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Edge-Orthogonal Patches through a Given Rational Bézier Curve

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Abstract. Applications in computational fluid dynamics (CFD) have led to the problem of finding a rational Bézier patch with a given edge parameter line k the way that the parameter lines of the other type intersect k orthogonally. This is what we call an *'orthogonal continuation of k'*. The variety of solutions to the problem is being investigated and a very geometric way for the construction of the solutions is being offered. Using some fundamental features of polynomials we can establish a link between the properties of the weight polynomial and the elevation of degree which is necessary to find non-trivial orthogonal continuations. For some cases which turn out to be unsolvable, and for cases where the solution existing has a very high degree, we can describe a Monte Carlo method providing surprisingly good approximations. This method is even capable of coping with tasks where the right angle is replaced by some arbitrary angle function.

Key Words: Bézier surfaces, orthogonal continuation, Monte Carlo method

MSC 1994: 53A05, 68U05

1 Preliminaries

The following definitions and properties are generally known. They are only noted for the sake of clarity with respect to the notation used here.

Be given a rational Bézier control net in the r-dimensional space $\mathbb{R}^r$ with the control points $\mathbf{p}^{ij}$, $i = 0,\dots,m$, $j = 0,\dots,n$ with $\mathbf{p}^{ij} = (p_1^{ij}, p_2^{ij}, \dots, p_r^{ij})^t$ and the weights γ^{ij}. We build up a corresponding net in $(r+1)$-space $\mathbb{R}^{r+1}$ given by the recipe

(1)
$$\mathbf{q}^{ij} = (q_0^{ij}, q_1^{ij}, q_2^{ij}, \dots, q_r^{ij})^t \text{ with}$$
$$q_0^{ij} := \gamma^{ij}, \quad q_1^{ij} := \gamma^{ij} p_1^{ij}, \quad q_2^{ij} := \gamma^{ij} p_2^{ij}, \dots, \; q_r^{ij} := \gamma^{ij} p_r^{ij}.$$

The integral Bézier patch in $\mathbb{R}^{r+1}$ defined by these points $\mathbf{q}^{ij}$ can be represented with the help of the so-called *'shift operators'* E and F, providing $E^i F^j \mathbf{q}^{00} = \mathbf{q}^{ij}$. The parameter representation of the patch can be written as:

(2)
$$\mathbf{y}(u,v) = (1 - u + uE)^m (1 - v + vF)^n \mathbf{q}^{00}.$$

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Let the coordinates of (2) — which is a surface in the $(r+1)$-space — be

$$\mathbf{y}(u,v) = (y_0(u,v), y_1(u,v), y_2(u,v), \ldots, y_r(u,v))^t.$$

We define a central projection f with the centre $O \ldots (0,0,\ldots,0)^t$ mapping the set $\mathbb{R}^{r+1} \setminus [y_0 = 0]$ into the image space $H_1 \ldots [y_0 = 1]$, which is an (r-dimensional) hyperplane in the affine space $\mathbb{R}^{r+1}$:

$$(3) \qquad \begin{array}{rccl} f: & \mathbb{R}^{r+1} \setminus [y_0 = 0] & \longrightarrow & H_1 \ldots [y_0 = 1] \\ & (y_0, y_1, y_2, \ldots, y_r)^t & \mapsto & \left(1, \dfrac{y_1}{y_0}, \dfrac{y_2}{y_0}, \ldots, \dfrac{y_r}{y_0}\right)^t =: (1, x_1, x_2, \ldots, x_r)^t. \end{array}$$

This projection maps the control net $(\mathbf{q}^{ij})$, $i = 0,\ldots,m$, $j = 0,\ldots,n$ in $(r+1)$-space into the control net $\mathbf{p}^{ij}$, $i = 0,\ldots,m$, $j = 0,\ldots,n$ we started with. The integral Bézier patch (2) is mapped into some surface

$$(4) \qquad \mathbf{x}(u,v) = (X_1(u,v), X_2(u,v), \ldots, X_r(u,v))^t = \left(\frac{Y_1(u,v)}{Y_0(u,v)}, \frac{Y_2(u,v)}{Y_0(u,v)}, \ldots, \frac{Y_r(u,v)}{Y_0(u,v)}\right)^t$$

called the *'parameter representation of a rational Bézier patch'* defined by the control points $\mathbf{p}^{ij}$, $i = 0,\ldots,m$ $j = 0,\ldots,n$ with $\mathbf{p}^{ij} = (p_1^{ij}, p_2^{ij}, \ldots, p_r^{ij})^t$ and the weight matrix (γ^{ij}).

The curve in $\mathbb{R}^{r+1}$ represented by

$$(5) \qquad \mathbf{y}(u,0) = (1 - u + uE)^m \mathbf{q}^{00}$$

is a border curve of the surface (2), which belongs to the parameters $(u,0)$, $u \in [0,1]$. So its projection k represented by $f(\mathbf{y}(u,0)) = \mathbf{x}(u,0)$ is the edge parameter line of (4) belonging to $v = 0$. It is a rational Bézier curve with the control points $\mathbf{p}^{i0}$, $i = 0,\ldots,m$ and the weights γ^{i0}.

