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On the Arc Length of Parametric Cubic Curves

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Abstract. We are seeking cubic parametric curves whose arc length can be expressed in a closed form. Based on the control points of a Bézier representation of parametric cubics, we provide a criterion to determine whether their arc length has a closed form or not.

Key Words: arc length, parametric cubic curve, Bézier curve

MSC 1994: 53A04, 68U05

1. Introduction

In this paper we examine when the arc length of cubic parametric curves, which are widely used in CAGD, can be expressed in a closed form. By cubic parametric curves we mean curves that have a parametric representation of the form

$$\mathbf{r}\,(u) = \mathbf{r}_3 u^3 + \mathbf{r}_2 u^2 + \mathbf{r}_1 u + \mathbf{r}_0\,, \quad u \in [u_0, u_1] \subseteq \mathbb{R}.$$

These curves form a subset of cubic curves since not all cubic curves have a polynomial parametric representation (see e.g. [3]). Any arc of such a curve on the domain $[a,b] \subseteq [u_0,u_1]$ can be described by a cubic Bézier curve

$$\mathbf{b}(t) = \sum_{i=0}^{3} B_i^3(t)\mathbf{b}_i\,, \quad t \in [0,1], \tag{1}$$

where $B_i^3(t) = \binom{3}{i} t^i (1-t)^{3-i}$, $i = 0, \ldots, 3$, is the i-th cubic Bernstein polynomial and

$$\mathbf{b}_0 = \mathbf{r}(a), \quad \mathbf{b}_1 = \mathbf{r}(a) + \tfrac{1}{3}\,\dot{\mathbf{r}}(a), \quad \mathbf{b}_2 = \mathbf{r}(b) - \tfrac{1}{3}\,\dot{\mathbf{r}}(b), \quad \mathbf{b}_3 = \mathbf{r}(b).$$

The choice of $[0,1]$ is not a restriction, since Bézier curves are invariant under affine parameter transformations (cf. [1], [4]). Due to their easy to handle nature and the direct geometric meaning of their defining data, from here on we use the Bernstein-Bézier representation of

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parametric cubics. Starting from this representation we examine under which conditions the arc length of a parametric cubic can be expressed in a closed form, and what the consequences of this property are to the control points of the Bézier curve.

In [7] there is a discussion of algebraically rectifiable polynomial plane parametric curves, with the study of their cubic, quartic and quintic special cases. In [2] Pythagorean-hodograph spatial polynomial curves are examined, with a special attention to the cubic case, and [10] extends these results to construct G^1 interpolation.

In this paper we study cubic parametric spatial curves; planar cubics of the previous type are obtained as special cases. At first we examine the hodograph of these cubics. Then we provide a criterion, based on the control polygon of the cubic's Bézier representation, to determine whether the arc length of a cubic has a closed form. Our approach is rather elementary, geometric and constructive. As a result of this approach we could find new characteristics of both the hodograph and the control polygon, additional to those which have already been published by other authors.

2. The arc length of cubic Bézier curves

The arc length of the Bézier curve (1) can be specified by means of its hodograph

$$\dot{\mathbf{b}}(t) = 3\sum_{i=0}^{2}\binom{2}{i}t^i(1-t)^{2-i}\mathbf{a}_i\,, \quad \mathbf{a}_i := \mathbf{b}_{i+1} - \mathbf{b}_i\,, \tag{2}$$

in the form

$$s = 3\int_0^1 \sqrt{\mathbf{a}_0^2(1-t)^4 + \mathbf{a}_2^2t^4 + (4\mathbf{a}_1^2 + 2\mathbf{a}_0\mathbf{a}_2)t^2(1-t)^2 + 4\mathbf{a}_0\mathbf{a}_1t(1-t)^3 + 4\mathbf{a}_1\mathbf{a}_2t^3(1-t)}\,dt.$$

In this formula there is the square root of a quartic polynomial in t. This is why the arc length in general leads to an elliptic integral, which can not be expressed in a closed form (cf. [5]). Our objective is to find those special cases, and their geometric meaning, when the evaluation of the integral above results in a closed form.

The detailed form of the expression (2) is

$$\dot{\mathbf{b}}(t) = 3\left(t^2(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0) + t(2\mathbf{b}_2 - 4\mathbf{b}_1 + 2\mathbf{b}_0) + (\mathbf{b}_1 - \mathbf{b}_0)\right). \tag{3}$$

In general, the hodograph of a cubic Bézier curve is a parabola. As a special case the coefficient of t^2 may be zero, i.e., the hodograph may be linear. In this case the original Bézier curve is a parabolic arc and its control points satisfy the equality

$$\mathbf{b}_3 - \mathbf{b}_0 = 3(\mathbf{b}_2 - \mathbf{b}_1).$$

If the coefficients of both t^2 and t vanish then the hodograph is constant and the original Bézier curve is a straight line segment, i.e., its control points are collinear. In the above special cases the arc length has a closed form. We exclude these special cases from our later investigation, i.e., we assume that the coefficient of t^2 differs from zero.

The integrand in the arc length's formula is difficult to handle. In order to simplify this expression we change the coordinate system and perform a parameter transformation. Based on expression (3), we introduce a new coordinate system $(\widehat{x}, \widehat{y}, \widehat{z})$: The origin coincides with the previous origin, the $\widehat{y}$-axis has the direction of $3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0)$, which is exactly the coefficient of the hodograph's second order term, and the $\widehat{x}$-axis is chosen to ensure either the parallelism or the coincidence of the plane spanned by the points $\mathbf{a}_i$ and the $\widehat{xy}$-plane (see Fig. 1).

