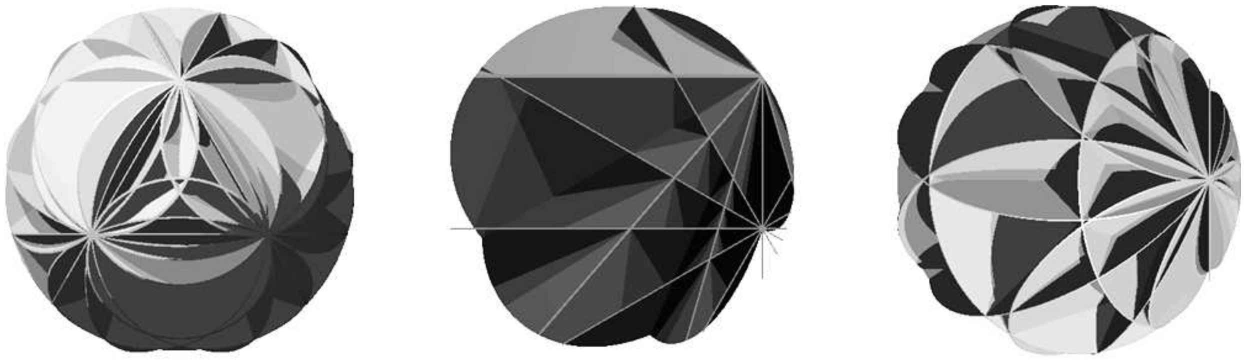

Figure 7: Deformation grid in 3-D

these solutions poses a further problem: not only is it so complex that the human eye and mind cannot grasp it and therefore reads it as a single more or less homogenous mass, but it could only really be assimilated in its 3-D form. It is therefore useless, unfortunately, to illustrate it in this context.

The process as a whole was not, however, a failure. It has after all been established that it is indeed possible to transfer a set of geometric constraints from the plane into space. The fact that this particular attempt was successful does not unfortunately guarantee the universal success of the process. It is quite likely, in fact, that several conditions to the existence of a solution happen to have been met.

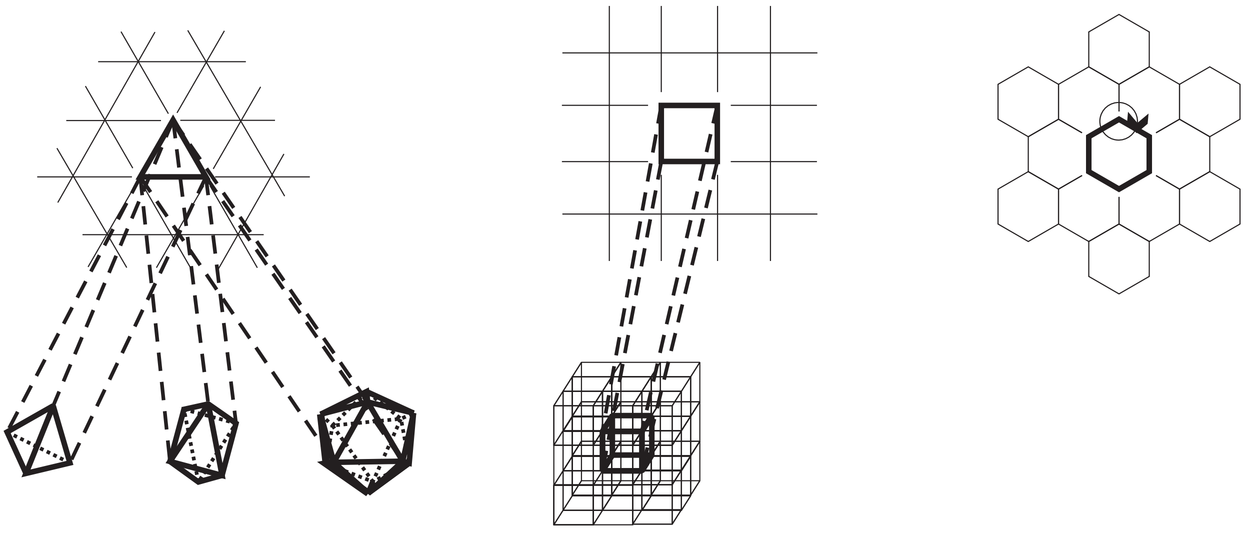

Figure 8: Regular grids and their equivalences in 3-D

For example, it happens to be the case that the preceding set of constraints are all based on the regular square grid. This structure is known to have an equivalent in 3-D. But what of the other two regular subdivisions of the plane? The equilateral triangle (Fig. 8, left) has three regular equivalents in 3-D, the tetrahedron, the octahedron, and the icosahedron. None of these fill space by themselves (although the tetrahedron and the octahedron can do so together). As for the hexagon (Fig. 8, right), it has no regular equivalent in 3-D since it takes at least three faces to meet at any vertex, and three hexagons cover 360° when placed corner to corner. There are probably other conditions like this one that could stand in the way of a solution to the transfer into 3-D of other sets of geometrical constraints. There may also be ways around these. It would also be interesting to try this process between two different spaces, for example 3-space and 4-space!

From the point of view of the designer, there are other drawbacks. First there is a fact that has been illustrated beautifully in this example; the 2-D set of constraints resulted in an object with just the right amount of complexity to make it interesting, while its equivalent in 3-D reached a complexity that becomes unreadable and therefore estetically irrelevant. Another point concerns the way visual design is perceived. In the case of a 2-D design, the esthetic decisions concern the inside, the whole breadth and width of the object. In 3-D, in most cases, the design decisions concern the shape and finish of the *outside*; what happens inside the thickness of the object is irrelevant. The 3-D object is therefore still, in a way, a 2-D design, except that the surface has been stretched over a 3-D shape.

## Acknowledgments

I would hereby like to thank the research group GRCAO of the Université de Montréal School of Architecture for letting me use their experimental CAD software, SGDL, as well as Janos Baracs for his advice and Willy Knoll and Roger Tobie for their inspiration.

## References

[1] H.J. Albrecht, R. Koella: *Hans Hinterreiter.* Waser, Buchs, Zürich 1982.

[2] H. Hinterreiter: *Die Kunst der reinen Form.* Ediciones Ebusus, Ibiza 1948.

[3] D. Hilbert, S. Cohn-Vossen: *Geometry and the Imagination.* Chelsea, New York 1952.

Received August 14, 1998