2 The general problem

We start with a given rational Bézier curve k in $\mathbb{R}^r$ defined by the control points $\mathbf{p}^{i0}$, $i = 0,\ldots,m$ and the weights γ^{i0}, $i = 0,\ldots,m$. It is our goal to find a rational Bézier patch as regarded above satisfying the additional *'orthogonality condition'*

$$(6) \qquad \mathbf{x}_u(u,0) \cdot \mathbf{x}_v(u,0) = 0.$$

Definition 2.1 *Be given a rational Bézier curve* $k \ldots \mathbf{x} = \mathbf{x}(u)$. *Any rational Bézier patch* $\Phi \ldots \mathbf{x} = \mathbf{x}(u,v)$ *satisfying* $\mathbf{x}(u,0) = \mathbf{x}(u)$ *and (6) will be called an* 'orthogonal continuation of k'.

The task can be set for the plane case and for cases in higher dimensions as well. Though the results worked out here are generally valid, we emphasize the plane case because of two reasons: The problems, which applicants brought about until now, have been of 2-dimensional nature and, some of the scopes may even turn out to be trivial in the spatial case. At least it can be noted, that the case of higher dimensions — which is not excluded in our considerations — would suggest additional conditions such as the concept of 'differentiable stripes' and orthogonal continuation of patches (see also M. Peternell, H. Pottmann [8], G. Geise, U. Pohl [3]).

Remark:

- In the plane case of integral Bézier curves there exists a 2-parameter set of integral patches representing an orthogonal continuation of a given starting curve (see A. Gehrer et al. [2] and J. Lang [7]).
- If $\mathbf{p}^{ij},\ \ i=0,\ldots,m,\ j=0,\ldots,n$ with $\mathbf{p}^{ij}=(p_1^{ij},p_2^{ij},\ldots,p_r^{ij})^t$ and the weight matrix $\gamma^{ij},\ \ i=0,\ldots,m,\ j=0,\ldots,n$ defines an orthogonal continuation of the curve k, any other patch with the same threads number 0 and number 1 is as well an orthogonal continuation, because the angle, under which the parameter lines meet the border curve k only depends on these threads 0 and 1. So we can easily restrict ourselves to the case $n=1$.
- If an orthogonal continuation is given we can change the weights of thread 1 by multiplying them with a unique real number $\varphi\neq 0$ without affecting the orthogonal continuation property.
- The choice $\mathbf{p}^{i1}=\mathbf{p}^{i0}$ and $\gamma^{i1}=\lambda\cdot\gamma^{i0}$ for any $\lambda\neq 0$ will of course provide a trivial *'solution'* as we have $\mathbf{x}_v(u,0)\equiv 0$.

Of course the rational Bézier patch (4) has a parameter representation[1] with the shape

$$\mathbf{x}(u,v)=\frac{\sum_{i=0}^m\sum_{j=0}^n u^iv^j\mathbf{r}^{ij}\delta^{ij}}{\sum_{i=0}^m\sum_{j=0}^n u^iv^j\delta^{ij}}. \tag{7}$$

We will use the following abbreviations (see also H.-P. Schröcker [9]), where $\mathbf{Z}(u,v)$ is a vector function and $N(u,v)$ is a scalar function of the two real variables u,v:

$$\mathbf{Z}(u,v):=\sum_{i=0}^m\sum_{j=0}^n u^iv^j\mathbf{r}^{ij}\delta^{ij},\quad N(u,v):=\sum_{i=0}^m\sum_{j=0}^n u^iv^j\delta^{ij}. \tag{8}$$

We can easily calculate the explicit formulae for $\mathbf{Z}_u(u,v),\mathbf{Z}_v(u,v),N_u(u,v),N_v(u,v)$ and so we omit displaying them here. In order to find all rational patches with the property (6), it is necessary to pose the *'orthogonality condition'*

$$(\mathbf{Z}_uN-\mathbf{Z}N_u)\cdot(\mathbf{Z}_vN-\mathbf{Z}N_v)\ =\ 0\quad\text{for all } u\in[0,1]\text{ and } v=0. \tag{9}$$

This is a polynomial with the variable u, which is generally of order[2] $4m-2$. So (9) has to turn out to be the zero polynomial. This yields $4m-1$ linear homogenous equations for the $(r+1)(m+1)$ unknowns (being the coordinates of the points[3] $\delta^{i1}\mathbf{r}^{i1}$ and the weights δ^{i1}).

3 A general solution

Be given a rational Bézier curve k of order m with monomial control points $\mathbf{r}^{i0}$ and weights δ^{i0}. The equations resulting from (9) give us the possibility to find a variety of orthogonal continuations of k. By allowing the orthogonal continuation to be of order $(m+h,1)$, $h\in\mathbf{N}_0$ we gain additional degrees of freedom, that may be useful[4].

[1] This representation is called *'monomial representation'* and for the moment it does a better job than (4). The vectors $\mathbf{r}^{ij}$ and the coefficients δ^{ij} can easily be calculated out of (4).

[2] As shown in H.-P. Schröcker [9] the leading coefficient of the highest order $4m-1$ vanishes!

[3] Note that the points $\mathbf{r}^{i1}$ are not the control points of the rational Bézier patch, because (7) is the monomial representation instead of the standard parameter representation (4).

[4] If k is a plane curve of order $m\geq 3$ an elevation of degree is even necessary to produce a *'non trivial'* orthogonal continuation (see H.-P. Schröcker [9]), which can be seen by counting the number of conditions and the number of variables.