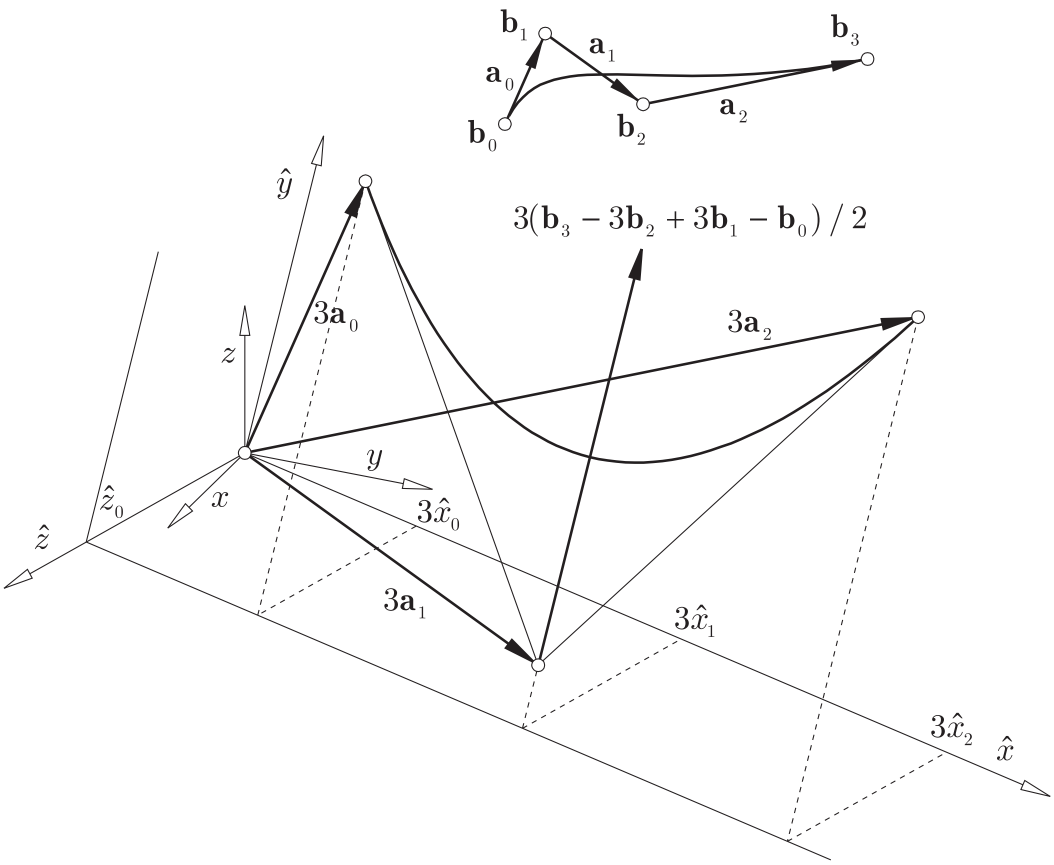

Figure 1: The new $(\widehat{x}, \widehat{y}, \widehat{z})$ coordinate system

The hodograph (2) is a parabolic arc. Therefore, if we project the arc parallel to its axis onto a line, the projection of the intersection point of its end-points' tangents bisects the projection of the arc. Consequently, denoting the abscissa of $\mathbf{a}_i$ in the new coordinate system by $\widehat{x}_i$, then $\widehat{x}_1 = \frac{1}{2}(\widehat{x}_0 + \widehat{x}_2)$ (see Fig. 1) and the $\widehat{x}$ coordinate function of the hodograph (3) is

$$\widehat{x}(t) = 3\left((1-t)^2\widehat{x}_0 + 2t(1-t)\frac{\widehat{x}_0 + \widehat{x}_2}{2} + t^2\widehat{x}_2\right) = 3\left(\widehat{x}_0(1-t) + \widehat{x}_2 t\right).$$

Carrying out the parameter transformation

$$t = \frac{\widehat{x} - 3\widehat{x}_0}{3(\widehat{x}_2 - \widehat{x}_0)}, \quad \widehat{x} \in [3\widehat{x}_0, 3\widehat{x}_2],$$

the hodograph's parametric form in the new coordinate system becomes

$$\widehat{y} = a\widehat{x}^2 + b\widehat{x} + c\,, \quad \widehat{z} = \widehat{z}_0\,, \tag{4}$$

where $a > 0$ and $\widehat{z}_0$ is constant. Introducing the notation $H := 3(\widehat{x}_2 - \widehat{x}_0)$ for the arc length of the Bézier curve (1), we gain the formula

$$s = \frac{1}{H} \int_{\widehat{x}_a}^{\widehat{x}_b} \sqrt{\widehat{x}^2 + \widehat{y}^2(\widehat{x}) + \widehat{z}_0^2}\, d\widehat{x} \tag{5}$$

which is easier to handle than its original form. In expression (5) H is the length of the hodograph's perpendicular projection on the $\widehat{x}$-axis, the limits of the integral are the new abscissas of the hodograph's points that correspond to the end-points of the arc to be rectified.

3. Hodographs of cubic Bézier curves with a closed form arc length

According to expressions (4) and (5), the integrand is the square root of the expression

$$\widehat{x}^2 + \left(a^2\widehat{x}^4 + 2ab\widehat{x}^3 + b^2\widehat{x}^2 + 2ac\widehat{x}^2 + 2bc\widehat{x} + c^2\right) + \widehat{z}_0^2\,, \tag{6}$$

which is a real quartic polynomial in $\widehat{x}$ due to our previous assumption. This integral has a closed form, if either

- expression (6) has multiple zeros, i.e., (6) is a complete square of the form $(e\widehat{x}^2+g\widehat{x}+h)^2$, or
- it is possible to factor out of (6) a quadratic term

$$(\widehat{x} + g)^2 = \widehat{x}^2 + 2g\widehat{x} + g^2. \tag{7}$$

3.1. Conditions for a complete square

The expression (6) is a complete square if the equality

$$\left(e\widehat{x}^2 + g\widehat{x} + h\right)^2 = \widehat{x}^2 + a^2\widehat{x}^4 + 2ab\widehat{x}^3 + b^2\widehat{x}^2 + 2ac\widehat{x}^2 + 2bc\widehat{x} + c^2 + \widehat{z}_0^2$$

holds, therefore

$$e = \pm a, \quad g = \pm b, \quad 1 + b^2 + 2ac = g^2 + 2eh \;\text{ and }\; 2bc = 2gh.$$

The last two equalities are contradictory unless $g = b = 0$, from which we gain $h = \pm\left(1/2a + c\right)$ and $c^2 + \widehat{z}_0^2 = h^2$.