Theorem 3.1 *Be given a rational Bézier curve $k \dots \mathbf{x} = \mathbf{x}(u)$ of order m in $\mathbb{R}^r$ and a non-negative integer p. In general there exists an $((m+p+1)r - 3m + 2)$-parameter variety of orthogonal continuations of order $(m+p, 1)$ of k.*

Proof: According to chapter 1 we view k as the f-image of an integral Bézier curve $l \dots \mathbf{y} = \mathbf{y}(u)$ in $\mathbb{R}^{r+1}$ (f is the central projection (3)). We choose a polynomial $P(u) \in \mathbb{R}[u]$ of degree p. Multiplying all coordinate functions of $\mathbf{y} = \mathbf{y}(u)$ by $P(u)$ we apply the *'formal degree elevation of the rational Bézier curve k'*. This has effects on the general orthogonality equation (9) as we have[5]

$$(10) \qquad 0 = (\mathbf{Z}_u N - \mathbf{Z} N_u) \cdot (\mathbf{Z}_v N - \mathbf{Z} N_v) = P^3 (\widetilde{\mathbf{Z}}_u \widetilde{N} - \widetilde{\mathbf{Z}} \widetilde{N}_u) \cdot (\mathbf{Z}_v \widetilde{N} - \widetilde{\mathbf{Z}} N_v)$$

where $\widetilde{\mathbf{Z}}(u) := P^{-1}(u)\mathbf{Z}(u,0)$ and $\widetilde{N}(u) := P^{-1}(u)N(u,0)$. The right hand side of (10) is a polynomial of degree $4m + 4p - 2$ which has to vanish for all $u \in \mathbb{R}$. This gives us $4m + 4p - 1$ linear homogenous equations for the $r(m+p+1)$ unknown coefficients of the vectors $\delta_{i1}\mathbf{r}_{i1}$ and the real numbers δ_{i1} ($i = 0, 1 \dots m+p$). Taking into account that only $(4m + p - 1)$ of them are relevant as we have the common factor P^3 on the right hand side of (10), we have $(m+p+1)r - 3m + 2$ degrees of freedom. □

For practical usage it may be a problem to solve the system of equations resulting from (10). The coefficients are cumbersome to calculate, the number of equations is very high. This is why we will present a convenient method of constructing orthogonal continuations without much calculation in the *plane* case. For this purpose we note the following lemma, which is easy to prove:

Lemma 3.1 *Be given two integral Bézier curves*

$$(11) \qquad k \dots \mathbf{y}(u) = \sum_{i=0}^{m} u^i \mathbf{a}^i, \quad l \dots \mathbf{z}(u) = \sum_{j=0}^{n} u^j \mathbf{b}^j.$$

There always exists a unique integral Bézier patch $\Phi \dots \mathbf{Y} = \mathbf{Y}(u,v)$ of order $(M,1)$ with $M := \max(m,n)$ satisfying $\mathbf{Y}(u,0) = \mathbf{y}(u)$ and $\mathbf{Y}_v(u,0) = \mathbf{z}(u)$.

Now we are ready to start with the construction of orthogonal continuations: Be given an integral Bézier curve $c \dots \mathbf{y}(u) = (Y^0(u), Y^1(u), \dots, Y^r(u))$ of order $m \geq 2$ in E^3. Its coordinate functions $Y^i(u)$ are polynomials of degree m or lower. $k = f(c) \dots \mathbf{x}(u) = \mathbf{y}(u)/Y^0(u)$ is the corresponding rational Bézier curve in the plane $H_1 \dots [y_0 = 1]$. The derivative vector $\mathbf{x}_u$ of k is given by

$$(12) \qquad \mathbf{x}_u(u) = \frac{\mathbf{y}_u(u) Y^0(u) - \mathbf{y}(u) Y^0_u(u)}{\left(Y^0(u)\right)^2}.$$

Now we define

$$(13) \qquad \hat{\mathbf{z}}(u) := \mathbf{D}\mathbf{x}_u(u) \left(Y^0(u)\right)^2,$$

where $\mathbf{D}$ is the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

[5]Here we have to take into account that the 0-thread, but not necessarily the 1-thread are degree-elevated. This is why P need not be a factor of $\mathbf{Z}_v$ and N_v.

The vector $\hat{\mathbf{z}}$ is parallel to H_1 and perpendicular to $\mathbf{x}_u$. Furthermore $\hat{\mathbf{z}}$ is a polynomial vector function of degree $2m-2$ or lower as the leading coefficients vanish. The vector

$$\mathbf{z}(u) := L(u)\hat{\mathbf{z}}(u) - Q(u)\mathbf{y}(u). \tag{14}$$

with $L(u), Q(u) \in \mathbb{R}[u]$, $\deg(L) = l$, $\deg(Q) \leq m + l - 2$ is of order $2m + l - 2$. According to Lemma 3.1 we can build an integral Bézier patch $\Phi \ldots \mathbf{Y} = \mathbf{Y}(u, v)$ satisfying $\mathbf{Y}(u, 0) = \mathbf{y}(u)$ and $\mathbf{Y}_v(u, 0) = \mathbf{z}(u)$. Its projection into H_1 produces an orthogonal continuation of k (see Fig. 1).

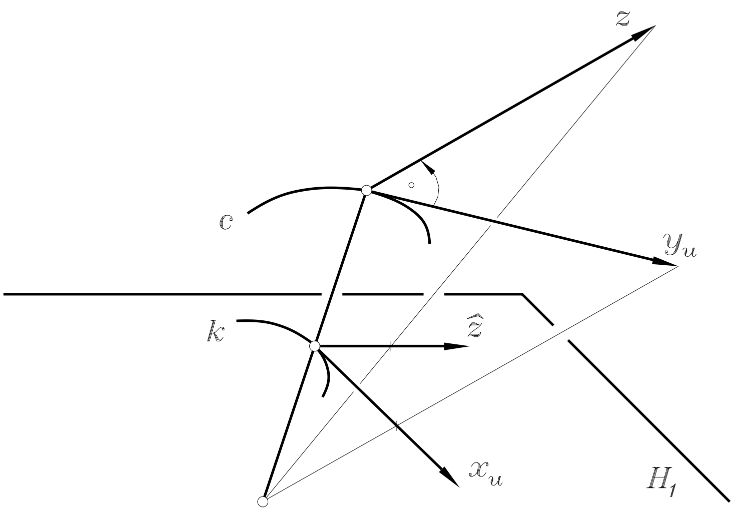

Figure 1: Constructing an orthogonal continuation to plane rational Bézier curve.