Therefore the coordinate functions are

$$\widehat{y} = a\widehat{x}^2 + c\,, \quad \widehat{z} = \widehat{z}_0 \;\text{ with }\; c^2 + \widehat{z}_0^2 = \left(\frac{1}{2a} + c\right)^2.$$

Denoting the parameter of the parabola, i.e., of the hodograph, by $p := \dfrac{1}{2a}$ we obtain the functions

$$\widehat{y} = \frac{\widehat{x}^2}{2p} + c\,, \quad \widehat{z} = \widehat{z}_0 = \pm\sqrt{p^2 + 2pc}\,. \tag{8}$$

Because of the previous coordinate system and parameter transformations p is positive, thus the value of $\widehat{z}_0$ is real, provided $p + 2c \geq 0$.

In order to find the parabolas specified above, let us draw a circle in the $\widehat{yz}$-plane with the center at the origin of the coordinate system and with the radius $r = \sqrt{c^2 + \widehat{z}_0^2}$. Let

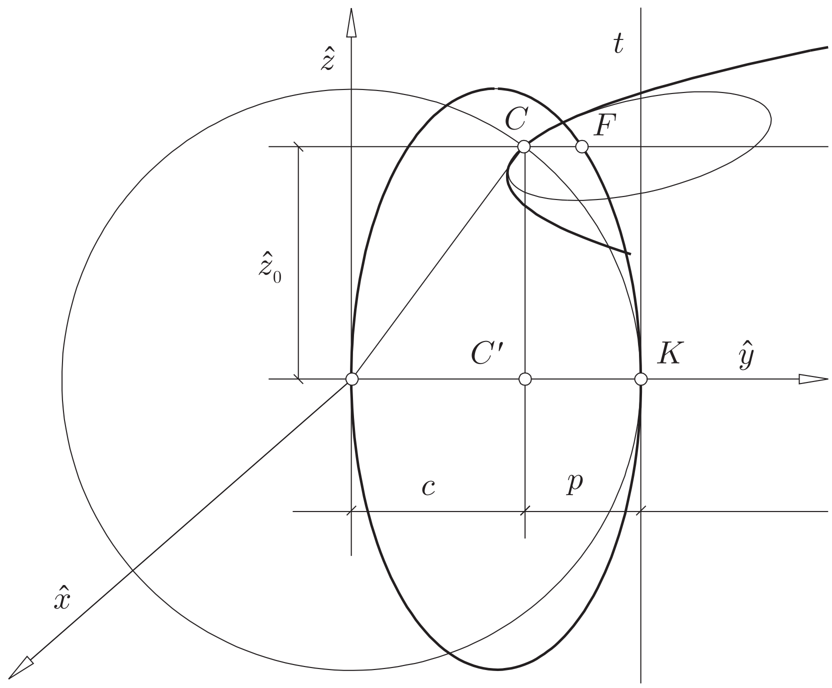

Figure 2: Construction of parabolas specified by (8)

the arbitrarily chosen point $C = [0, c, \widehat{z}_0]^T$ on this circle be the vertex of the parabola we are looking for, and C' its perpendicular projection on the $\widehat{y}$-axis (see Fig. 2). The distance from C' to $K = [0, r, 0]^T$ is equal to the parameter p of the parabola, thus the line t passing through K and parallel to $\widehat{z}$ is the axis of revolution of the osculating circle of the parabola at the point C. The parabola's focus is the midpoint of the straight line segment bounded by C and the foot point of the perpendicular from C to t. As C moves along the circle, the osculating circle forms the parallel circles of the torus, which is generated by the circle with radius r under the rotation about the line t. The foci of these parabolas form an ellipse which can also be gained from the original circle by the orthogonal axial affinity with the axis t and the scaling factor $\frac{1}{2}$.

Note, that O is on the focal conic (with focus C and directrix t) of the hodograph since $OC = OK$. Therefore the hodograph's position vectors form a cone of revolution, thus the angle between the axis of this cone and the tangents of the cubic corresponding to the hodograph is constant, i.e., the tangents of the cubic are of constant slope. Such a curve is called a cubic helix in [11]. Vice versa, the hodograph of a cubic helix is on a cone of revolution the vertex of which is the origin, therefore the origin is on the focal conic of the hodograph, thus in a suitable coordinate system the equation of the hodograph is of the form (8). It is also obvious, that $\sphericalangle(\mathbf{a}_1, \mathbf{a}_0) = \sphericalangle(\mathbf{a}_1, \mathbf{a}_2)$ since $\mathbf{a}_1$ represents the line of intersection between the tangent planes of the previous cone of revolution along its generators $\mathbf{a}_0$ and $\mathbf{a}_2$, respectively. This property was revealed in [2] and it was utilized in [10].

On the other hand, the expression

$$c^2 + \widehat{z}_0^2 = (p + c)^2$$

also implies that the hodograph above could be gained by intersecting the plane $\widehat{z} = \widehat{z}_0$ and the paraboloid of revolution with the axis $\widehat{y}$, the focus at the origin and the parameter p. Vice versa, if we intersect such a paraboloid of revolution with the plane $\widehat{z} = \widehat{z}_0$, then the obtained parabola is of the form (8).

We now examine the special case $\widehat{z}_0 = 0$, i.e., the case when the Bézier curve is a plane curve. The case $p = 0$ is not relevant here, since then the Bézier curve is a straight line segment. In the case $c = -p/2$ the hodograph is a parabola with its focus at the origin, and the cubic Bézier curve has a loop. A parametric cubic for which expression (6) is a complete square is also called a Pythagorean-hodograph (PH) cubic. Their planar case is investigated in [7] and the spatial one in [2]. In [10] it is shown, that PH cubics (cubic helices) represent the simplest, yet most important set of rational Frenet-Serret motion curves.

3.2. Conditions for a common quadratic factor

Here we examine when expression (7) can be factored out of (6). Expression (6) can be subdivided by (7) without a remainder if the system

$$(1 + b^2 + 2ac - 4abg + 3a^2g^2)g^2 = c^2 + \widehat{z}_0^2 \tag{9}$$

$$(bc - abg^2 + a^2g^3)g = c^2 + \widehat{z}_0^2 \tag{10}$$

of parametric equations is solvable for g.