Remark:

- This construction demands an elevation of degree from m to $2m + l - 2$ and gives us freedom to choose the $m+2l$ components of the polynomials $L(u)$ and $Q(u)$. According to Theorem 3.1 we cannot expect a greater variety in the general case. Our construction is much more convenient than solving (10).
  And now we even know that it can provide the full variety of solutions.
- The choice $L(u) \equiv 0$ and $Q(u) \in \mathbb{R}[u]$ of degree $q < m-2$ produces a $(q+1)$-parameter variety of (trivial) orthogonal continuations of order $(m+q, 1)$. This is even more than we can expect, as $-m + 2q + 4 \leq q + 1$ if $q < m - 2$ (compare with Theorem 3.1). In case of $q < m - 3$ the equations (10) are obviously not independent.
- The non-trivial solution of the lowest possible degree $2m-2$ is given by $L(u) = \lambda \in \mathbb{R} \setminus 0$.

Remark: We can generalize our construction to rational Bézier curves in r-dimensional Euclidean space $\mathbb{R}^r$, if we replace the matrix $\mathbf{D}$ in (13) by a skew-symmetric matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{(r+1)\times(r+1)}$ with $a_{0i} = 0$, $i = 0, 1 \ldots r$. This choice garantuees that the vector $\hat{\mathbf{z}}(u)$ in (13) possesses all relevant properties: It is polynomial of degree $2m - 2$ or lower, parallel to H_1 and perpendicular to $\mathbf{x}_u(u)$. However in this case we do not get the full variety of solutions.

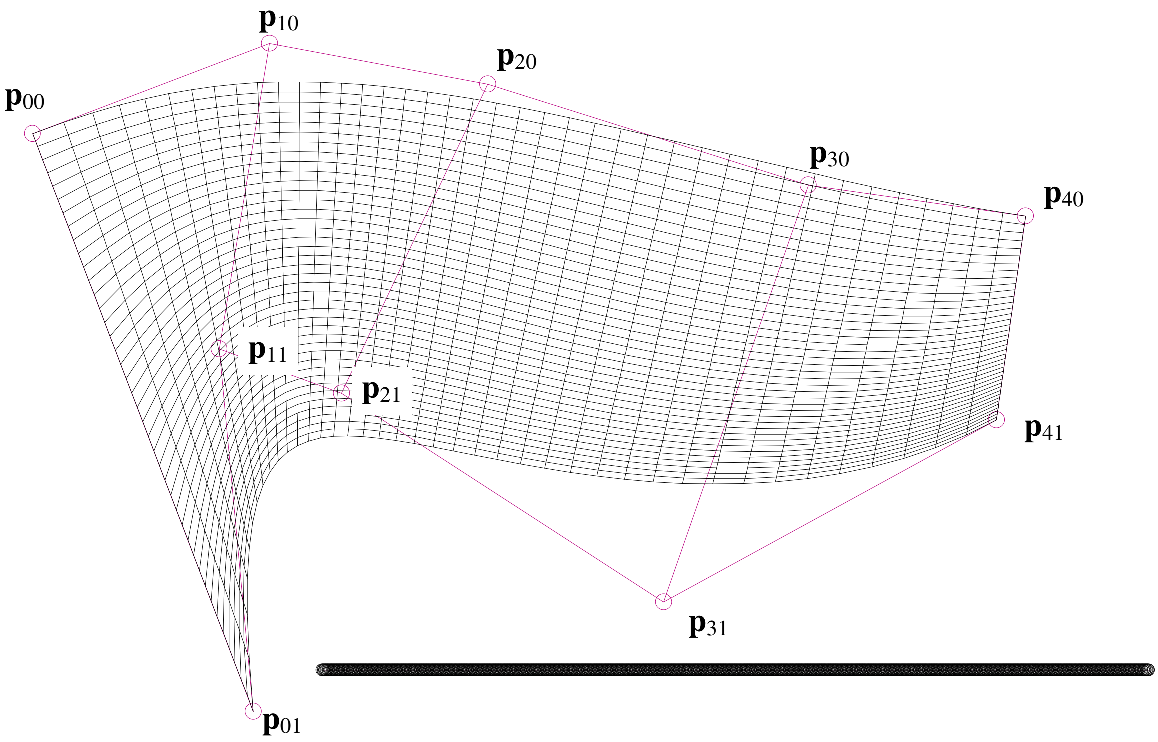

Figure 2: One exact solution gained after the necessary degree elevation.

4 A lower degree gained by special choice of weights

A rational Bézier curve $k \subset E^2$ in general cannot be continued orthogonally without elevation of degree while the integral Bézier curve $\tilde{k} \subset E^2$ defined by the same control points as k permits even a 2-paramter variety of *integral* orthogonal continuations *without* degree elevation. So obviously choosing weights for certain control points has an effect on the variety and order of possible orthogonal continuations. In this section we will use special relations between the weigths of a given rational Bézier curve to improve the construction of chapter 3. The following lemma (see P. Borwein, T. Erdely [1]) provides the basic idea:

Lemma 4.1 *Be given a polynomial $P(u) \in \mathbb{R}[u]$ in its factorized form*

$$P(u) = c \prod_{\sigma=0}^{s} (u - u_\sigma)^{\nu_\sigma}$$

with pairwise different $u_\sigma \in \mathbf{C}$ (the zeros of P), $c \in \mathbb{R} \setminus \{0\}$ (the leading coefficient of P) and $\nu_\sigma \in \mathbf{N}$ (the algebraic multiplicity of the zero u_σ). Then we have

$$\frac{\dot{P}(u)}{P(u)} = \sum_{\sigma=0}^{s} \frac{\nu_\sigma}{u - u_\sigma}.$$

Let us return to our weight polynomial Y^0

$$Y^0(u) = c \prod_{\sigma=0}^{s} (u - u_\sigma)^{\nu_\sigma}. \tag{15}$$

In (13) we have defined the polynomial vector $\hat{\mathbf{z}}(u)$ and used it to build an orthogonal continuation of k. According to Lemma 4.1 we can use the vector

$$\widetilde{\mathbf{z}}(u) := \mathbf{D}\dot{\mathbf{x}}(u) \cdot Y^0(u) \prod_{\sigma=0}^{s}(u - u_\sigma) \tag{16}$$

instead of $\hat{\mathbf{z}}(u)$ as $\widetilde{\mathbf{z}}(u)$ has the following properties:

- It is a real vector,
- it is parallel to H_1,
- it is perpendicular to $\mathbf{x}_u(u)$ and
- it is a *polynomial* vector (which is not trivial but can be seen with the help of Lemma 4.1).

Proceeding as in chapter 3 we can build an orthogonal continuation of order $(M, 1)$, where M denotes the degree of the polynomial vector $\widetilde{\mathbf{z}}(u)$. (If M happens to be lower than m we must of course multiply $\widetilde{\mathbf{z}}(u)$ by an arbitrary polynomial $R(u)$ of degree $m - M$ before constructing the orthogonal continuation patch.)

Lemma 4.2 *The vector $\widetilde{\mathbf{z}}(u)$ is of order $m + s - 1$ at most. If the weight polynomial Y^0 is of no lower degree than m the order of $\widetilde{\mathbf{z}}(u)$ is even $m + s - 2$ or less. (s is the number of different zeros of Y^0.)*

Proof: The first part of our assumption is obvious by definition (16). If the weight polynomial $Y^0(u)$ is actually of degree m, the leading coefficient of $\widetilde{\mathbf{z}}(u)$ is given by

$$\left(0, -cmy_{2,m} + cy_{2,m}\sum_{\sigma=0}^{s}\nu_\sigma, cmy_{1,m} - cy_{1,m}\sum_{\sigma=0}^{s}\nu_\sigma\right)^t,$$

where $y_{1,m}$ and $y_{2,m}$ denote the leading coefficients of $Y^1(u)$ and $Y^2(u)$, respectively. It vanishes in any case, as the algebraic multiplicities ν_σ add up to m. □

Because of Lemma 4.2 we have achieved a significant improvement as the degree of our patch is lower than indicated in chapter 3 in all cases where the weight polynomial has fewer than m different zeros.

However in general a rather high elevation of degree is still necessary in order to achieve sensible results. It can be avoided by admitting *'near orthogonal continuations'*. In the following chapter we try such an approach. We restrict ourselves to the case $r = 3$ presenting the method in a more lucid way. The whole considerations could easily be adapted to any other dimension.

5 The Monte Carlo method in 4-space

We make another try whose target is not to find an exact solution, but to handle the problem in a numerical way. It is interesting with respect to several properties:

- We can ask for continuations, where the cosine of the intersection angle between the starting curve k and the parameter lines is not necessarilly constant or even zero along k. We can even choose a function (see (17)) displaying the cosine which we want to achieve.

- For many functions (17) it may not be possible to find a patch matching to it in a mathematical exact way: The set of functions which may occur as the cosine of two polynomial vector functions, is of course a proper subset of the set of all functions $\eta : [0,1] \to [-1,1]$.
- The method shown in this section will provide sufficiently good *'solution patches'* of our problem.
- The dimension of the space does have great influence to the variety or even the existence of solutions, as has been shown in Theorem 3.1. The method suggested in this section will not substantially depend on the dimension. So, focussing the plane case of our problem, we can apply the Monte Carlo method in 3-space as well.

Step 1: We start with a given rational Bézier curve k defined by the Bézier control polygon $\mathbf{p}^{i0}, i = 0,\dots,m$ and weights γ^{i0}, $i = 0,\dots,m$. We expand it to a control net $\mathbf{p}^{ij}$, $i = 0,\dots,m$, $j = 0,\dots,n$ and a weight matrix γ^{ij}, $\imath = 0,\dots,m$, $j = 0,\dots,n$ in an arbitrary way: This is our starting net.

Step 2: Here we can afford to set a goal for the cosines $\eta(u) := \cos(\mathbf{x}_u(u)\cdot\mathbf{x}_v(u))$ of the angles between the parameter lines and the curve k. The function

$$\text{(17)} \qquad \begin{array}{rccl} \eta: & [0,1] & \to & [-1,1] \\ & u & \mapsto & \eta(u) \end{array}$$

can be chosen arbitrarily and will be called the *'performance map'* of our continuation[6].