We are looking for those values of the parameters $a, b, c, \widehat{z}_0$, for which the equations (9) and (10) have a common solution in g. A trivial solution is $g = 0$, which implies $c^2 + \widehat{z}_0^2 = 0$. Vice versa, if $c^2 + \widehat{z}_0^2 = 0$, then the only solution is $g = 0$, since otherwise from (10) $g = b/a$ could be deduced, which leads to a contradiction when it is substituted in (9).

To find a non-trivial solution, we assume that $g \neq 0$. Dividing the difference of the equations (9) and (10) by g, we obtain the equation

$$c(2ag - b) = -2a^2g^3 + 3abg^2 - (1 + b^2)g \tag{11}$$

which is linear in c. Assuming $2ag \neq b$, we get from (11)

$$c = -ag^2 + bg - \frac{g}{2ag - b}\,. \tag{12}$$

Inserting (12) into (10), we obtain the equality

$$0 = g^2 + \frac{g^2}{(2ag - b)^2} + \widehat{z}_0^2$$

which can only be satisfied with the trivial solution $g = \widehat{z}_0 = 0$.

Now let us examine the case $2ag = b$. Dividing the right hand side of equation (11) by g, we gain the equality

$$2a^2g^2 - 3abg + 1 + b^2 = 0\,,$$

which leads to the contradiction $1 = 0$, since $2ag = b$.

Summarising our results, we can state that the only solution is the trivial one $g = c = \widehat{z}_0 = 0$, and in this case the parabola, which contains the hodograph, passes through the origin of the coordinate system, since its equations are

$$\widehat{y} = a\widehat{x}^2 + b\widehat{x}, \quad \widehat{z}_0 = 0\,.$$

4. The arc length and properties of curves

In this section we determine the arc length of cubic curves with hodographs of Section 3, and examine the properties of such cubics.

4.1. Case of a complete square

Taking into account the results of Subsection 3.1, the equations of the hodograph's parabola are

$$\widehat{y} = \frac{\widehat{x}^2}{2p} + c\,, \quad \widehat{z} = \widehat{z}_0 \ \text{ with } \ c^2 + \widehat{z}_0^2 = (p+c)^2 .$$

The arc length of the cubic curve is

$$s = \frac{1}{H}\int\limits_{\widehat{x}_a}^{\widehat{x}_b} \sqrt{\widehat{x}^2 + \left(\frac{\widehat{x}^2}{2p} + c\right)^2 + p^2 + 2pc}\, dx = \frac{1}{H}\int\limits_{\widehat{x}_a}^{\widehat{x}_b} \left|\frac{\widehat{x}^2}{2p} + c + p\right| dx .$$

According to our earlier assumption $p + 2c \geq 0$, we have

$$s = \frac{1}{H}\left[\frac{\widehat{x}^3}{6p} + (c+p)\widehat{x}\right]_{\widehat{x}_a}^{\widehat{x}_b} . \tag{13}$$

If the arc to be rectified coincides with our original curve, i.e., $\widehat{x}_a = 3\widehat{x}_0$ and $\widehat{x}_b = 3\widehat{x}_2$, then after some simplification we obtain

$$s = \frac{3(\widehat{x}_2^2 + \widehat{x}_0\widehat{x}_2 + \widehat{x}_0^2)}{2p} + (c+p).$$

In order to determine the coordinate functions of the corresponding parametric cubic curve, we reparametrise the hodograph in such a way that the new parameter u differs only in an additive constant from the original one. This is to ensure that the parabola's vertex belongs to the parameter value $u = 0$. The resulting coordinate functions of the hodograph are

$$\widehat{x}(u) = Hu, \quad \widehat{y}(u) = \frac{(Hu)^2}{2p} + c\,, \quad \widehat{z}(u) = \widehat{z}_0 = \pm\sqrt{p^2 + 2pc}\,.$$

Now we determine the coordinate functions of the cubic parametric curve by indefinite integration in the coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ which arises from the system $(\widehat{x}, \widehat{y}, \widehat{z})$ by a translation. This gives

$$\tilde{x}(u) = \frac{Hu^2}{2} + \tilde{x}_0\,, \quad \tilde{y}(u) = \frac{H^2u^3}{6p} + cu + \tilde{y}_0\,, \quad \tilde{z}(u) = \widehat{z}_0 u + \tilde{z}_0\,.$$

The constants of the integrals above mean only a translation of the curve, thus we omit them, i.e., we set $\tilde{x}_0 = \tilde{y}_0 = \tilde{z}_0 = 0$. In this manner the coordinate functions of the parametric cubic are

$$\tilde{x}(u) = \frac{Hu^2}{2}\,, \quad \tilde{y}(u) = \frac{H^2u^3}{6p} + cu\,, \quad \tilde{z}(u) = \widehat{z}_0 u\,. \tag{14}$$

One can see that $\tilde{x}(-u) = \tilde{x}(u)$, $\tilde{y}(-u) = -\tilde{y}(u)$ and $\tilde{z}(-u) = -\tilde{z}(u)$, i.e., the curve is symmetric with respect to the $\tilde{x}$-axis. In Fig. 3 the curve is illustrated by its orthographic projections in the coordinate planes. After the elimination of the parameter u the equations of the projections are

$$\tilde{x} = \frac{H\tilde{z}^2}{2\widehat{z}_0^2}\,, \quad \tilde{y} = \frac{H^2\tilde{z}^3}{6p\widehat{z}_0^3} + \frac{c\tilde{z}}{\widehat{z}_0}\,, \quad \tilde{y}^2 = 2\tilde{x}\left(\frac{\sqrt{H}\tilde{x}}{3p} + \frac{c}{\sqrt{H}}\right)^2 ,$$

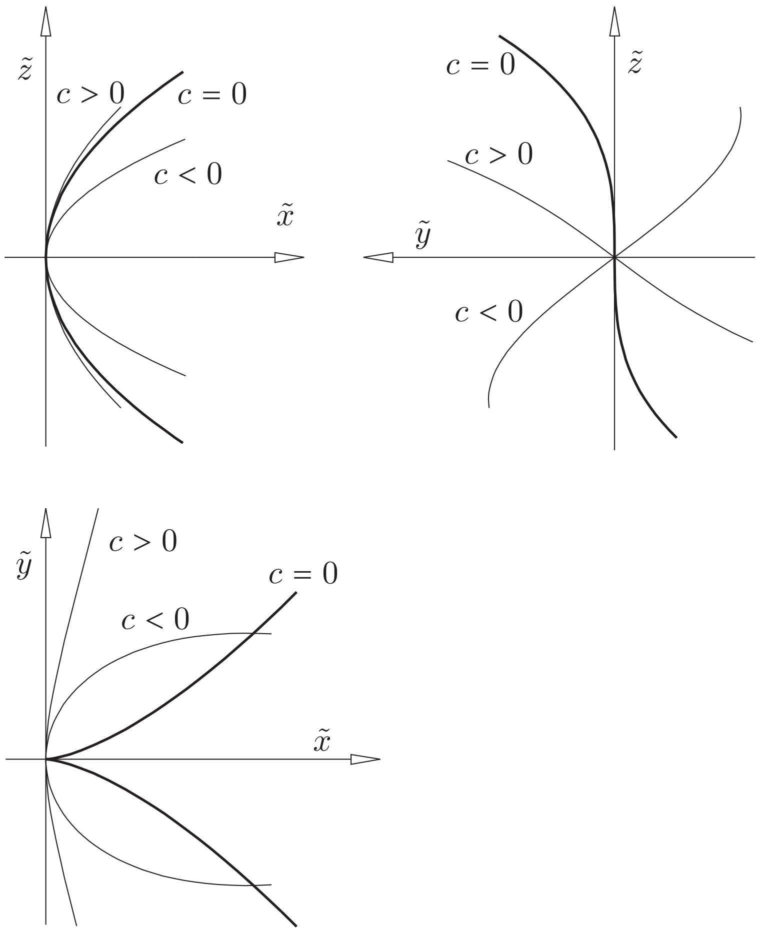

Figure 3: Orthographic projections of cubic (14) in the coordinate planes

with $\widehat{z}_0 = \pm\sqrt{p^2 + 2pc}$. Such curves are similar if their hodographs are similar with the centre of similarity at the origin, i.e., if the ratios of the corresponding values of H, c, p, and $\widehat{z}_0$ are identical.

Under $c = 0$ the parametric form becomes

$$\tilde x_N(u) = \frac{Hu^2}{2}, \quad \tilde y_N(u) = \frac{H^2u^3}{6p}, \quad \tilde z_N(u) = \widehat{z}_0 u\,. \tag{15}$$

The equations of the curve's projections are

$$\tilde x_N = \frac{H}{2\widehat{z}_0^2}\,\tilde z_N^2, \quad \tilde y_N = \frac{H^2}{6p\widehat{z}_0^3}\,\tilde z_N^3, \quad \tilde y_N^2 = \frac{2H}{9p^2}\,\tilde x_N^3,$$

with $\widehat{z}_0 = \pm p$, i.e., its projection in the $\tilde x\tilde z$-plane is a second-order parabola, in the $\tilde z\tilde y$-plane is a cubic curve and in the $\tilde x\tilde y$-plane is an O'Neill's parabola. The functions of the generic case can be obtained from the special case $c = 0$ by the transformation

$$\tilde x = \tilde x_N\,, \quad \tilde y = \tilde y_N + \frac{c}{p}\,\tilde z_N\,, \quad \tilde z = \frac{\widehat{z}_0}{p}\,\tilde z_N\,.$$

According to the equations above, the generic case $c \neq 0$ can be obtained from the special case $c = 0$ by applying a spatial axial affinity whose axial plane is the $\tilde x\tilde y$-plane, whose direction is

$[0, c, \widehat{z}_0 - p]^T$ and ratio is $\widehat{z}_0/p$. In a suitable affine coordinate system $(\breve{x}, \breve{y}, \breve{z})$ this curve has the parametric form

$$\breve{x} = k_2 u^2, \quad \breve{y} = k_3 u^3, \quad \breve{z} = k_1 u$$

(cf. [9]). According to Seydewitz's classification of cubic curves this is called a cubic parabola.

4.1.1. The focus of the hodograph is at the origin

We get a special case of the hodograph in 4.1 when

$$\widehat{z}_0 = 0, \quad a = \frac{1}{2p} \quad \text{and} \quad c = -\frac{p}{2},$$

since the equation of the hodograph is $\widehat{y} = \dfrac{\widehat{x}^2}{2p} - \dfrac{p}{2}$, i.e., the hodograph is a parabola with parameter p and its focus is at the origin. By substitution into (13), the arc length of the corresponding Bézier curve is

$$s = \frac{1}{2H}\left[\frac{\widehat{x}^3}{3p} + p\widehat{x}\right]_{\widehat{x}_a}^{\widehat{x}_b}. \tag{16}$$

For $\widehat{x}_a = 3\widehat{x}_0$ and $\widehat{x}_b = 3\widehat{x}_2$ the arc length becomes

$$s = \frac{3(\widehat{x}_2^2 + \widehat{x}_2\widehat{x}_0 + \widehat{x}_0^2)}{2p} + \frac{p}{2}.$$

The parametric cubic curve (14) is reduced to

$$\tilde{x}(u) = \frac{Hu^2}{2} + \tilde{x}_0\,, \quad \tilde{y}(u) = \frac{H^2u^3}{6p} - \frac{pu}{2} + \tilde{y}_0$$

and this plane curve is also symmetric with respect to the $\tilde{x}$-axis. Such a cubic is also called a Tschirnhausen's cubic (cf. [6], [7]).