Step 3: With the help of (1) we get the points $\mathbf{q}^{ij} = (q_0^{ij}, q_1^{ij}, q_2^{ij}, q_3^{ij})^t$ in 4-space. In order to be able to improve our net, we have to score the quality. The score function shall be:

$$\text{(18)} \qquad \int_0^1 \Big(\frac{\mathbf{x}_u(u,0)\cdot\mathbf{x}_v(u,0)}{|\mathbf{x}_u(u,0)|\cdot|\mathbf{x}_v(u,0)|}\Big)^2 - \eta^2(u)du \;=:\; S((\mathbf{q}^{ij})_{i=0,\dots,m,j=0,\dots 1}).$$

It measures the square of the deviation from the function η, which is our target performance map.

Step 4: We now apply a Monte Carlo method in order to improve the net $\mathbf{q}^{ij}$:
Thread 0 is not concerned by our action. We replace the first point $\mathbf{q}^{01} \in \mathbb{R}^4$ of thread 1 by a random point $\mathbf{q}^{01}_\star = \mathbf{q}^{01} + \mathbf{r}$, adding a random vector $\mathbf{r}$. Now we compare the new score (18) with the score before the replacement. Of course we have to apply (3) in order to gain (4), because it is the patch in $\mathbb{R}^3$ we are interested in; and this patch also gives us the score for the valuation of the effect.
If the score increases, the replacement is cancelled, otherwise we keep the new point $\mathbf{q}^{01}_\star$ instead of $\mathbf{q}^{01}$. Then we continue with $\mathbf{q}^{11}$ and so on until we get to the last point $\mathbf{q}^{m1}$ of thread 1.

In fact first of all we have to decide upon the maximum size of our random vector $\mathbf{r}$. This size has to be adapted with respect to the size of the whole net; let us call it the *'pace'*. We shall not only make one single try, but several random attempts. And in case that all of them fail we shall reduce the pace and go on with another set of attempts. This shall be repeated until the pace is smaller than the desired accuracy limit.

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[6] At this point the title of the paper may not seem appropriate: We do not only look for *'orthogonal continuations'*, we can choose the angles along k by ourselves. The orthogonal continuation is a special case characterised by $\eta(u) \equiv 0$.

After having worked through with thread 1 we begin with point $\mathbf{q}^{01}$ again and start with the second cycle and so on. Whenever we remove a point and replace it by another one we gain a better score (18). At every random step described above we change our points $\mathbf{q}^{ij} = (q_0^{ij}, q_1^{ij}, q_2^{ij}, q_3^{ij})^t$ in 4-space by adding a random vector: $\mathbf{q}_\star^{ij} = \mathbf{q}^{ij} + \mathbf{r}$.

We can choose between three different strategies:

1. The random vector $\mathbf{r}$ is any vector out of $\mathbb{R}^4$ whose length is limited by the pace valid at that moment. All its four coordinates are treated equally.
2. The random vector is a vector $(r_0, r_1, r_2, r_3)^t$ with $r_1 = r_2 = r_3 = 0$, where the only random number r_0 is limited by the pace.
3. We choose a real number $\lambda > 0$ and add the vector $(\lambda\, r_0, r_1, r_2, r_3)^t$, where the random vector $|(r_0, r_1, r_2, r_3)^t|$ is, as above, limited by the pace. As a consequence, the effect of the random process on the weights is less if $\lambda < 1$ and is gradually larger if $\lambda > 1$.

The number of cycles which is necessary may vary. According to our experience 5 to 20 cycles provided very good performance.

We keep in mind that a score (18) equal to zero would say that our net represents an exact solution to our target performance map $\eta = \eta(u)$. Recall that the plane orthogonal continuation problem for a rational starting curve k of order m in general does not have an exact solution unless the degree is elevated to order $2m-2$. Applying the Monte Carlo method shown above we can find a solution with surprisingly high accuracy — without elevating the degree.

Of course, the way of sweeping across our threads of the patch is *'lop-sided'*, because we always start with the point on the *'left hand side'*. A *'zig-zag-strategy'* can easily be applied without any additional adaption. It acknowledges the symmetry of the patch properly.

The following example deals with the plane case. In 3-space finding a patch meeting our orthogonality conditions (or any condition given (see (17))) is also interesting and our method provides solutions. We have to admit, however, that in many cases of applications, there is an additional condition, demanding that the new patch be tangent to an existing one or to a given differentiable *'stripe'*. The Monte Carlo approach is capable of managing this case in a powerful way. Considering this task is the scope of another paper (see [7]).

6 An interesting example

We are going to look for an orthogonal continuation of a *plane* rational Bézier curve of order 3. The same problem has been solved exactly in Fig. 2 after applying the necessary elevation of degree by 1. Here we refuse to elevate the degree keeping in mind that, as a consequence, the problem does not have an exact solution. We are going to apply the Monte Carlo method as described in section 5, setting $\eta(u) \equiv 0$ (Figures 3, 4, 5). The example has shown that we gain a solution to our problem with an appropriate performance. The calculation time involved does not exceed sensible limits. According to our experience the Monte Carlo approach provides surprisingly good results even in cases, where the non-existence of an exact solution can easily be proved.

7 Further remarks

In this paper we set ourselves two goals: On one hand we dealt with the problem in the mathematically exact way, on the other hand we tried a numerical approach.