If $u = \pm p/H$, then

$$\dot{\tilde{y}}(u) = \frac{(Hu)^2}{2p} - \frac{p}{2} = 0 \quad \text{and} \quad \ddot{\tilde{y}}(u) = \frac{Hu}{p} = \pm 1\,;$$

therefore the $\tilde{y}$-coordinate function has an extreme value with

$$\tilde{x} = \frac{p^2}{2H}, \quad \tilde{y} = \mp\frac{p^2}{3H}.$$

At $u = 0$ there is a maximum of the curvature, and the radius of curvature at this point is $\varrho = p^2/4H$. Eliminating the parameter, the implicit form of the curve is

$$\tilde{y}^2 = \frac{\tilde{x}}{2H}\left(\frac{2H\tilde{x}}{3p} - p\right)^2.$$

One can see that $\tilde{x} = 0$ is a single and $\tilde{x} = \dfrac{3p^2}{2H}$ is a double root of this function. The latter is a crunode of the curve, since it belongs to two different parameter values, namely to $u = \pm\sqrt{3}p/H$. At this point we get $\dot{\tilde{x}}(u) = \pm\sqrt{3}p$ and $\dot{\tilde{y}}(u) = p$, consequently the angle between the two tangents is $\pi/3$. This curve is shown in Fig. 4.

All parametric cubics whose hodograph is a parabola of the form $\widehat{y} = \dfrac{\widehat{x}^2}{2p} - \dfrac{p}{2}$ with the focus at the origin are similar and their proportionality constant is p^2/H.

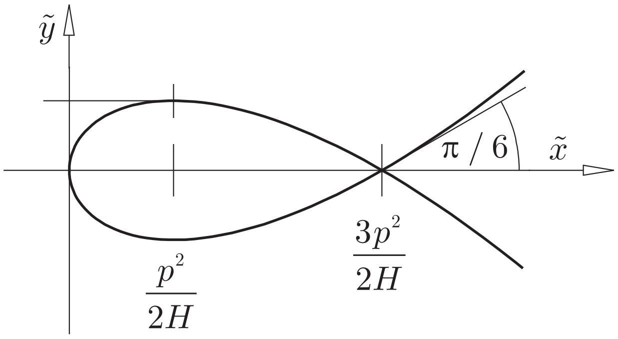

Figure 4: The plane cubic curve whose hodograph's focus is the origin

4.2. Case of a common quadratic factor

If $g = c = \widehat{z}_0 = 0$, the equation of the hodograph is

$$\widehat{y} = a\widehat{x}^2 + b\widehat{x}\,,$$

i.e., the hodograph passes through the origin, thus the parametric cubic curve is cuspidal. Vice versa, the hodograph of a cuspidal cubic curve passes through the origin.

The arc length of this cubic curve is

$$s = \frac{1}{H}\int_{\widehat{x}_a}^{\widehat{x}_b} \sqrt{\widehat{x}^2 + a^2\widehat{x}^4 + 2ab\widehat{x}^3 + b^2\widehat{x}^2}\, d\widehat{x}\,.$$

Supplementing the right hand side with a^2 and factoring out $\widehat{x}^2$ we obtain

$$s = \frac{1}{a^2H}\int_{\widehat{x}_a}^{\widehat{x}_b} a\,|\widehat{x}|\,\sqrt{(a\widehat{x}+b)^2+1}\,a\,d\widehat{x}\,.$$

If $\widehat{x}$ changes its sign, i.e., the arc contains the cusp, the arc has to be split at this point.

Assuming $\widehat{x} > 0$ and carrying out the substitutions $a\widehat{x} + b = \sinh u$, we get because of $a\,d\widehat{x} = \cosh u\,du$

$$s = \frac{1}{a^2H}\int_{\widehat{x}_a}^{\widehat{x}_b} (\sinh u - b)\cosh^2 u\,du\,,$$

from which follows

$$s = \frac{1}{a^2H}\left[\frac{((a\widehat{x}+b)^2+1)^{\frac{3}{2}}}{3} - \frac{b}{2}\left((a\widehat{x}+b)\left((a\widehat{x}+b)^2+1\right)^{\frac{1}{2}} + \operatorname{arsinh}(a\widehat{x}+b)\right)\right]_{\widehat{x}_a}^{\widehat{x}_b}.$$

In order to find the parametric cubic curve, we reparametrise the hodograph as we did in Subsection 4.1. Thus we obtain the coordinate functions

$$\widehat{x}(u) = Hu, \quad \widehat{y}(u) = a(Hu)^2 + bHu.$$

We determine the parameter functions of the curve by indefinite integration in a new coordinate system $(\tilde{x}, \tilde{y})$, which is obtained from the system $(\widehat{x}, \widehat{y})$ by a translation. In this manner we get

$$\tilde{x}(u) = \frac{H}{2} u^2 + \tilde{x}_0 \quad \text{and} \quad \tilde{y}(u) = \frac{aH^2}{3} u^3 + \frac{bH}{2} u^2 + \tilde{y}_0 \, .$$

We omit the integration constants $\tilde{x}_0, \tilde{y}_0$, i.e., we shift the cusp to the origin.

If $b = 0$, the cubic curve is an O'Neill's parabola, since the parameter transformation $u = \sqrt{2/H}\, v$ gives

$$\tilde{x}_N = v^2, \quad \tilde{y}_N = \frac{a\sqrt{8H}}{3} v^3,$$

which is the parametric form of O'Neill's parabola.

The case $b \neq 0$ can be obtained from the previous one by using the transformation

$$\tilde{x} = \tilde{x}_N \, , \quad \tilde{y} = \tilde{y}_N + b\tilde{x}_N \, ,$$

i.e., by means of an affine elation. The axis of the elation (shearing transformation) is the $\tilde{y}$-axis of the coordinate system, and its ratio is b (see Fig. 5).

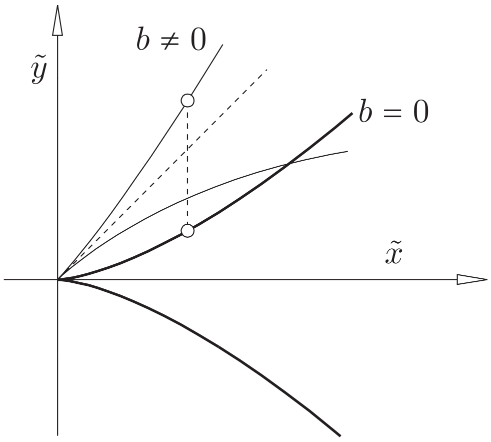

Figure 5: Cuspidal cubics

5. Consequences to the control points of the Bézier curve

In this section we answer the question how one can determine whether the arc length of a parametric cubic curve has a closed form if only the control points of the curve's Bézier representation are known.