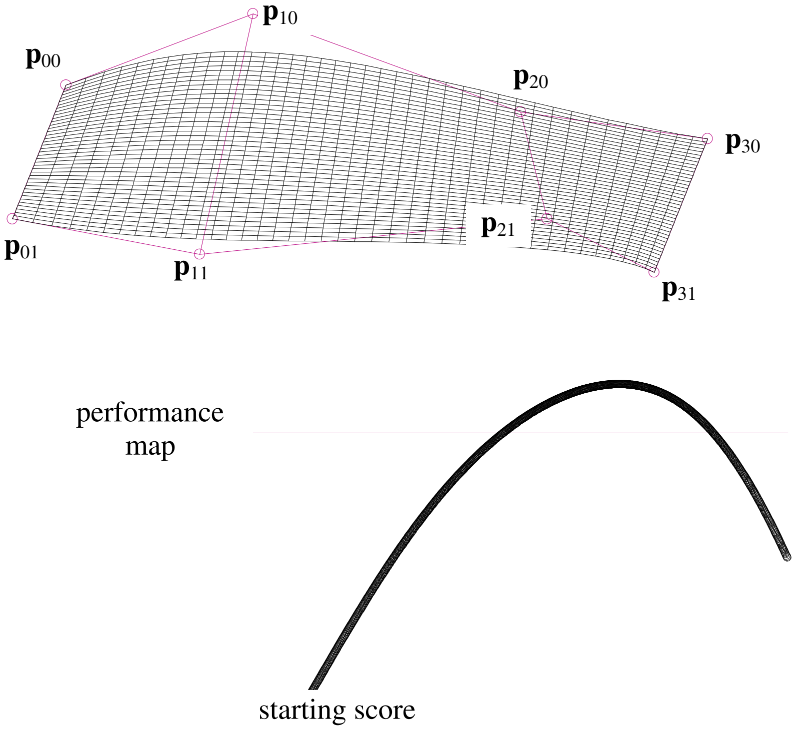

Figure 3: The starting net and the starting score before beginning with the Monte Carlo optimisation.

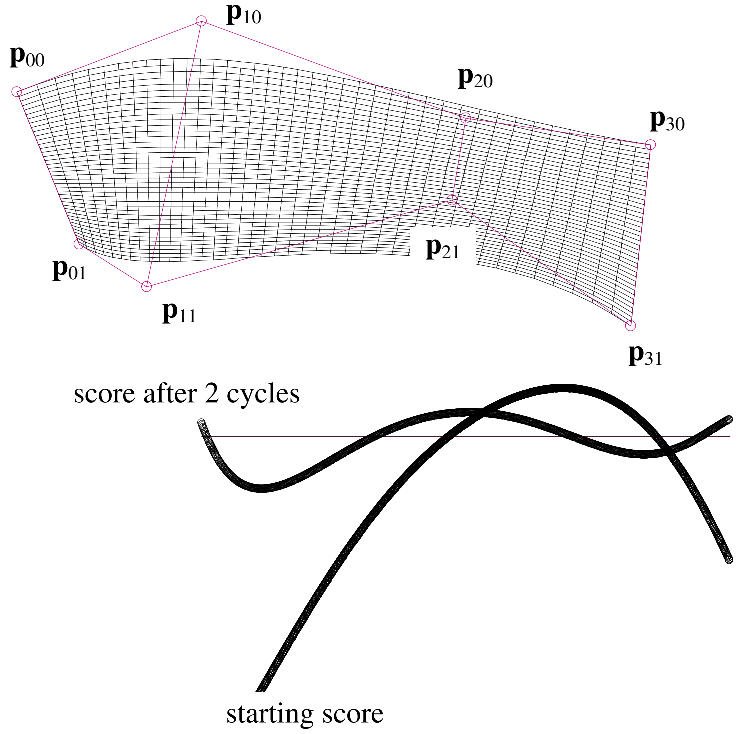

Figure 4: The net has been changed essentially after 2100 random attempts (2 cycles). The score has been improved to 0.002.

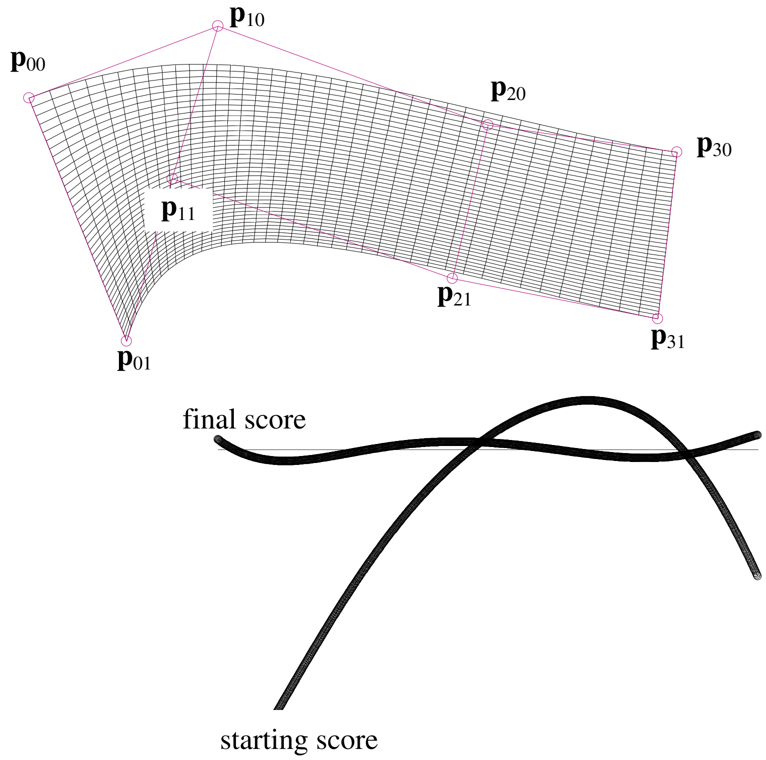

Figure 5: After 19050 random attempts (20 cycles) the score has been improved to 0.00001. The maximal angle deviation is below 1.5 degrees. The angle cosines function is sufficiently close to the target performance map $\eta \equiv 0$.

The first view led to a system of linear equations showing, that the number and degree of solutions depends on the elevation of degree applied to the starting curve and on the dimension of the space, but also on the properties of its weight polynomial. For the plane case we were able to provide a straightforward solution which helps a lot in practical application.

The second view allowed us to deal with all the cases, where an exact solution does not exist or is of a very high degree so that it does not seem to make sense to look for it. For these cases a Monte Carlo method is obviously very appropriate.

Finally, in Fig. 6 we show an example of a calculation grid, provided by a CFD-application[7] using the methods from above. Some details: In order to meet the demands of the boundary layer calculation, the grid is densified in the neighbourhood of the boundary. The calculation is done in the plane (2D). The picture shows a section through a turbine blade.

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[7]This figure has been provided by A. Gehrer and H. Jericha, and it is part of their paper on *"External Heat Transfer Predictions in a Highly-Loaded Transonic Linear Turbine Guide Vane Cascade Using an Upwind-Biased Navier-Stokes Solver"*, ASME-Paper 98-GT-238, Transactions of the ASME.

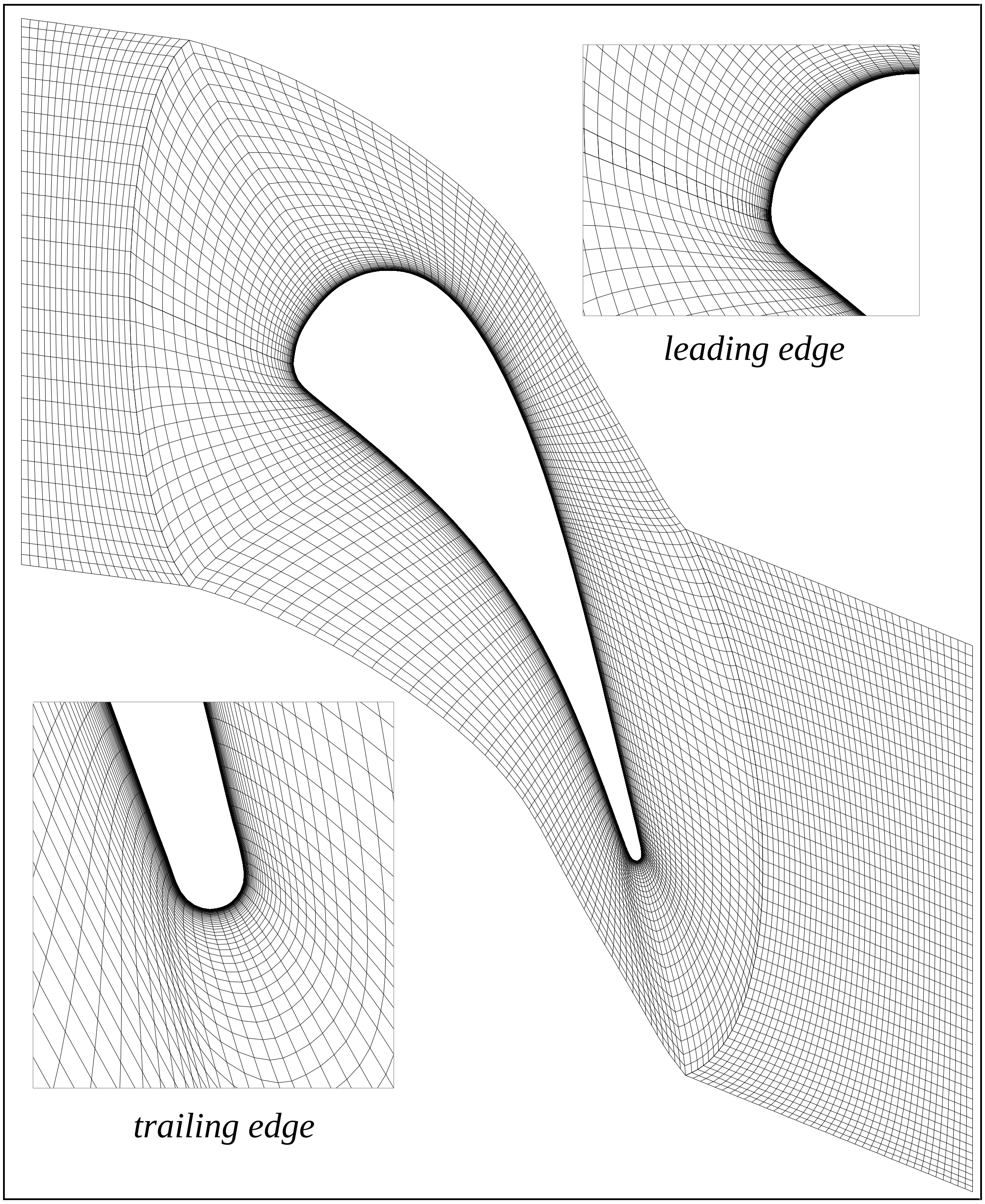

Figure 6: Grid generation: A glance at a CFD-application: Periodic O-type grid (175×59 cells) with inlet (19×30 cells) and outlet (39×49 cells) patches (in total: 12806 cells).

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