5.1. Case of a complete square

Theorem 1 *The hodograph of form* (8) *is the intersection of the plane* $\widehat{z} = \widehat{z}_0$ *with the paraboloid of revolution* $\widehat{x}^2 + \widehat{z}^2 = 2p\left(\widehat{y} + \frac{p}{2}\right)$, *whose axis is* $\widehat{y}$, *parameter is* p *and focus is at the origin. Furthermore, at any point of the hodograph the angle between the position vector and the tangent is equal to the angle between the tangent and the axis of the paraboloid.*

Proof: According to expression (8), any position vector of the hodograph is of the form $\mathbf{r} = \left[\widehat{x}, \dfrac{\widehat{x}^2}{2p} + c, \widehat{z}_0\right]^T$. Its norm is $|\mathbf{r}| = \left|\dfrac{\widehat{x}^2}{2p} + c + p\right| = \widehat{y} + p$, thus the hodograph is on the paraboloid of revolution specified above. At any point of the paraboloid of revolution the position vector and the line passing through the point and parallel to the axis are symmetric with respect to the tangent plane, consequently the angle between these lines and the hodograph's tangent line, lying in the tangent plane, are equal. □

Now we prove a converse of the previous theorem.

Theorem 2 *If, at two points of a parabola, the angles between the position vectors and the tangent lines are equal to the angles between the tangent lines and the axis of the parabola, respectively, and if moreover the absolute value of the difference of the position vectors' norms is equal to the perpendicular projection of the chord (joining the points) onto the parabola's axis, then there exists a coordinate system, whose origin coincides with the given coordinate system's origin, in which the parabola's equations are of the form* (8).

Proof: Firstly we show: If the position vectors satisfy the conditions then the origin is on the focal conic of the given parabola.

Let us denote the position vectors of the parabola-arc by $\mathbf{a}_0$ and $\mathbf{a}_2$, and the position vector of the tangents' intersection point by $\mathbf{a}_1$ (this notation refers to the hodograph of the Bézier curve), as in Fig. 6. The direction of the parabola's axis is $\mathbf{u} = \frac{1}{2}(\mathbf{a}_2 + \mathbf{a}_0) - \mathbf{a}_1$. In order to find the locus of the required origin, we first take into consideration the angular equalities. If we take lines through the given points parallel to the axis and rotate them about the tangent lines respectively, we obtain two cones of revolution. The intersection of these cones is the locus of points whose position vectors satisfy the given conditions. The point $\mathbf{a}_1$ bisects the distance between the lines taken through the points $\mathbf{a}_0$, $\mathbf{a}_2$ and parallel to the axis. Thus the two cones have an inscribed sphere with the centre $\mathbf{a}_1$ in common. Therefore, the intersection of the cones splits into two conics. One part is a parabola due to the parallel outline generators, the other part, drawn in dashed line in Fig. 6, is an ellipse. The non-parallel outlines of the constructed cones are intersecting at the point F, which is the focus of the given parabola. Let us mirror this focus with respect to the tangent lines, through the foot points D_0 and D_2, and denote their reflections, which are on the directrix of the given parabola, respectively by F_0 and F_2. The foot points D_0 and D_2 are on the tangents of the given parabola drawn at its vertex, and these are the centres of Dandelin's spheres of the intersected parabola. For this reason, the focus of the intersected parabola is the vertex C of the given parabola, the vertex of the intersected parabola is the focus F of the given parabola and the planes of these two parabolas are perpendicular, consequently the two parabolas are focal conics of each other.

If we choose a new origin O on the intersected parabola and rotate the position vectors of the points $\mathbf{a}_0$ and $\mathbf{a}_2$ respectively about the tangent lines into the image plane, we obtain the points O_0 and O_2. The triangles OO_0O_2 and FF_0F_2 are congruent with parallel corresponding sides, thus the difference between the magnitude of the position vectors of $\mathbf{a}_0$ and $\mathbf{a}_2$ is equal to the perpendicular projection of the chord $\mathbf{a}_0$, $\mathbf{a}_2$ on the axis of the given parabola. If we have chosen a point O on the intersected ellipse, it would not share this property.

We set up a coordinate system whose origin O is on the intersected parabola, $\widehat{y}$-axis is parallel to $\mathbf{u}$ and $\widehat{x}$-axis is parallel to the plane of the hodograph. In this system, if we denote the $\widehat{y}$-component of the hodograph's vertex by c and its $\widehat{z}$-component by $\widehat{z}_0$, we obtain $c^2 + \widehat{z}_0^2 = (c+p)^2$, i.e., the equations of the hodograph are of the form (8). □

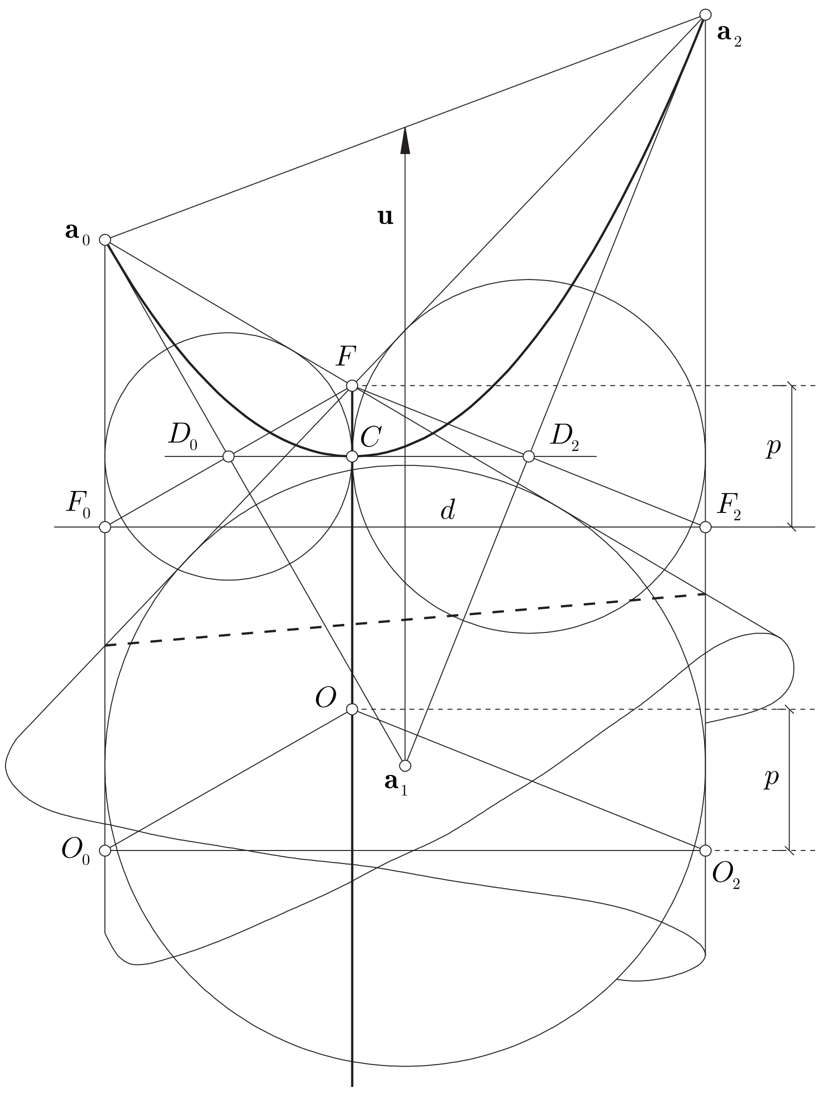

Figure 6:

In [2] one can find a different criterion for the Bézier points of Pythagorean-hodograph twisted cubics, which are extended in [10] in order to construct G^1 interpolation to given points and tangent directions.

5.2. Case of a common quadratic factor

As we have shown in Subsection 3.2, in this case the hodograph passes through the origin. The hodograph, i.e., a parabola, is given by the control points of its Bézier representation. An option to determine whether such a parabola passes through the origin is to convert the curve into implicit form.

Introducing the notations

$$\mathbf{e}_2 := \mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0, \quad \mathbf{e}_1 := 2\mathbf{b}_2 - 4\mathbf{b}_1 + 2\mathbf{b}_0 \quad \text{and} \quad \mathbf{e}_0 := \mathbf{b}_1 - \mathbf{b}_0\,,$$

the implicit form of the hodograph is

$$e_{2y}^2 x^2 - 2e_{2x}e_{2y}xy + e_{2x}^2 y^2 + (-2e_{0x}e_{2y}^2 + e_{1x}e_{1y}e_{2y} - e_{2x}e_{1y}^2 + 2e_{2x}e_{0y}e_{2y})x +$$

$$+(-2e_{0y}e_{2x}^2 + e_{1y}e_{1x}e_{2x} - e_{2y}e_{1x}^2 + 2e_{2y}e_{0x}e_{2x})y + \det \begin{bmatrix} e_{2x} & e_{1x} & e_{0x} & 0 \\ 0 & e_{2x} & e_{1x} & e_{0x} \\ e_{2y} & e_{1y} & e_{0y} & 0 \\ 0 & e_{2y} & e_{1y} & e_{0y} \end{bmatrix} = 0$$

as shown in [8]. This equation has to be satisfied by the point $[0,0]^T$, i.e., the constant term has to vanish.

5.3. Summary of conditions

Now we summarise how one can decide whether a parametric cubic curve's arc length has a closed form or not, provided the control points of its Bézier representation are given.

Let $\mathbf{b}_i$, $i = 0, \ldots, 3$, be the control points of the Bézier curve, and $\mathbf{a}_i := \mathbf{b}_{i+1} - \mathbf{b}_i$, $i = 0,1,2$. *The arc length of this Bézier curve has a closed form if and only if one of the following conditions is fulfilled:*

1. the control points are collinear (the Bézier curve is degenerated into a straight line segment);
2. $\mathbf{b}_3 - \mathbf{b}_0 = 3(\mathbf{b}_2 - \mathbf{b}_1)$ (the cubic Bézier curve is degenerated into a quadratic parabola);
3. denoting the direction of the hodograph's axis by $\mathbf{u} = \frac{1}{2}(\mathbf{a}_2 + \mathbf{a}_0) - \mathbf{a}_1$, the equalities

$$\begin{aligned} \sphericalangle(\mathbf{a}_0 - \mathbf{a}_1, \mathbf{a}_0) &= \sphericalangle(\mathbf{a}_0 - \mathbf{a}_1, \mathbf{u}), \\ \sphericalangle(\mathbf{a}_2 - \mathbf{a}_1, \mathbf{a}_2) &= \sphericalangle(\mathbf{a}_2 - \mathbf{a}_1, \mathbf{u}), \\ \left| |\mathbf{a}_2| - |\mathbf{a}_0| \right| &= \left| (\mathbf{a}_2 - \mathbf{a}_0) \cdot \mathbf{u} / |\mathbf{u}| \right| \end{aligned}$$

   are valid (the hodograph is of the form (8));
4. for the components of the vectors $\mathbf{e}_2 := \mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0$, $\mathbf{e}_1 := 2\mathbf{b}_2 - 4\mathbf{b}_1 + 2\mathbf{b}_0$ and $\mathbf{e}_0 := \mathbf{b}_1 - \mathbf{b}_0$ the equality

$$\det \begin{bmatrix} e_{2x} & e_{1x} & e_{0x} & 0 \\ 0 & e_{2x} & e_{1x} & e_{0x} \\ e_{2y} & e_{1y} & e_{0y} & 0 \\ 0 & e_{2y} & e_{1y} & e_{0y} \end{bmatrix} = 0$$

   holds, (the parabola of the hodograph passes through the origin, i.e., the parametric cubic is a cuspidal plane curve).

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